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CONTINUITY OF TRANSLATION OPERATORS

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ABSTRACT. For a Radon measure μ on $\mathbb{R}$, we show that $L^\infty(\mu)$ is invariant under the group of translation operators $T_t(f)(x) = f(x-t)$ $(t \in \mathbb{R})$ if and only if μ is equivalent to the Lebesgue measure m. We also give necessary and sufficient conditions for $L^p(\mu)$, $1 \le p < \infty$, to be invariant under the group $\{T_t\}$ in terms of the Radon-Nikodým derivative w.r.t. m.

1. INTRODUCTION

One of the celebrated theorems of analysis is the existence of a unique measure m on $\mathbb{R}$ such that $m([0,1]) = 1$ and $m(E+t) = m(E)$ for any Borel subset $E \subset \mathbb{R}$, for all $t \in \mathbb{R}$. This theorem was later extended by Haar ([2]) to the setting of locally compact groups.

We could rephrase the theorem as follows: there is a unique Radon measure m on $\mathbb{R}$ for which $m([0,1]) = 1$ and translation acts as an isometry on $L^p(m)$, for some p, $1 \le p < \infty$. That is, for all $f \in L^p(m)$, $||T_t(f)||_p = ||f||_p$, for all $t \in \mathbb{R}$, where $T_t(f)(x) = f(x-t)$.

In this paper, we ask the question: which Radon measures μ have the property that T_t maps $L^p(\mu)$ into itself, for all $t \in \mathbb{R}$? We make no assumption of continuity of the T_t.

We show that for $1 \le p < \infty$ if the translation operators T_t map $L^p(\mu)$ into itself, for all $t \in \mathbb{R}$, then each T_t is strongly continuous, and moreover for any compact neighborhood $\mathcal{N}$ of 0, there is a constant C so that $||T_t|| \le C$ for all $t \in \mathcal{N}$ (Corollary 2) and thus the group T_t acts as a C_0-group on $L^p(\mu)$, i.e. $\lim_{t\to 0} ||T_t(f) - f||_p = 0$ (Theorem 1). On the other hand, if T_t acts on $L^\infty(\mu)$ for all $t \in \mathbb{R}$, the action is not even weakly continuous (Theorem 2). In Theorems 3 and 4 we give necessary and sufficient conditions for T_t to map $L^p(\mu)$ to itself in the cases $p = \infty$ and $p < \infty$ respectively.

A related problem was treated by D. Bell ([1]). He called a Radon measure μ *quasi-translation-invariant* if the null sets of μ are translation-invariant. He showed that with additional assumptions on μ, μ must be equivalent to Lebesgue measure. Bell's results are valid in the context of locally compact abelian groups. We deduce Bell's results from our results for the group $\mathbb{R}$. Lamperti [3] considered a general

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class of isometries of $L^p(X, \mu)$, for $p < \infty$. Our results do not overlap with his since the mappings we consider are not isometries, except for the case of $L^\infty(\mu)$.

While it is not difficult to see that some of our results may be recast in a more general setting, beginning with Lemma 9 we are restricted to the setting of $\mathbb{R}$. For thematic unity, we have presented all results in the context of $\mathbb{R}$.

1.1. **Background and notation.** A Radon measure is a positive Borel measure μ on $\mathbb{R}$ such that $\mu([-n, n]) < \infty$ for all $n \in \mathbb{N}$. The vector space $C_{00}(\mathbb{R})$ of continuous functions with compact support has as its dual space the vector space of signed Radon measures. The space $C_{00}(\mathbb{R})$ is dense in $L^p(\mu),\ 1 \le p < \infty$. The space $L^\infty(\mu)$ is the dual space of $L^1(\mu)$.

If f is an extended real-valued function defined on $\mathbb{R}$, $T_t(f)$ will denote the function $T_t(f)(x) = f(x - t),\ t \in \mathbb{R}$.

Certain hypotheses will appear often, and so it is convenient to state them here.

Hypotheses.

H1 μ is a Radon measure with the property that if $f,\ g$ are extended real-valued Borel functions such that $f = g$ a.e.$[\mu]$, then for all $t \in \mathbb{R},\ T_t(f) = T_t(g)$ a.e.$[\mu]$.

H2(a) Given a Radon measure μ and for some $p,\ 1 \le p < \infty$ if f is a Borel function such that

$$\int_{\mathbb{R}} |f(x)|^p \, d\mu(x) < \infty, \text{ then } \int_{\mathbb{R}} |T_t(f)(x)|^p \, d\mu(x) < \infty, \text{ for all } t \in \mathbb{R}.$$

H2(b) Given a Radon measure μ, an extended real-valued Borel function f is essentially $[\mu]-$bounded if and only if $T_t(f)$ is essentially $[\mu]-$bounded, for all $t \in \mathbb{R}$.

In [1] Bell made the following definition.

Definition 1. A Radon measure μ is called *quasi-translation-invariant (q.t.i.)* if

$$\forall E \in \mathfrak{B},\ \mu(E) = 0 \iff \mu(E + t) = 0 \text{ for all } t \in \mathbb{R},$$

where $\mathfrak{B}$ is the σ-algebra of Borel sets.

While the following result is obvious, it is nonetheless worth recording.

