

HOLOMORPHIC AUTOMORPHISMS OF DANIELEWSKI SURFACES I — DENSITY OF THE GROUP OF OVERSHEARS

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ABSTRACT. We define the notion of shears and overshears on a Danielewski surface. We show that the group generated by shears and overshears is dense (in the compact open topology) in the path-connected component of the identity of the holomorphic automorphism group.

1. INTRODUCTION

In the early 90's, Andersén and Lempert started intensive studies of the automorphisms group $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ of \mathbb{C}^n , [3] and [4]. They showed (thus answering a question by Rosay and Rudin [18]) that the subgroup generated by shears and overshears $\text{OS}(\mathbb{C}^n)$ is dense in the holomorphic automorphism group $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$, although it is a proper subgroup of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ in all dimensions $n \geq 2$. A special feature in dimension 2 is that the group $\text{OS}(\mathbb{C}^2)$ possesses the structure of a free amalgamated product of the affine automorphisms and of the Jonquiere automorphisms (proved by Ahern and Rudin [2]). This structure theorem is analogous to the classical structure theorem for the polynomial automorphism group of \mathbb{C}^2 due to van der Kulk [19] and Jung [10].

In recent years Varolin, [20], extended the notion of shears and generalized the Andersén-Lempert theory for automorphisms of \mathbb{C}^n to a larger class of Stein manifolds, namely manifolds with the density property; see [21] and [22]. Continuing Varolin's research, the first author and Kaliman ([11] and [12]) showed that this class of manifolds is quite large and that it is an important source of examples concerning many challenging problems on affine n -space; see also the overview article [13].

The present paper deals with a class of affine algebraic manifolds which is well known in affine algebraic geometry but has received relatively little attention from the complex analytic point of view. It is the class of Danielewski surfaces, which are hypersurfaces in $\mathbb{C}_{(x,y,z)}^3$ defined by $D_p := \{xy = p(z)\}$, where $p \in \mathbb{C}[z]$ is a polynomial of degree at least 2 and with simple zeros (if the degree of p equals 1 the Danielewski surface is obviously just \mathbb{C}^2). They have been introduced by Danielewski (see [6]) in connection with the cancellation property. Namely, the

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surfaces have the following remarkable property: If we denote by D_p^n the hypersurfaces in $\mathbb{C}_{(x,y,z)}^3$ defined by $D_p^n := \{x^n y = p(z)\}$, then these manifolds, for different n , are not homeomorphic, [8], but their products with the affine line \mathbb{C} are algebraically isomorphic. Their polynomial automorphism groups have been determined by Makar-Limanov [16] and [17]. It turns out that the polynomial automorphism groups of D_p^n are small for $n \geq 2$, but for $n = 1$ their automorphism groups are very similar to the polynomial automorphism group of \mathbb{C}^2 .

In this paper we concentrate on the latter ones with $n = 1$ and show that it is possible to define a notion of shears and overshers on these surfaces whose behavior is analogous to the behavior of the overshers on \mathbb{C}^2 . Our main result is the following generalization of the main result of Andersén-Lempert theory of \mathbb{C}^n to Danielewski surfaces

Theorem 1.1. *The group $\text{OS}(D_p)$ generated by overshers is dense in the path component of the identity of the holomorphic automorphism group $\text{Aut}(D_p)$ of a Danielewski surface D_p .*

It is already known that all D_p have the density property (see [11], Theorem 1). In that proof so-called hyperbolic vector fields are used, which in the case of \mathbb{C}^2 are known to be (and so far are the only known) concrete examples of holomorphic automorphisms which are not contained in the group generated by overshers (see [15], [3] and [4]). Thus the main part of our proof consists in proving the fact that the corresponding hyperbolic fields are contained in the Lie algebra generated by overshers fields; see Proposition 3.9.

This paper is organized as follows: In section 2, we define shears and overshers on Danielewski surfaces and recall some important results from [11] which will be used further on. In section 3, we construct some explicit Lie brackets of overshers fields which lead to certain hyperbolic fields. These are used to formulate the missing part in the proof of the main theorem as a purely algebraic problem. In the same section, 3, we solve this problem. The proof of the main theorem comes in section 4. In the last section, section 5, we comment on the number of connected components of $\text{Aut}(D_p)$ and formulate two questions.

In a forthcoming paper the authors will prove a structure theorem for the group $\text{OS}(D_p)$ in the spirit of the classical van-der Kulk-Jung theorem and analogously to the structure theorem for the algebraic automorphism group of D_p indicated by Makar-Limanov in [16]. A similar result for \mathbb{C}^2 has been proven by Ahern and Rudin in [2].