Lemma 1. *For a Radon measure μ, hypothesis* (H1) *is equivalent to q.t.i.*

Remark 1. It is easy to see that each of the hypotheses (H2(a)), (H2(b)) implies (H1), but we find it useful to state the conditions separately.

Remark 2. (H1) implies that the group of translation operators $\{T_t\}_{t\in\mathbb{R}}$ not only acts on functions, but acts as a group of linear operators on equivalence classes of functions. By that we mean that each T_t is a linear operator, and $T_s \circ T_t = T_{s+t}$ for all $s,\ t \in \mathbb{R}$, and $T_0 = I$. (H1) and (H2(a)) (resp., (H1) and (H2(b))) together imply that the group $\{T_t\}_{t\in\mathbb{R}}$ acts on $L^p(\mu)$ (resp., $L^\infty(\mu)$).

Conversely, if $\{T_t\}_{t\in\mathbb{R}}$ acts as a group of linear operators on some $L^p(\mu)$, then μ must satisfy (H1), or equivalently q.t.i., because if $\mu(E) > 0$ but $\mu(E + t) = 0$ for some $t \in \mathbb{R}$, then $\chi_E = T_0(\chi_E) = T_{-t} \circ T_t(\chi_E) = T_{-t}(0) = 0$. This contradiction shows that the group property of the linear operators $\{T_t\}$ on $L^p(\mu)$ is equivalent to q.t.i.

From now on m will denote Lebesgue measure.

2. Continuity of translation

Lemma 2. *Let μ be a Radon measure satisfying either* (H1) *and* (H2(a)) *or* (H1) *and* (H2(b)). *Then for each $t \in \mathbb{R}$, T_t is continuous.*

Proof. Let $\mathcal{V} = L^p(\mu)$. By Remark 2, the maps $\{T_t\}$ act as a group on $\mathcal{V}$. Thus $T_t(\mathcal{V}) = \mathcal{V}$ so that T_t is onto and in particular has closed range. It now follows from the Closed Graph Theorem ([5]) that for each $t \in \mathbb{R}$, T_t is strongly continuous on $L^p(\mu)$. □

Remark 3. The simple application above of the Closed Graph Theorem yields that the maps T_t are individually continuous. A more difficult question is whether the *group* $\{T_t\}_{t\in\mathbb{R}}$ is continuous in some sense. We will see that there is a sharp distinction between the cases $p < \infty$ and $p = \infty$.

Definition 2. A Radon measure μ is *continuous* if for all $x \in \mathbb{R}$, $\mu(\{x\}) = 0$.

Lemma 3. *Let μ satisfy* (H1) *and* (H2(a)) *or* (H1) *and* (H2(b)). *Then μ is a continuous measure, and support$(\mu) = \mathbb{R}$.*

Proof. Now suppose that $\mu(\{x_0\}) > 0$. Since a Radon measure can have at most countably many points with positive measure, there is some $y \in \mathbb{R}$ with $\mu(\{y\}) = 0$. But then a translation operator maps the characteristic function of the point x to that of y, contradicting (H1).

If *support*$(\mu) \neq \mathbb{R}$, then by definition there is a nonempty open set $\mathcal{O} \subset \mathbb{R}$ such that $\mu(\mathcal{O}) = 0$. Let $I = (a, b)$ be an interval containing $\mathcal{O}$. Set $d = \frac{b-a}{2}$ and cover $\mathbb{R}$ by intervals $I + nd$, $n \in \mathbb{Z}$. Since this is a countable open cover of $\mathbb{R}$ and the measure $\mu \neq 0$, for some n we must have that $\mu(I + nd) > 0$. But then

$$||T_t(\chi_I)||_p \neq 0, \text{ but } ||\chi_I||_p = 0,$$

for $t = -nd$, contradicting (H1). □

The statement that supports $\mu = \mathbb{R}$ was proved in [1]; we included a proof here for completeness.

Lemma 4. *Let μ be a q.t.i. Radon measure on $\mathbb{R}$. Then given a bounded Borel set E, a sequence $\{t_n\}$ in $\mathbb{R}$ which converges to zero, and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,*

$$\mu(\bigcup_{k=n}^{\infty}(E + t_k)) < \mu(E) + \epsilon.$$

Proof. Let I be a compact interval in $\mathbb{R}$ such that E, $E + t_n \subset I$ for all $n \in \mathbb{N}$. Since $\mu(E) \leq \mu(I) < \infty$, there is an increasing sequence $\{C_n\}$ of compact subsets of E such that $E \setminus \bigcup_{n=1}^{\infty} C_n$ is a μ-null set. But then

$$\bigcup_{k=1}^{\infty}(E + t_k) \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}(C_n + t_k)$$
$$\subset \bigcup_{k=1}^{\infty}((E \setminus \bigcup_{n=1}^{\infty} C_n) + t_k)$$

Since μ is q.t.i., $(E \setminus \bigcup_{n=1}^{\infty} C_n) + t_k$ is a μ-null, as is the countable union of these sets.

Thus ([6], Theorem 1.19) there is a compact set $C = C_n$ such that

$$\mu(\bigcup_{k=1}^{\infty}(E+t_k) \setminus \bigcup_{k=1}^{\infty}(C+t_k)) < \epsilon.$$

Let $F = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}(C+t_k)$. Thus if $x \in F$, there is a subsequence t_{k_j} and $x_j \in C$ such that $x = x_j + t_{k_j}$ for all j. Since $\{t_k\}$ converges to zero, that implies that x is a limit point of $\{x_j\}$, hence in C. Now there is $N \in \mathbb{N}$ such that for $n \geq N$,

$$\mu(\bigcup_{k=n}^{\infty}(C+t_k) \setminus F) < \epsilon.$$

Thus, for $n \geq N$,

$$\begin{aligned}\mu(\bigcup_{k=n}^{\infty}(E+t_k) \setminus F) &\leq \mu(\bigcup_{k=n}^{\infty}(E+t_k) \setminus \bigcup_{k=n}^{\infty}(C+t_k)) \\ &\quad + \mu(\bigcup_{k=n}^{\infty}(C+t_k) \setminus F) \\ &< \epsilon + \epsilon\end{aligned}$$

In other words,

$$\begin{aligned}\mu(\bigcup_{k=n}^{\infty}(E+t_k)) &< \mu(F) + 2\epsilon \\ &< \mu(E) + 2\epsilon.\end{aligned}$$

□

Lemma 5. *Let μ be a q.t.i. Radon measure. Let g be a bounded Borel function with bounded support, and let $\{t_n\}$ be a sequence which converges to 0. Then $\{T_{t_n}(g)\}$ converges in measure to g. Hence there is a subsequence which converges a.e.$[\mu]$.*

Proof. Given $\epsilon > 0$, by Lusin's Theorem ([6], Theorem 2.23) there is a continuous function h with compact support, $||h||_\infty \leq ||g||_\infty$, and such that if

$$E = \{x : \ |g(x) - h(x)| \geq \epsilon\},$$

then $\mu(E) < \epsilon$.

By Lemma 4, there is $N \in \mathbb{N}$ such that for $n \geq N$,

$$\mu(E \bigcup_{k=n}^{\infty}(E+t_k)) < 2\epsilon.$$

Now choose $j \geq N$ such that for all $x \in \mathbb{R}$,

$$|h(x-t_j) - h(x)| < \epsilon.$$

Then for $x \notin E\bigcup_{k=N}^{\infty}(E+t_k)$,

$$\begin{aligned}|g(x) - g(x-t_j)| &\leq |g(x) - h(x)| \\ &\quad + |h(x) - h(t-t_j)| + |h(x-t_j) - g(x-t_j)| \\ &< 3\epsilon.\end{aligned}$$

It follows that $\{T_{t_k}(g)\}$ converges in measure to g; hence there is a subsequence converging a.e. $[\mu]$ ([6], p. 73). □

Remark 4. Note that if μ is a continuous Radon measure which is not q.t.i., neither of the conclusions of Lemmas 4 or 5 need hold.

Let C be the Cantor set, and $\varphi : C \to [0,1]$ be defined as follows: if $x \in C,\ x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$ where $x_n \in \{0,2\}$, let $\varphi(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^{n+1}}$. Then the Cantor measure μ can be defined by $\mu(E) = m(\varphi(E \cap C))$. (Recall that m is Lebesgue measure.)

Now $\mu(C) = 1,\ \mu(C + \frac{1}{3^n}) = 0$, so that $T_{\frac{1}{3^n}}(\chi_C)$ does not converge to χ_C in measure. So the conclusion of Lemma 5 fails.

If $E = \bigcup_{n=1}^{\infty}(C + \frac{1}{3^n})$, then $\mu(E) = 0$, but $\mu(E + t_n) = 1$ for all n if $t_n = -\frac{1}{3^n}$, so that for all n, $\mu(\bigcup_{k=n}^{\infty}(E + t_k)) = 1$. Thus the conclusion of Lemma 4 fails.

Corollary 1. *Let μ satisfy* (H1) *and* (H2(a)). *Let $f \in L^p(\mu)$ and $\{t_k\}$ be a sequence converging to t_0. Then there is a subsequence S of $\{t_n\}$ such that $T_{t_k}(f)$ converges to $T_{t_0}(f)$ a.e.$[\mu]$.*

Proof. Let

$$f_n(x) = \begin{cases} f(x) \text{ if } |f(x)| \le n \text{ and } |x| \le |n|; \\ n \text{ if } f(x) > n \text{ and } |x| \le n; \\ -n \text{ if } f(x) < -n \text{ and } |x| \le |n|; \\ 0 \text{ if } |x| > n. \end{cases}$$

Now each f_n is equal a.e.$[\mu]$ to a bounded Borel function with bounded support. Thus, by Lemma 5 there is a subsequence of $\{t_k\}$ for which $T_{t_k}(f_1)$ converges pointwise a.e.$[\mu]$ to $T_{t_0}(f_1)$ and a subsequence of that subsequence for which $T_{t_k}(f_2)$ converges to $T_{t_0}(f_2)$, and so forth. The standard diagonal argument gives a subsequence S for which $T_{t_k}(f_n)$ converges to $T_{t_0}(f_n)$ a.e.$[\mu]$ for all n, and hence $T_{t_k}(f)$ converges to $T_{t_0}(f)$ for $t_k \in S$, a.e.$[\mu]$. $\square$

Lemma 6. *Let μ satisfy* (H1) *and* (H2(a)). *Let E be a set with $\mu(E) < \infty$. For any positive C, the set*

$$\{t \in \mathbb{R} :\ \mu(E + t) \le C\mu(E)\}$$

is closed.