2. THE DEFINITION OF SHEARS AND OVERSHERS

Recall that a holomorphic vector field $V \in \text{VF}_{\text{hol}}(X)$, on a complex manifold X , is complete (or globally integrable) if for any initial value $z \in X$ there is a global holomorphic solution of the ordinary differential equation

$$(2.1) \quad \dot{\gamma}(t) = V(\gamma(t)), \quad \gamma(0) = z.$$

In this case, the phase flow (i.e. the map $\mathbb{C} \times X \rightarrow X$ given by $(t, z) \mapsto \gamma_z(t)$) is a holomorphic action of the additive group \mathbb{C}_+ on X , where the subscript z in γ_z denotes the dependence on the initial value. It is worth mentioning that this action is not necessarily algebraic in the case of an algebraic vector field $V \in \text{VF}_{\text{alg}}(X)$ on an affine algebraic manifold X .

We will consider globally integrable holomorphic vector fields on Danielewski surfaces of a very special form, called shear and overshear fields. There are more globally integrable holomorphic vector fields on Danielewski surfaces. To determine all of them (say up to conjugation by holomorphic automorphisms) is a very difficult task. Even in the simplest case, \mathbb{C}^2 , this seems almost out of reach. It is for instance not known whether a globally integrable holomorphic vector field on \mathbb{C}^2 can have more than one isolated zero.

From now on let $p \in \mathbb{C}[z]$ be a polynomial with simple zeros. Consider the Danielewski surface

$$D_p = \{(x, y, z) \in \mathbb{C}^3 : xy = p(z)\}.$$

Since p has only simple zeros, the surface D_p is smooth and is therefore a smooth affine algebraic hypersurface in \mathbb{C}^3 . Furthermore, D_p is an affine modification of $\mathbb{C}_{x,z}^2$ along the divisor $D \cong \mathbb{C} = \{x = 0\}$ with center $C_x = \{x = 0, p(z) = 0\}$. By symmetry in x and y it is also an affine modification of $\mathbb{C}_{y,z}^2$ along the divisor $D \cong \mathbb{C} = \{y = 0\}$ with center $C_y = \{y = 0, p(z) = 0\}$. Thus D_p contains two copies of $\mathbb{C}^2 \setminus D \cong \mathbb{C} \times \mathbb{C}^*$ as Zariski open subsets. Our definition of shears and overshears on D_p will be such that we consider shears and overshears on these subsets (defined as usual shears and overshears on $\mathbb{C} \times \mathbb{C}^*$) and impose conditions on the functions involved so that our shears and overshears extend holomorphically (not only meromorphically) to D_p :

Consider $V = zf(x)\frac{\partial}{\partial z}$, $f \in \mathcal{O}(\mathbb{C})$, which is an overshear field on $\mathbb{C}_x^* \times \mathbb{C}_z$. Define a biholomorphic mapping $\pi: \mathbb{C}^* \times \mathbb{C} \rightarrow D_p^* := D_p \setminus \{x = 0\}$ by

$$\pi(x, z) = \left(x, \frac{p(z)}{x}, z\right).$$

Pushing forward V to D_p^* via π gives us a complete vector field on D_p^* , since V is complete and π is biholomorphic. Determine the flow φ_t of $\pi_*(V)$, and consider the time-one map φ_1 . This is now a mapping $D_p^* \rightarrow D_p^*$. It extends to a mapping φ_1 on D_p iff $f(0) = 0$. Thus we have to consider overshear fields of the form $V = zxf(x)\frac{\partial}{\partial z}$, and these give us the definition of an overshear on D_p :

$$(2.2) \quad O_f(x, y, z) = \left(x, y + \frac{p(ze^{xf(x)}) - p(z)}{x}, ze^{xf(x)}\right), f \in \mathcal{O}(\mathbb{C}).$$

The same procedure with the shear field $xf(x)\frac{\partial}{\partial z}$ (also here the corresponding coefficient is divisible by x to make the field extend holomorphically to D_p) produces a shear on D_p defined by

$$(2.3) \quad S_f(x, y, z) = \left(x, y + \frac{p(z + xf(x)) - p(z)}{x}, z + xf(x)\right), f \in \mathcal{O}(\mathbb{C}).$$

Remark 1. Polynomial shears on D_p have been considered previously by Makar-Limanov [16], where he determined the polynomial automorphism group of D_p in terms of generators. In that paper, polynomial shears are called triangular mappings and correspond to our S_f in the case where $f \in \mathbb{C}[z]$ is a polynomial. They are time-one maps of locally nilpotent derivations. Our shears can thus be viewed as time-one maps of “locally nilpotent derivations with holomorphic coefficients”. Other automorphisms that participate in the generation of the group $\text{Aut}_{\text{pol}}(D_p)$ are the involution $I(x, y, z) = (y, x, z)$ and hyperbolic mappings $H_\lambda(x, y, z) = (\lambda x, \lambda^{-1}y, z)$, where $\lambda \in \mathbb{C}^*$. \square

Remark 2. In the holomorphic case, we have more hyperbolic mappings. In fact, if $f \in \mathcal{O}(\mathbb{C})$ is an arbitrary holomorphic function of one variable, then we have hyperbolic mappings of the form $H_f(x, y, z) = (xe^{f(z)}, ye^{-f(z)}, z)$. These H_f are holomorphic automorphisms of D_p . We will show in a forthcoming paper that these maps are not contained in the overshear group $OS(D_p)$ (see the definition below), in spite of the fact that their corresponding vector fields are Lie combinations of the shears fields, as we prove in Proposition 3.9 below. There is no contradiction between these two facts, since being a Lie combination of shear fields implies an approximation result in the compact-open topology only. \square

As mentioned before, there is a second way to view D_p as an affine modification of \mathbb{C}^2 , namely, by interchanging x and y . This interchange leads to maps of the form $IO_f I$, where I is the involution automorphism from Remark 1 above.