Proof. Let $\{t_k\}$ be a sequence with $\mu(E + t_k) \le C\mu(E)$ and suppose $t_0 = \lim t_n$. By Corollary 1 there is a subsequence such that χ_{E+t_k} converges to χ_{E+t_0} *a.e.*$[\mu]$. Thus, replacing the sequence by the subsequence and applying Fatou's Lemma we have

$$\begin{aligned} \mu(E + t_0) &= \int \chi_{E+t_0}(x)\, d\mu(x) \\ &= \int \liminf \chi_{E+t_n}(x)\, d\mu(x) \\ &\le \liminf \int \chi_{E+t_n}(x)\, d\mu(x) \\ &\le C\mu(E). \end{aligned}$$ $\square$

Lemma 7. *Let μ satisfy* (H1) *and* (H2(a)). *Let $\mathcal{N} = [a,b],\ a,\ b \in \mathbb{R}$. Then there is a constant $C > 0$ such that for all $t \in \mathcal{N}$ and for all measurable sets E with $\mu(E) < \infty$,*

$$\mu(E + t) \le C\mu(E).$$

Proof. For $n = 1, 2, \ldots$, set $A_n = \{t \in \mathcal{N} : \ \mu(E+t) \le n\mu(E)\}$ for all measurable sets $E \subset \mathbb{R}$ with $\mu(E) < \infty$. By Lemma 6, A_n is a closed subset of $\mathcal{N}$. Since for $n \ge ||T_t||,\ t \in A_n$ we have that

$$\bigcup_{n=1}^{\infty} A_n = \mathcal{N}.$$

By the Baire Category Theorem ([6]), some A_N has interior.

Say $(a', b') \subset \operatorname{int} A_N$. If $d = \frac{b'-a'}{2}$, then we have

$$\mu(E+t) \le C'\mu(E)$$

for all $t \in (-d, d)$, where $C' = ||T_{-d}||C_N$. Let k be such that $\mathcal{N} \subset (-kd, kd)$. Then we can choose the constant C of the lemma to be $C = (C')^k$. □

Corollary 2. *Let μ satisfy* (H1) *and* (H2(a))*. Then there is a constant C_p such that*

$$||T_t(f)||_p \le C_p||f||_p$$

for all $t \in \mathcal{N},\ f \in L^p(\mu)$.

Proof. It is enough to prove that the conclusion holds for all f in a dense subset of $L^p(\mu)$. Let C be as in Lemma 7 and let

$$f = \sum_n a_n \chi_{E_n}$$

be a simple function; in particular, the sum is finite and the E_n are disjoint, $\mu(E_n) < \infty$. Then

$$\begin{aligned} ||T_t(f)||_p^p &= \sum_n |a_n|^p \mu(E_n + t) \\ &\le \sum_n |a_n|^p C\mu(E_n) \\ &\le C||f||_p^p. \end{aligned}$$

□

Now that we know that the translation operators are uniformly bounded in a compact neighborhood of 0, the proof of strong continuity of translation on $L^p(\mu)$ mimics that on $L^p(m)$.

Theorem 1. *Let μ be a Radon measure satisfying* (H1) *and* (H2(a))*. Then $\{T_t\}$ is a C_0-group acting on $L^p(\mu)$. That is,*

$$\lim_{t\to 0} ||T_t(f) - f||_p = 0$$

for all $f \in L^p(\mu)$.

Proof. Let C and $\mathcal{N}$ be as in Corollary 2, and let $f \in L^p(\mu)$ and $\epsilon > 0$ be given. Let g be a continuous function with compact support such that $||f - g||_p < \delta$, where δ satisfies $(C+1)\delta < \epsilon/2$. Since $T_t(g)$ converges to g uniformly as $t \to 0$, it also converges in $L^p(\mu)$. Let $(-\eta, \eta) \subset \mathcal{N}$, where η is sufficiently small so that $||T_t(g) - g||_p < \epsilon/2$. Then for $|t| < \eta$,

$$\begin{aligned} ||T_t(f) - f||_p &\le ||T_t(f-g)||_p + ||T_t(g) - g||_p + ||f - g||_p \\ &\le C\delta + \epsilon/2 + \delta \\ &< \epsilon. \end{aligned}$$

□

Next we give a negative result for $L^\infty(\mu)$. Obviously translation is not norm continuous on $L^\infty(\mu)$, but it is not even weakly continuous.

Theorem 2. *Let μ be a Radon measure satisfying* (H1) *and* (H2(b))*. Then there exists a bounded function $f \in \mathcal{V} = L^\infty(\mu)$ and a bounded linear functional $\rho : \mathcal{V} \to \mathbb{R}$ such that*

$$\lim_{t\to 0} \rho(T_t(f)) \text{ exists but } \neq \rho(f).$$

In particular, T_t is not weakly continuous on $\mathcal{V}$.

Proof. Claim: By Lemma 3, μ is a continuous measure. There exists a point $a \in \mathbb{R}$ such that for all $\delta > 0$, $\mu([a, a+\delta)) > 0$. Suppose to the contrary. Let $I = [c, d]$ be a closed interval with $\mu(I) > 0$. Then for each $x \in \mathbb{R}$ there is a $\delta_x > 0$ such that $\mu([x, x+\delta_x)] = 0$. Since I is compact, I can be covered by finitely many intervals of the form $(x, x+\delta_x)$, $x \in \mathbb{R}$, say $(x_i, x_i+\delta_i)$, $1 = 1, \dots, n$. But then

$$0 < \mu(I) \leq \sum_{i=1}^{n} \mu(x_i, x_i + \delta_i) = 0.$$

This contradiction proves the claim.

Let $a \in \mathbb{R}$ be such that $\mu[a, a+\delta) > 0\ \forall \delta > 0$, and let δ_n be a sequence decreasing to 0. Define a bounded linear functional ρ_n by

$$\rho_n(f) = \frac{1}{\mu[a, a+\delta_n)} \int_{[a,a+\delta_n)} f \, d\mu.$$

Note that ρ_n has norm 1 on $L^\infty(\mathbb{R}, \mu)$. By the Alaoglu Theorem ([5], 3.15) the set $\{\rho_n\}$ has a limit point, say ρ.

Let $h = \chi_{[a,\infty)}$. Then $\rho_n(h) = 1$ for all n so that $\rho(h) = 1$. On the other hand, for $\delta_n < t$,

$$\rho_n(T_t(h)) = 0; \text{ hence } \rho(T_t(h)) = 0.$$

Thus, $\rho(h) \neq \lim_{t\to 0} \rho(T_t(h))$, completing the proof. $\square$

3. The measure μ

In this section we characterize those measures μ for which $L^p(\mu)$ admits translation operators.

Definition 3. Let $\mathcal{B}$ be the space of bounded, Borel functions on $\mathbb{R}$ with bounded support. Let μ be a Radon measure on $\mathbb{R}$. We say that $T_t \to T_0 = I$ as $t \to 0$ in the *wb*-topology if

$$(T_t(f), g)_\mu := \int f(x-t)g(x)\, d\mu(x) \to (f, g)_\mu := \int f(x)g(x)\, d\mu(x)$$

as $t \to 0$, for all $f,\ g \in \mathcal{B}$.

Remark 5. If $\{T_t\}$ is a C_0-group acting on $L^p(\mu)$ for some $1 \leq p < \infty$, then clearly $T_t \to T_0$ as $t \to 0$ in the *wb*-topology.

Lemma 8. *With notation as above, $T_t \to T_0 = I$ as $t \to 0$ in the wb-topology on $\mathcal{B}$ if and only if μ is absolutely continuous with respect to the Lebesgue measure m.*

Proof. Assume that $(T_t(f), g)_\mu \to (f,g)_\mu$ as $t \to 0$ for all $f,\ g \in \mathcal{B}$. We need to show that μ is absolutely continuous with respect to Lebesgue measure.

Fix $0 < n < \infty$, and let $\mu_{t,n}(A) = \mu(A \cap [-n,n] - t)$ for $A \in \mathfrak{B}$, the σ-algebra of Borel sets. Take $f = \chi_{A\cap[-n,n]},\ g = \chi_{[-n-1,n+1]}$. Note that $f,\ g \in \mathcal{B}$, and that for all $A \in \mathfrak{B}$,

$$\mu_{t,n}(A) = \int \chi_{A\cap[-n,n]}(x+t)\chi_{[-n-1,n+1]}(x)\,d\mu(x) \to \mu_{0,n}(A) \tag{1}$$

as $t \to 0$.

For $\epsilon > 0$, set

$$\lambda_\epsilon(A) = \frac{1}{\epsilon}\int_0^\epsilon \mu_{t,n}(A)\,dt,\ A \in \mathfrak{B}.$$

By (1),

$$\lambda_\epsilon(A) \to \mu_{0,n}(A) \text{ as } \epsilon \to 0. \tag{2}$$

By Fubini's Theorem,

$$\begin{aligned}\lambda_\epsilon(A) &= \frac{1}{\epsilon}\int_0^\epsilon (\int_{\mathbb{R}} \chi_{A\cap[-n,n]-t}(x)\,d\mu(x))\,dt \\ &= \int_{\mathbb{R}} (\frac{1}{\epsilon}\int_0^\epsilon \chi_{A\cap[-n,n]-t}(x)\,dt)\,d\mu(x).\end{aligned}$$

If $m(A) = 0$, then for all $\epsilon > 0,\ x \in \mathbb{R},\ \int_0^\epsilon \chi_{A\cap[-n,n]-t}(x)\,dt = 0$ since $m(A \cap [-n,n]-t) = m(A\cap[-n,n]) \le m(A) = 0$ using the translation invariance of m. Thus, $\lambda_\epsilon(A) = 0\ \forall \epsilon > 0$.

By (2), $\mu_{0,n}(A) = 0$, for all n. Since $\mu(A) = \lim_n \mu_{0,n}(A)$, it follows that $\mu(A) = 0$. Thus $\mu \ll m$.

Now suppose that μ is absolutely continuous with respect to m. We need to show that T_t is continuous at 0 in the *wb*-topology. Fix $f,\ g \in \mathcal{B}$ and let $h = \frac{d\mu}{dm}$ be the Radon-Nikodým derivative.