Definition 2.1. The overshear group $OS(D_p)$ on the Danielewski surface D_p is the group generated by the maps O_f, S_g and $IO_f I, IS_g I$, where $f, g \in \mathcal{O}(\mathbb{C})$ are arbitrary holomorphic functions of one variable and I is the involution $I(x, y, z) = (y, x, z)$.

Remark 3. If p is a linear polynomial, e.g. $p(z) = z$, the Danielewski surface D_p is isomorphic to \mathbb{C}^2 with coordinates x and y . The shears and overshears on D_p defined in equations (2.2) and (2.3) are in this case exactly the usual shears and overshears on \mathbb{C}^2 along the coordinate lines. The group $OS(D_p)$ in this case is exactly that used by Andersén and Lempert, and thus our main result includes their ground-breaking result from [4]. \square

We will use the following notation for the vector fields whose time-one maps are our shears and overshears:

- (V1) $OF_i^x := zp'(z)x^i \frac{\partial}{\partial y} + zx^{i+1} \frac{\partial}{\partial z}$ (Overshear field)
- (V2) $OF_i^y := zp'(z)y^i \frac{\partial}{\partial x} + zy^{i+1} \frac{\partial}{\partial z}$ (Overshear field)
- (V3) $SF_i^x := p'(z)x^i \frac{\partial}{\partial y} + x^{i+1} \frac{\partial}{\partial z}$ (Shear field)
- (V4) $SF_i^y := p'(z)y^i \frac{\partial}{\partial x} + y^{i+1} \frac{\partial}{\partial z}$ (Shear field)
- (V5) $HF_f := f(z) \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right)$ (Hyperbolic field)

where $i \geq 0$ is an integer and $f \in \mathbb{C}[z]$.

For simplicity of notation we will write SF^x and SF^y for SF_0^x and SF_0^y respectively from now on. The same notation applies to overshear fields.

The algebraic density property for Danielewski surfaces was established by Kaliman and the second author in [11]; see Proposition 2.3. For the convenience of the reader let us recall the definition due to Varolin:

Definition 2.2. A complex manifold X has the density property if, in the compact-open topology, the Lie algebra $\text{Lie}_{\text{hol}}(X)$ generated by globally integrable holomorphic vector fields on X is dense in the Lie algebra $\text{VF}_{\text{hol}}(X)$ of all holomorphic vector fields on X . An affine algebraic manifold has the algebraic density property if the Lie algebra $\text{Lie}_{\text{alg}}(X)$ generated by globally integrable algebraic vector fields on X coincides with the Lie algebra $\text{VF}_{\text{alg}}(X)$ of all algebraic vector fields on X .

Clearly the algebraic density property implies the density property for any affine algebraic manifold X . Indeed, since the tangent sheaf of an affine algebraic manifold X can be generated by a finite number of global algebraic sections, Oka-Weill approximation shows that the algebraic density property for such X implies the density property for X ; in particular, this holds for Danielewski surfaces.

In [11] the following is proven (see Proposition 2.3 and its proof):

Proposition 2.3. *D_p has the algebraic density property. More precisely: Any algebraic vector field on D_p is a Lie combination of (finitely many) vector fields of the form (V1) – (V5) above, i.e., of shear fields, overshear fields and hyperbolic fields.*

Corollary 2.4. *If we can write any hyperbolic vector field as a Lie combination of shear and overshear vector fields, then any algebraic vector field can be written as a Lie combination of shear and overshear vector fields.*

3. HYPERBOLIC FIELDS ARE LIE COMBINATIONS OF SHEARS

In this section we will prove that all hyperbolic fields HF_f , $f \in \mathbb{C}[z]$, are Lie combinations of the fields $SF_i^x, SF_i^y, OF_i^x, OF_i^y$.

Straightforward calculations show:

Lemma 3.1. *For every integer $i \geq 0$,*

- (a) $[SF_i^x, SF_i^y] = HF_{(p^{i+1})''/(i+1)}$,
- (b) $[SF_i^x, OF_i^y] - [SF_i^y, OF_i^x] = HF_{2z(p^{i+1})''/(i+1)+p^i p^i}$,
- (c) $[OF_i^x, OF_i^y] = HF_{(z^2(p^{i+1})''+z(p^{i+1})')/(i+1)}$.

To make calculations for more sophisticated brackets easier, let us remember from [11] that there is a unique (up to a constant) algebraic volume form on ω on D_p . For every volume-preserving (algebraic) vector field, θ holds (since ω is a closed form):

$$0 = L_\omega(\theta) = di_\theta\omega + i_\theta d\omega = di_\theta\omega.$$

Since D_p is simply connected we get $i_\theta\omega = df$ for some function $f \in \mathbb{C}[D_p]$. Thus we have a bijection between algebraic volume preserving fields and polynomial functions modulo constants.

Shear fields and hyperbolic fields are volume preserving. Using the fact that in the coordinates $(x, z) \in \mathbb{C}^* \times \mathbb{C}$ the form ω equals $\frac{dx \wedge dz}{x}$ (thus this expression extends regularly to D_p from $\mathbb{C} \times \mathbb{C}^*$), one easily calculates the functions corresponding to shear fields and hyperbolic fields:

Lemma 3.2. *The following holds for all $f \in \mathcal{O}(\mathbb{C})$:*

$$i_{SF^x}\omega = -dx, \quad i_{SF^y}\omega = dy, \quad i_{HF_{f'(z)}}\omega = df(z).$$

The following is useful for calculations of brackets.

Lemma 3.3. *If θ is a volume-preserving field with $i_\theta\omega = df$ and ψ is another volume preserving field, then*

$$i_{[\psi, \theta]}\omega = dL_\psi(f).$$

The following lemma produces our important Lie combinations.

Lemma 3.4. *If f_1, f_2, \dots, f_k are finitely many holomorphic functions of the variable z , then*

- (1a) $[HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^x]] \dots]] = -xHF_{(\prod f_\alpha)'} + (\prod f_\alpha)SF^x,$
- (1b) $[HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^y]] \dots]] = (-1)^k \{yHF_{(\prod f_\alpha)'} + (\prod f_\alpha)SF^y\},$
- (2a) $[SF^y, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^x]] \dots]] = HF_{-(p \prod f_\alpha)''},$
- (2b) $[SF^x, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^y]] \dots]] = HF_{(-1)^k (p \prod f_\alpha)''},$
- (3a) $[SF^y, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, OF^x]] \dots]] = HF_{-z(p \prod f_\alpha)'' - p(\prod f_\alpha)'} + y(\prod f_\alpha)SF^x,$
- (3b) $[SF^x, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, OF^y]] \dots]] = HF_{(-1)^k z(p \prod f_\alpha)'' + (-1)^k p(\prod f_\alpha)'} + (-1)^k x(\prod f_\alpha)SF^y,$
- (3c) $(-1)^k [SF^x, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, OF^y]] \dots]] - [SF^y, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, OF^x]] \dots]] = HF_{2z(p \prod f_\alpha)'' + 2p(\prod f_\alpha)' + p'(\prod f_\alpha)},$
- (4) $[OF^y, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, OF^x]] \dots]] = HF_{-z^2(p \prod f_\alpha)'' - z(p \prod f_\alpha)'},$
- (5) $(-1)^k [OF^x, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^y]] \dots]] - [OF^y, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^x]] \dots]] = HF_{2z(p \prod f_\alpha)'' + p'(\prod f_\alpha)}.$

Proof. (1) By Lemma 3.2 we have $i_{SF^x}\omega = -dx$. Denote by β the left hand side of (1a). Lemma 3.3 gives

$$\begin{aligned} i_\beta\omega &= -d(L_{HF_{f_k}} \dots L_{HF_{f_1}}(x)) = -d(f_k(z) \dots f_1(z)x) \\ &= -xd \left(\prod f_\alpha \right) - \left(\prod f_\alpha \right) dx. \end{aligned}$$

Thus by Lemma 3.2, $\beta = -xHF_{(\prod f_\alpha)'} + (\prod f_\alpha)SF^x$, which is (1a). The proof of (1b) is similar.

(2) Denote by γ the left hand side of (2a). By the same logic as above

$$\begin{aligned} i_\gamma\omega &= -d(L_{SF^y} \cdot L_{HF_{f_k}} \dots L_{HF_{f_1}}(x)) = -d \left(L_{SF^y} \left(x \left(\prod f_\alpha \right) \right) \right) \\ &= -d \left(xy \left(\prod f_\alpha \right)' + p' \left(\prod f_\alpha \right) \right) = -d \left(p \prod f_\alpha \right)', \end{aligned}$$

using the fact that $xy = p$. Again Lemma 3.2 yields $\gamma = HF_{-(p \prod f_\alpha)'}$, which is (2a). The proof of (2b) is similar.