Let M be such that the supports of $f,\ T_t(f),\ g$ are all contained in $[-M, M]$ for $|t| < 1$. Let $\epsilon > 0$ be given. Clearly,

$$\int_{|x|\le M} h(x)\,dm < \infty \text{ implies } \lim_{N\to\infty}\int_{|x|\le M,\ h(x)>N} h(x)\,dm = 0.$$

Thus, there exists N such that

$$\int_{h(x)>N,\ |x|\le M} h(x) < \epsilon.$$

Since $T_t(f) \to f$ as $t \to 0$ in $L^1(m)$, there is a $\delta > 0$ such that for $|t| < \delta$,

$$\int |f(x-t) - f(x)|\,dm(x) < \epsilon/N.$$

So for $|t| < \delta$,

$$|(T_t(f), g)_\mu - (f, g)_\mu| \leq \int_{h(x)>N,\ |x|\leq M} |f(x-t) - f(x)||g(x)|h(x)\, dm(x)$$
$$+ N||g||_\infty \int |f(x-t) - f(x)|\, dm(x)$$
$$\leq 2||f||_\infty ||g||_\infty \int_{h(x)>N,\ |x|\leq M} h(x)\, dm(x) + N||g||_\infty \int |f(x-t) - f(x)|\, dm(x)$$
$$\leq 2||f||_\infty ||g||_\infty \epsilon + ||g||_\infty \epsilon.$$

This shows that T_t is *wb*-continuous at $t = 0$, completing the proof. □

Lemma 9. *Let μ be a continuous Radon measure on the σ-algebra of Borel sets. Let E be a Borel set with $\mu(E) < \infty$ and $0 < m(E) < \infty$. Let $\mathcal{O}$ be an open set with $E \subset \mathcal{O}$ and $m(\mathcal{O})$ finite. Write $\mathcal{O} = \bigcup_i (a_i, b_i)$, where the (a_i, b_i) are disjoint intervals. Then there exists i such that*

$$\mu((a_i, b_i)) \geq \frac{\mu(\mathcal{O})}{m(\mathcal{O})}\,(b_i - a_i),$$

and there exists i such that

$$\mu((a_i, b_i)) \leq \frac{\mu(\mathcal{O})}{m(\mathcal{O})}\,(b_i - a_i).$$

In particular, if $\mu(E) = 0$, then given $\epsilon > 0$, $\mathcal{O}$ can be chosen so that

$$\mu((a_i, b_i)) < \epsilon\,(b_i - a_i)$$

for some i.

Proof. Let $I_i = (a_i, b_i)$. Suppose the first statement fails; then

$$\text{for all } i,\ \mu(I_i) < \frac{\mu(\mathcal{O})}{m(\mathcal{O})}\, m(I_i).$$

Hence,

$$\begin{aligned}\mu(\mathcal{O}) &= \sum_i \mu(I_i)\\ &< \sum_i \frac{\mu(\mathcal{O})}{m(\mathcal{O})}\, m(I_i)\\ &< \mu(\mathcal{O})\frac{m(\mathcal{O})}{m(\mathcal{O})}\\ &< \mu(\mathcal{O}),\end{aligned}$$

a contradiction.

The proof of the second assertion is similar and is omitted. The final statement follows from the regularity of the measure μ. □

Recall that two measures μ and ν are *equivalent* if μ is absolutely continuous with respect to ν, and ν is absolutely continuous with respect to μ.

Lemma 10. *Suppose that μ is a nonzero Radon measure which is q.t.i. Then μ is equivalent to Lebesgue measure.*

Proof. Let $\{t_n\}$ be a sequence in $\mathbb{R}$ which converges to 0, and $f,\ g$ bounded Borel functions with bounded support. Since $\{T_{t_n}(f)\}$ converges in measure to f by Lemma 5, then in the notation of Definition 3,

$$(T_{t_n}(f), g)_\mu \to (f, g)_\mu$$

so that $T_t \to I$ in the wb-topology as $t \to 0$. Thus by Lemma 8, $\mu \ll m$.

Let $h = \frac{d\mu}{dm}$, and set $F = \{x : \ h(x) = 0\}$. Since μ is q.t.i., F is translation-invariant. Define a Radon measure ν by

$$\nu(E) = m(E \cap F).$$

Then

$$\begin{aligned}\nu(E + t) &= m((E + t) \cap F)\\ &= m((E + t) \cap (F + t))\\ &= m(E \cap F + t)\\ &= m(E \cap F)\\ &= \nu(E).\end{aligned}$$

As Lebesgue measure is the unique translation-invariant measure, up to a scalar multiple, if ν is nonzero there is a positive constant c such that $\nu(E) = c\, m(E)$ for any Borel set E. In particular,

$$\nu(F^c) = m(F \cap F^c) = 0$$

and hence $m(F^c) = 0$. This implies that μ is the zero measure, contrary to assumption. Thus it must be that ν is the zero measure, so that $m(F) = 0$. But then $m \ll \mu$. □

Theorem 3. *Let μ be a Radon measure on $\mathbb{R}$. Then the following conditions are equivalent:*