(3) Since $OF^x = zSF^x$ and the function z is in the kernel of hyperbolic fields, we get

$$\begin{aligned} &[SF^y, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, OF^x]] \dots]] \\ &= [SF^y, z[HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^x]] \dots]]. \end{aligned}$$

Using (2a) and (1a) the above equation equals

$$\begin{aligned} &zHF_{-(p \prod f_\alpha)''} + SF^y(z) (-xHF_{(\prod f_\alpha)'} + (\prod f_\alpha)SF^x) \\ &= HF_{-z(p \prod f_\alpha)''} + y (-xHF_{(\prod f_\alpha)'} + (\prod f_\alpha)SF^x), \end{aligned}$$

which using $p = xy$ again is (3a). (3b) is proved the same way. (3c) follows immediately from (3a) and (3b) using the fact that $xSF^y - ySF^x = HF_{p'}$.

(4) Since z is in the kernel of all hyperbolic fields, we get

$$\begin{aligned} & [OF^y, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, OF^x]] \dots]] \\ &= [zSF^y, z[HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^x]] \dots]] \\ &= z^2[SF^y, [HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^x]] \dots]] \\ &+ zSF^y(z)[HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^x]] \dots]] \\ &- z[HF_{f_k}, [HF_{f_{k-1}}, [\dots, [HF_{f_1}, SF^x]] \dots]](z)SF^y. \end{aligned}$$

Now one applies (1a) and finally uses the fact that $xSF^y - ySF^x = HF_{p'}$ to get (4).

(5) is proved in the same manner as (3); we leave it as an exercise for the reader. □

These brackets give us concrete ways to produce hyperbolic fields as Lie combinations of shear and overshear fields. We can now reformulate our problem using Lemma 3.1 and Lemma 3.4(2a), (3c), (4) and (5) subtracted from (3c) as follows:

Let $W \subset \mathbb{C}[z]$ be a vector subspace such that

- (C1) (i) $(p^{i+1})''$, (ii) $2z(p^{i+1})'' + (p^{i+1})'$, and (iii) $z^2(p^{i+1})'' + z(p^{i+1})' \in W$ for each integer $i \geq 0$.

If finitely many functions $f_1, f_2, \dots, f_k \in W$, then

- (C2) $(p \prod_{\alpha=1}^k f_{\alpha})'' \in W$,
- (C3) $2z(p \prod f_{\alpha})'' + 2p(\prod f_{\alpha})' + p'(\prod f_{\alpha}) \in W$,
- (C4) $z^2(p \prod f_{\alpha})'' + z(p \prod f_{\alpha})' \in W$ and
- (C5) $p(\prod f_{\alpha})' \in W$.

Show that $W = \mathbb{C}[z]$.

The strategy of our proof is first to show that the algebra \mathcal{A}_W generated by the elements of a vector subspace W satisfying properties (C1) to (C5) is equal to $\mathbb{C}[z]$.

Lemma 3.5. *There is no point $a \in \mathbb{C}$ such that all elements in \mathcal{A}_W have a common zero at a .*

Proof. Suppose to the contrary that such a point a exists. Then $p''(a) = 0$, from (C1)(i). Furthermore, using (C1)(ii), we get $2ap''(a) + p'(a) = 0$. Thus $p'(a) = 0$. If the degree of p is one, this yields a contradiction. From now on let $n \geq 2$ be the degree of p . Since p and p' have no common zeros, $p(a) \neq 0$. Define inductively functions $f_k \in \mathbb{C}[z]$ by $f_0 = p''$ and $f_{k+1} = pf'_k$. By (C5) they are in W . Since $p(a) \neq 0$ we have $f'_k(a) = 0$ for all k . Using $f'_k = p'f'_{k-1} + pf''_{k-1}$ we find $f''_{k-1}(a) = 0$ for all k . Continuing by induction we find that all derivatives of the functions f_k at the point a are zero, which is absurd since $f_0^{(n-2)} = p^{(n)}$ is a nonzero constant. □

Lemma 3.6. *There is no point $b \in \mathbb{C}$ such that the derivatives of all elements in \mathcal{A}_W have a common zero at b .*

Proof. Assume that such b exists. Since $p'' \in W$ by (C1)(i), we get that $p'''(b) = 0$. From (C1)(ii) we get $(2zp''(z) + p'(z))'(b) = 0$, which implies that $p''(b) = 0$. Furthermore, (C1)(iii) gives that $(z^2p''(z) + zp'(z))'(b) = 0$, which implies that $p'(b) = 0$. Thus $p(b) \neq 0$, since p only has simple zeros. Define again inductively functions $f_k \in \mathbb{C}[z]$ by $f_0 = p''$ and $f_{k+1} = pf'_k$. By (C5) they are in W . Since $p(b) \neq 0$ and $f'_k = p'f'_{k-1} + pf''_{k-1}$, we find $f''_{k-1}(b) = 0$ for all k .

Now $f_k'' = p' f_{k-1}'' + p f_{k-1}''' + p'' f_{k-1}'$, so again $p(b) \neq 0$ implies $f_{k-1}'''(b) = 0$ for all k . Continuing inductively with higher derivatives we find that all derivatives of the functions f_k at the point b are zero, which is a contradiction since $f_0^{(n-2)} = p^{(n)}$ is a nonzero constant. \square

Lemma 3.7. *There is no pair of points $c_1 \neq c_2 \in \mathbb{C}$ such that all elements of \mathcal{A}_W take the same values at c_1 and c_2 .*

Proof. Suppose to the contrary that such a pair of points exists.

Step 1:

By (C1)(i) we have $p''(c_1) = p''(c_2) =: A$. We get from (C1)(ii), (iii) for $i = 0$ that

$$\begin{aligned} 2(c_1 - c_2)A &= p'(c_2) - p'(c_1), \\ (c_1^2 - c_2^2)A &= c_2 p'(c_2) - c_1 p'(c_1). \end{aligned}$$

Eliminating A from these equations we find $(c_1 - c_2)(p'(c_1) + p'(c_2)) = 0$. Hence $-p'(c_1) = p'(c_2)$, and using this fact in the first equation yields

$$p'(c_2) = A(c_1 - c_2).$$

Step 2:

Suppose for a moment that $A \neq 0$. Property (C1)(i) with $i = 1$ implies that $(p^2)'' = 2(p')^2 + 2pp''$ takes the same value B at c_1, c_2 . Thus $p(c_1) = p(c_2)$ by virtue of $A \neq 0$. Using (C1)(ii) with $i = 1$: $2c_1 B + p(c_1)p'(c_1) = 2c_2 B + p(c_2)p'(c_2)$; thus $2B(c_1 - c_2) = 2p'(c_2)(p(c_1) + p(c_2)) = 2A(c_1 - c_2)2p(c_2)$ and thus $B = 2Ap(c_2)$. Remembering the definitions of A and B , this yields $2pp'' + 2(p')^2 = 2p''p$ at c_2 . Thus $p'(c_2) = 0$, which implies that $A = 0$. Thus our assumption that $A \neq 0$ led to $A = 0$ and therefore $p''(c_1) = p''(c_2) = p'(c_1) = p'(c_2) = 0$.

Step 3:

As in Step 1 we use the fact (following from (C2), (C3) minus (C5), (C4) respectively) that for any $f \in W$ the functions

$$(pf)'', \quad 2z(pf)'' + (pf)', \quad z^2(pf)'' + z(pf)'$$

take the same values at c_1, c_2 to conclude that $(pf)'(c_1) = -(pf)'(c_2)$ for all $f \in W$. Since p' vanishes at these points we get $p(c_1)f'(c_1) = -p(c_2)f'(c_2)$. By (C5) $p(c_1)f'(c_1) = p(c_2)f'(c_2)$. Thus for all $f \in W$ there holds $p(c_1)f'(c_1) = 0$. By Lemma 3.6 this implies $p(c_1) = 0$, contradicting $p'(c_1) = 0$. \square

Proposition 3.8. *The subalgebra \mathcal{A}_W of $\mathbb{C}[z]$ generated by W coincides with $\mathbb{C}[z]$.*

Proof. Lemmas 3.6 and 3.7 yield the existence of a finite number of polynomials $q_1, q_2, \dots, q_K \in \mathcal{A}_W$ such that the map $\mathbb{C} \rightarrow \mathbb{C}^K$ given by

$$z \mapsto ((q_1(z), q_2(z), \dots, q_K(z)))$$

is an injective immersive embedding. Now the subalgebra generated by q_1, q_2, \dots, q_K and constants is isomorphic to $\mathbb{C}[z]$. Thus the subalgebra \mathcal{A}_0 generated by q_1, q_2, \dots, q_K (without constants) is an (in view of Lemma 3.6 maximal) ideal in $\mathbb{C}[z]$. Lemma 3.5 implies that $\mathcal{A}_0 = \mathbb{C}[z]$. Since by construction \mathcal{A}_0 contains \mathcal{A}_W , the proof is complete. \square

Proposition 3.9. $W = \mathbb{C}[z]$.

Proof. Since by Proposition 3.8 the subalgebra generated by elements of W is the whole polynomial ring, we get by (C2) – (C5) that the polynomials $(pg)''$, $2z(pg)'' + 2p(g)' + p'(g)$, $z^2(pg)'' + z(pg)'$, pg' are contained in W for all $g \in \mathbb{C}[z]$.

Looking at polynomials of the same degree $k + n - 2$, the following polynomials (for different g being powers of z) are contained in W :

$$(z^k p)'', \quad z^2(z^{(k-2)} p)'' + z(z^{(k-2)} p)', \quad z^{(k-2)} p.$$

The above polynomials are all linear combinations of the three polynomials $z^{(k-2)} p$, $z^{(k-1)} p'$ and $z^k p''$. Since the matrix of coefficients

$$\begin{bmatrix} k(k-1) & 2k & 1 \\ (k-2)^2 & 2k-3 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

has determinant 3, the polynomials $z^{k-2} p$, $z^{k-1} p'$ and $z^k p''$ are in W . This means (if we do it for all k) that the principal ideal (p) generated by p and the principal ideals (p') and (p'') are contained in W . Since p and p' have no common zeros, the Nullstellensatz implies that $W = \mathbb{C}[z]$. □