(1) *μ is q.t.i.;*
(2) *$\{T_t\}_{t\in\mathbb{R}}$ acts as a group of linear operators on $L^\infty(\mu)$;*
(3) *$\{T_t\}$ acts as a group of isometries on $L^\infty(\mu)$;*
(4) *μ is equivalent to Lebesgue measure.*

Proof. Suppose μ is q.t.i. and let $f \in L^\infty(\mu),\ f = \sum a_k E_k$ be a finite sum, with the sets E_k pairwise disjoint and of positive or infinite μ-measure. By hypothesis, the sets $E + t$ have positive or infinite μ-measure, for all $t \in \mathbb{R}$. Thus, $T_t(f) \in L^\infty(\mu)$ and $||T_t(f)||_\infty = ||f||_\infty$. Thus, T_t is an isometry on a dense subspace of $L^\infty(\mu)$, hence is an isometry on $L^\infty(\mu)$. Furthermore the $\{T_t\}$ act as a group (cf. Remark 2). This shows that (1) implies (3), and clearly (3) implies (2). Also by Remark 2, q.t.i. is equivalent to (2).

Lemma 10 proves the equivalence of (1) and (4). □

Remark 6. Recall that hypothesis (H1) for μ is equivalent to μ being q.t.i. (Lemma 1). The theorem now shows that (H1) is equivalent to (H2(b)).

Notation. If h is a Lebesgue measurable function and I an interval in $\mathbb{R}$, write $||h||_{I,\infty}$ to denote the essential supremum of h on I.

Theorem 4. *Let μ be a Radon measure on $\mathbb{R}$. Then μ satisfies* (H1) *and* (H2(a)) *if and only if μ is equivalent to Lebesgue measure, and the Radon-Nikodým derivative $\frac{d\mu}{dm} := h$ satisfies: For each real number $L > 0$ there is a constant C_L such that if I is any interval of length L, then*

$$||h||_{I,\infty}\,||1/h||_{I,\infty} \leq C_L.$$

Proof. Suppose T_t maps $L^p(\mu)$ to itself, for all $t \in \mathbb{R}$. Let $L > 0$ be given, and let $I = [a, b]$ be an interval of length L.

A fortiori there is no reason why $||h||_{I,\infty}$ could not be infinite, or similarly why $||1/h||_{I,\infty}$ could not be infinite. From Lemma 10 we know that μ is equivalent to Lebesgue measure, and hence $||h||_{I,\infty}$ cannot be zero, and $||1/h||_{I,\infty}$ cannot be zero. We will show in fact that both are finite. Let M and K be constants such that

$$0 < M < ||h||_{I,\infty}$$

and

$$0 < K < ||1/h||_{I,\infty}.$$

Let $E = \{x \in I : \ h(x) \geq M\}$ and $F = \{x \in I : \ 1/h(x) \geq K\}$. By definition of essential supremum, both of these sets have positive Lebesgue measure, not exceeding $L = m(I)$. Let $\mathcal{O}$ be an open cover of E and $\mathcal{P}$ an open cover of F, so that both $\mathcal{O}$, $\mathcal{P}$ are contained in the interval $[a-1, b+1]$. Furthermore, given $\epsilon > 0$, we can require that $m(\mathcal{O}) < m(E) + \epsilon$ and $\mu(\mathcal{P}) < \mu(F) + \epsilon$.

If $\mathcal{O} = \bigcup_i I_i$, where the I_i are pairwise disjoint and open, then by Lemma 9 there is an interval $I' = I_i$ for which $\mu(I') \geq \frac{\mu(\mathcal{O})}{m(\mathcal{O})}\, m(I')$. Hence,

$$\begin{aligned}\mu(I') &\geq \frac{\mu(E)}{m(\mathcal{O})} m(I') \\ &\geq \frac{M\, m(E)}{m(E) + \epsilon}\, m(I').\end{aligned}$$

Writing $\mathcal{P}$ as the disjoint union of pairwise disjoint open sets, there is, by the same lemma, an interval J in $\mathcal{P}$ for which $\mu(J) \leq \frac{\mu(\mathcal{P})}{m(\mathcal{P})}\, m(J)$. Furthermore, we may assume that the length of J is less than the length of I'. Indeed, if this is not the case, then bisect J so that $J = J_1 \cup J_2$. Then J_i satisfies $J_i \subset \mathcal{P}$ and $\mu(J_i) < \frac{\mu(\mathcal{P})}{m(\mathcal{P})}\, m(J_i)$ for at least one of $i = 1,\ 2$. Continue bisecting until an interval is obtained with length less than I'. Rename that interval J.

Hence,

$$\begin{aligned}\mu(J) &\leq \frac{\mu(F) + \epsilon}{m(F)}\, m(J) \\ &\leq \frac{K^{-1}\, m(F) + \epsilon}{m(F)}\, m(J).\end{aligned}$$

Cover I' by finitely many translates of J,

$$I' \subset \bigcup_{k=1}^{n} (J + t_k),$$

where $m(\bigcup_k (J + t_k)) = n\, m(J) \leq 2m(I')$. Also, since the intervals $I',\ J$ are subsets of an interval of length $L + 2$, we have that $|t_k| \leq L + 2,\ 1 \leq k \leq n$.