Thus we have an immediate consequence of Corollary 2.4 and Proposition 3.9, which is the core part of our main theorem:

Theorem 3.10. *Let D_p be the Danielewski surface with respect to a polynomial $p \in \mathbb{C}[z]$ with simple zeros. Then the Lie algebra generated by overshers equals the Lie algebra of all algebraic vector fields on D_p .*

4. THE DENSITY OF THE OVERSHEAR GROUP

To make the presentation self-contained we outline the proof of a version of the Andersén-Lempert theorem, Proposition 4.2 (see also Forstnerič-Rosay’s Theorem 1.1 in [9]). Let V be a holomorphic vector field on a complex manifold M . Assume that $K_t: M \rightarrow M$ is a family of holomorphic mappings which are \mathcal{C}^1 in $t \in [0, 1]$. Assume furthermore that $K_0 = \text{Id}$ and that $\frac{dK_t}{dt} \Big|_{t=0} = V$. Such a K is called a consistent algorithm for V . Assume that $\varphi_t(z)$ is the flow of V starting at z . Euler’s method gives

$$\lim_{N \rightarrow \infty} K_{t/N}^N(z) = \varphi_t(z),$$

locally uniformly on subsets of $\mathbb{R} \times M$, where either side is defined (see Theorem 2.1.26 in [1]).

Here are two consistent algorithms for the sum and for the Lie bracket of two vector fields; see [21].

Lemma 4.1. *Let V and W be two holomorphic vector fields on M with flows $\{\varphi_t\}$, respectively $\{\psi_t\}$. Then*

1. $\frac{d}{dt} \Big|_{t=0} \varphi_t \circ \psi_t = V + W,$
2. $\frac{d}{dt} \Big|_{t=0+} \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}} = [V, W].$

Proposition 4.2. *Assume that $\Omega \subset D_p$ is a domain. Let $\Phi: [0, 1] \times \Omega \rightarrow D_p$ be a \mathcal{C}^1 -family such that $\Phi_t: \Omega \rightarrow D_p$ is injective and holomorphic for each fixed*

$t \in [0, 1]$. For each time $t_0 \in [0, 1]$ and for $z \in \Phi_{t_0}(\Omega)$, define the time dependent vector field X_{t_0} by

$$X_{t_0}(z) := \left. \frac{d}{dt} \right|_{t=t_0} \Phi_t(\Phi_{t_0}^{-1}(z)).$$

Furthermore, assume that all vector fields X_{t_0} for $t_0 \in [0, 1]$ can be approximated (on compacts in their domain of definition) by (globally defined) polynomial vector fields on D_p .

If Φ_0 can be approximated by elements in $\text{OS}(D_p)$, then Φ_1 can also be approximated by elements in $\text{OS}(D_p)$.

Remark 4. By approximation, we mean uniform approximation on compact subsets. \square

Proof of Proposition 4.2. A first observation is that $\Phi_1 \circ \Phi_0^{-1}$ equals the result of integrating X_t from 0 to 1.

Let φ_s^t be the flow of X_t . Construct a partition of $[0, 1]$ by pieces of length $1/n$. By Euler's method, the flow of X_t is approximated by flowing the time independent field $X_{k/n}$ from time k/n to $(k+1)/n$ for $k = 0, \dots, n-1$. Now use Euler's method again to approximate each flow $\varphi_s^{k/n}$ for $k = 0, \dots, n$ as follows: By assumption we can approximate $X_{k/n}$ uniformly by polynomial vector fields. Using Theorem 3.10 we can write every polynomial vector field as a Lie combination of complete polynomial vector fields whose flows are in $\text{OS}(D_p)$. Their flows are shear and overshear automorphisms, so we can approximate $\Phi_1 \circ \Phi_0^{-1}$ using Lemma 4.1 by finite compositions of shears and overshears. Hence we can approximate $\Phi_1 = (\Phi_1 \circ \Phi_0^{-1}) \circ \Phi_0$ by elements in $\text{OS}(D_p)$. \square