Let C be the constant in Corollary 2 corresponding to the neighborhood $[L-2, L+2]$. Then

$$\mu(J+t_k) \le C\mu(J),\ 1 \le k \le n.$$

Hence,

$$\begin{aligned}\frac{m(E)}{m(E)+\epsilon}\, Mm(I') &\le \mu(I') \\ &\le \mu(\bigcup_{k=1}^{n} J+t_k) \\ &\le n\,C\,\mu(J) \\ &\le C\,\frac{K^{-1}\,m(F)+\epsilon}{m(F)}\,n\,m(J) \\ &\le C\,\frac{K^{-1}\,m(F)+\epsilon}{m(F)}\,2\,m(I').\end{aligned}$$

Cancelling $m(I')$ from both sides of the inequality, we obtain

$$\frac{m(E)}{m(E)+\epsilon}\,M\,\frac{m(F)}{K^{-1}\,m(F)+\epsilon} \le 2C.$$

Since this is true for all $\epsilon > 0$ we conclude that

$$M\,K \le 2C.$$

However, M was arbitrary in the interval $(0, ||h||_{I,\infty})$ and K was arbitrary in the interval $(0, ||1/h||_{I,\infty})$, so it follows that

$$||h||_{I,\infty}\,||1/h||_{I,\infty} \le 2C.$$

Thus, we can take the constant $C_L = 2C$.

For the converse, suppose that the Radon-Nikodým derivative $h = \frac{d\mu}{dm}$ satisfies the condition of the theorem. It is enough to show that each translation operator T_t is a bounded operator on $L^p(\mu),\ 1 \le p < \infty$. So fix $t \in \mathbb{R}$ and let $L = 2|t|$. So if E is any measurable subset of $\mathbb{R}$ contained in an interval of length $L/2$, then both $E,\ E+t$ are contained in an interval of length L. Hence, $\mu(E+t) \le C_L\,\mu(E)$. Let f be a simple function with compact support in $L^p(\mu)$ and write

$$f = \sum_k a_k\,E_k,$$

where each E_k is contained in an interval of length $L/2$. Then $f \in L^p(\mu)$. Also note that the set of such functions is dense in $L^p(\mu)$ for $1 \le p < \infty$. Then

$$\begin{aligned}||T_t(f)||_p{}^p &= \sum_k |a_k|^p \mu(E_k+t) \\ &\le \sum_k |a_k|^p C_L \mu(E_k) \\ &\le C_L ||f||_p{}^p.\end{aligned}$$

Since T_t is bounded by $C_L{}^{\frac{1}{p}}$ on a dense subset, it is bounded on $L^p(\mu)$. □

Corollary 3. *If μ is a Radon measure on $\mathbb{R}$ and if the group $\{T_t\}$ maps $L^p(\mu)$ into itself for some $p,\ 1 \le p < \infty$, then $\{T_t\}$ maps $L^p(\mu)$ into itself for all $p,\ 1 \le p \le \infty$.*

Example 1. Let $d\mu = h\,dm$, where $h(x) = \exp(c|x|^\alpha)$. Then μ satisfies the conditions of the theorem for any $c \in \mathbb{R}$ and $0 \leq \alpha \leq 1$.

4. Relation to the work of D. Bell

Given a Radon measure μ on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, define μ_t to be the translated measure, $\mu_t(A) = \mu(A - t)$. Note that μ is quasi-translation-invariant (Definition 1) if and only if μ and μ_t are equivalent measures, for all $t \in \mathbb{R}$.

Theorem 5 (Bell [1])**.** *The following are equivalent:*

(1) *μ is q.t.i. on $\mathbb{R}$, and the family of Radon-Nikodým derivatives*

$$h(t,x) = \frac{d\mu_t}{d\mu}$$

is separately continuous in t and x.

(2) *μ is equivalent to Lebesgue measure m with density $f = \frac{d\mu}{dm}$, and f is continuous and positive on $\mathbb{R}$.*

Proof. Suppose (1) holds. By Theorem 3, μ is equivalent to m. If $f = \frac{d\mu}{dm}$, then $f(x-t) = \frac{d\mu_t(x)}{dm(x)}$. Hence,

$$f(x-t) = \frac{d\mu_t(x)}{dm(x)} = \frac{d\mu_t(x)}{d\mu(x)}\,\frac{d\mu(x)}{dm(x)} = h(t,x)f(x).$$

Now for each $x \in \mathbb{R}$ there exists $t \in \mathbb{R}$ such that $f(x-t) > 0$. But then the formula implies $f(x) > 0$. Thus f is strictly positive. Since h is continuous in t for fixed x, that implies that $f(t) = h(-t,0)f(0)$ is continuous in t.

Now assume that (2) holds. By Theorem 3, μ is q.t.i., and the formula in the first part of the proof implies that h is separately (in fact, jointly) continuous in t and x. $\square$

Remark 7. Theorem 3 is not the same as Theorem 5. Theorem 3 says that μ is q.t.i. if and only if μ is equivalent to m, whereas Theorem 5 assumes more about μ and asserts more.

Remark 8. Added in proof: The authors would like to thank Arlan Ramsay and Paul Muhly for the reference to Lemma 3.3 in George Mackey's paper, *A theorem of Stone and von Neumann*, Duke Math. J. **16**, 1949, 313–326, in which he shows that a q.t.i. measure on a locally compact group is equivalent to Haar measure. That proof is different from what is done here.

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