Proof of Theorem 1.1. Let $K \subset D_p$ be a compact set, and let Ψ be an arbitrary automorphism in the path component of the identity. We need to show that we can approximate this automorphism on K . By assumption there is a continuous path $\Phi: [0, 1] \rightarrow \text{Aut}(D_p)$ of automorphisms such that $\Phi_0 = \text{Id}$ and $\Phi_1 = \Psi$. This family of mappings induces a mapping, continuous in t , defined on $[0, 1] \times D_p \rightarrow D_p$ given by $(t, (x, y, z)) \mapsto \Phi_t(x, y, z)$. With some abuse of notation, we will call this mapping Φ_t as well. Using sublevel sets of exhaustion functions we can choose a Runge domain $\Omega \Subset D_p$ containing the compact set $\bigcup_{t \in [0, 1]} \Phi_t(K)$. Furthermore, let $\tilde{\Omega} \Subset D_p$ be an open set such that $\bigcup_{t \in [0, 1]} \Phi_t(\Omega) \subset \tilde{\Omega}$. By a convolution of Φ_t , we get a smooth mapping $\tilde{\Phi}_t: [0, 1] \times D_p \rightarrow \mathbb{C}^3$. By differentiating under the integral sign we see that $\tilde{\Phi}_t$ is holomorphic for fixed t . Then use the tubular neighborhood theorem [7] to project $\tilde{\Phi}_t(z)$ back into the manifold D_p ; i.e. if $N(D_p)$ is the normal bundle of D_p in \mathbb{C}^3 and \mathcal{N} is a neighborhood of the zero section in $N(D_p)$ which is biholomorphic, via $\alpha: \mathcal{N} \rightarrow U$, to an open neighborhood U of D_p in \mathbb{C}^3 then define $\hat{\Phi}: [0, 1] \times D_p \rightarrow D_p$ by $\hat{\Phi}_t(z) = \pi \circ \alpha^{-1} \circ \tilde{\Phi}_t(z)$, where π is the projection. Since injectivity is an open condition in \mathcal{C}^1 -topology and approximation in values for holomorphic mappings automatically implies approximation of the derivatives (in a smaller compactum, by Cauchy's Integral Formula), we may as well assume that $\hat{\Phi}_t: \tilde{\Omega} \rightarrow D_p$ is injective. Thus the vector fields X_t , from the proof of Proposition 4.2, are well defined on Ω . Since Ω is Runge we can approximate X_t by polynomial vector fields, and now Proposition 4.2 implies that we can approximate $\hat{\Phi}_1$ arbitrarily close by elements in $\text{OS}(D_p)$. Also $\hat{\Phi}_1$ is by construction arbitrarily close to $\Phi_1 = \Psi$, which concludes the proof. \square

5. ON THE NUMBER OF COMPONENTS
OF THE AUTOMORPHISM GROUP

If the degree of the defining polynomial is one, the Danielewski surface D_p is \mathbb{C}^2 and its group of holomorphic automorphisms is path-connected. If the degree is at least 2, the situation is different: Since they are contained in a flow, the elements $H_f, S_f^x, S_f^y, O_f^x, O_f^y$ in $\text{Aut}(D_p)$ are contained in the path component of the identity of $\text{Aut}(D_p)$. On the other hand, the map $I(x, y, z) = (y, x, z)$ is not. It can be shown that it acts as $\alpha \mapsto -\alpha$ on $H^2(D_p, \mathbb{Z}) \cong \mathbb{Z}^{n-1}$, where $n = \deg p$. Thus $\text{Aut}(D_p)$ has at least two path components.

Remark 5. The authors do not know how to calculate the exact number of (path) components of $\text{Aut}(D_p)$. The results of Makar-Limanov show that the algebraic automorphism group has exactly two (path) components. Moreover, we do not know whether the path-connected component of the identity in $\text{Aut}(D_p)$ is equal to the connected component of the identity in $\text{Aut}(D_p)$. For the diffeomorphism group of a noncompact manifold they may differ (even if the contrary is stated in the literature), as the following example shows. \square

Example 1. Take a Riemann surface X of infinite genus, say a connected sum of countably many tori $T_i, i = 1, \dots, \infty$. We define a diffeomorphism φ_i of X by being an identity at all tori except the i th, and there we build in the Dehn twist in the following way: We cut the torus open along an S^1 , rotate one side by 2π and glue back. This rotation is smoothly cut down to zero rotation a bit from the place where we cut in order to extend it smoothly to the whole surface X .

Each of these φ_i acts nontrivially on the fundamental group, thus does not belong to the path component of the identity. However, in the compact open topology the sequence φ_i converges to identity since the support of φ_i is leaving any compact subset for $i \rightarrow \infty$. \square

We end the paper with two open problems:

Problem 5.1. In [14] it is shown that the Danielewski surfaces D_p admit the algebraic volume density property with respect to the unique algebraic volume form ω on them. Is it true that the group generated by shears is dense in the path component of the group $\text{Aut}^\omega(D_p)$ of volume-preserving holomorphic automorphisms of D_p ? The results of the present paper combined with the method from [14] do not suffice to prove this, since we use overshear fields (not volume preserving) in the Lie combinations from Lemma 3.4 that produce hyperbolic fields (volume preserving). This problem is also interesting in connection with the following question posed in [5]: If an affine algebraic manifold is flexible (the group generated by locally nilpotent derivations acts transitively on it), does it have the algebraic volume density property?

Problem 5.2. Determine the number of (path) components of the group $\text{Aut}(D_p)$. Are the path components also the connected components? More generally: Let $\text{Aut}(X)$ be the topological group of holomorphic automorphisms of a complex manifold X equipped with compact-open topology. Are the path components of $\text{Aut}(X)$ also the connected components?

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