

## MULTI-POINT VARIATIONS OF THE SCHWARZ LEMMA WITH DIAMETER AND WIDTH CONDITIONS

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ABSTRACT. Suppose that  $f$  is holomorphic in the unit disk  $\mathbb{D}$  and  $f(\mathbb{D}) \subset \mathbb{D}$ ,  $f(0) = 0$ . A classical inequality due to Littlewood generalizes the Schwarz lemma and asserts that for every  $w \in f(\mathbb{D})$ , we have  $|w| \leq \prod_j |z_j(w)|$ , where  $z_j(w)$  is the sequence of pre-images of  $w$ . We prove a similar inequality by replacing the assumption  $f(\mathbb{D}) \subset \mathbb{D}$  with the weaker assumption  $\text{Diam} f(\mathbb{D}) = 2$ . This inequality generalizes a growth bound involving only one pre-image, proven recently by Burckel et al. We also prove growth bounds for holomorphic  $f$  mapping  $\mathbb{D}$  onto a region having fixed horizontal width. We give a complete characterization of the equality cases. The main tools in the proofs are the Green function and the Steiner symmetrization.

### 1. INTRODUCTION

The classical lemma of Schwarz, in its standard form, appeared in a paper of Carathéodory in the first decade of the twentieth century. Since then, it has continuously been a source of inspiration for several generations of analysts and geometers. We refer to [1], [13], [17], [5] for nontechnical introductions, historical accounts, and numerous references to the extensions and variations of the Schwarz lemma.

In the present paper, we will prove some multi-point variations of the lemma. Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the unit disk in the complex plane  $\mathbb{C}$  and suppose that  $f : \mathbb{D} \rightarrow \mathbb{C}$  is a holomorphic function. If  $w \in \mathbb{C}$ , we denote by  $\{z_1(w), z_2(w), \dots\}$  the finite or countably infinite set of pre-images of  $w$  with the convention that each pre-image is repeated as many times as its multiplicity. One of the classical extensions of the Schwarz lemma, due to Littlewood (see e.g. [11, Theorem 214], [12, p. 52]), is the following: If  $f(0) = 0$  and  $|f(z)| < 1$  for all  $z \in \mathbb{D}$ , then for all  $w \in f(\mathbb{D}) \setminus \{0\}$ ,

$$(1.1) \quad |w| \leq \prod_j |z_j(w)|.$$

This inequality, expressed via the Nevanlinna counting function

$$N_f(w) := \sum_j \log \frac{1}{|z_j(w)|}, \quad w \in \mathbb{D} \setminus \{0\},$$

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takes the form

$$(1.2) \quad N_f(w) \leq \log \frac{1}{|w|}, \quad w \in \mathbb{D} \setminus \{0\}.$$

The inequality, in this latter form, plays an important role in the theory of composition operators; see [18].

The case of equality in (1.1) or (1.2) has been studied by Lehto [10] (see also [18]): Equality holds for some  $w \in \mathbb{D} \setminus \{0\}$  if and only if  $f$  is an inner function. In this case, equality holds for all  $w$  outside a subset of  $\mathbb{D}$  having logarithmic capacity zero. Recall that an inner function is a holomorphic function  $h : \mathbb{D} \rightarrow \mathbb{D}$  whose radial limits have modulus one at a.e. point of  $\partial\mathbb{D}$ .

Another variation of the Schwarz lemma was proved by Landau and Toeplitz (see [6], [14, Problem 239, p. 151]) and involves the diameter  $\text{Diam}f(\mathbb{D})$  of  $f(\mathbb{D})$ . It states that if  $f : \mathbb{D} \rightarrow \mathbb{C}$  is holomorphic and  $\text{Diam}f(\mathbb{D}) = 2$ , then for every  $0 < r < 1$ ,

$$(1.3) \quad \text{Diam}f(r\mathbb{D}) \leq 2r$$

and

$$(1.4) \quad |f'(0)| \leq 1.$$

Moreover, equality holds in (1.3) for some  $0 < r < 1$  or in (1.4) if and only if  $f(z) = a + cz$  for some constants  $a \in \mathbb{C}$  and  $c \in \partial\mathbb{D}$ . Here and below,  $r\mathbb{D} = \{rz : z \in \mathbb{D}\}$ .

Recently, Burckel, Marshall, Minda, Poggi-Corradini, and Ransford [6] (see also [4]) proved the stronger result that under the same assumptions, the function  $r \mapsto \text{Diam}f(r\mathbb{D})/(2r)$  is increasing and in fact it is strictly increasing unless  $f(z) = a + cz$ . Moreover, they proved a related modulus growth estimate: If  $\text{Diam}f(\mathbb{D}) = 2$ , then for all  $z \in \mathbb{D}$ ,

$$(1.5) \quad |f(z) - f(0)| \leq \frac{2|z|}{1 + \sqrt{1 - |z|^2}}.$$

Equality holds in (1.5) for some  $z \in \mathbb{D} \setminus \{0\}$  if and only if

$$f(z) = c \frac{z - b}{1 - \bar{b}z},$$

for some constants  $a \in \mathbb{C}$ ,  $b \in \mathbb{D} \setminus \{0\}$ ,  $c \in \partial\mathbb{D}$ . Our first theorem is a multi-point extension of (1.5).

**Theorem 1.** *Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a holomorphic function with  $\text{Diam}f(\mathbb{D}) = 2$ . Then*

$$(1.6) \quad \frac{4|w - f(0)|}{4 + |w - f(0)|^2} \leq \prod_j |z_j(w)|, \quad w \in f(\mathbb{D}) \setminus \{f(0)\}.$$

*Equality holds in (1.6) for some  $w_o \in f(\mathbb{D}) \setminus \{f(0)\}$  if and only if there exist an  $a \in \mathbb{D} \setminus \{0\}$  and an inner function  $h$  with  $h(0) = 0$  such that*

$$(1.7) \quad f(z) = \frac{h(z) + a}{1 + \bar{a}h(z)} + a + w_o, \quad z \in \mathbb{D}.$$

The paper [6] contains various other versions of the Schwarz lemma with geometric conditions involving the logarithmic capacity, the  $n$ -diameter, and the area

of  $f(\mathbb{D})$ . We will prove an analogous result involving the horizontal width. This quantity is defined by

$$W(f(r\mathbb{D})) = \sup_{z_1, z_2 \in r\mathbb{D}} |\operatorname{Re} f(z_1) - \operatorname{Re} f(z_2)|, \quad 0 < r \leq 1.$$

The horizontal width of the image domain was studied by Pólya and Szegő [14, Problem 238, p. 151; Problem 289, p. 162], who showed that if  $W(f(\mathbb{D})) = \frac{\pi}{2}$ , then

$$(1.8) \quad |f'(0)| \leq 1$$

with equality if  $f(z) = \tan^{-1} z$ . Note that the function  $\tan^{-1} z$  maps  $\mathbb{D}$  conformally onto the vertical strip

$$S_o = \{w \in \mathbb{C} : -\frac{\pi}{4} < \operatorname{Re} w < \frac{\pi}{4}\}.$$

Hayman [8, p. 129] gave another proof of (1.8) using Steiner symmetrization. We will also use Steiner symmetrization to prove the following theorem. Vuorinen [20, p. 143] has obtained similar results for quasiregular mappings.

**Theorem 2.** *Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a holomorphic function with  $W(f(\mathbb{D})) = \frac{\pi}{2}$ .*

(a) *For every  $r \in (0, 1)$ ,*

$$(1.9) \quad W(f(r\mathbb{D})) \leq 2 \tan^{-1} r.$$

*Equality holds for some  $r \in (0, 1)$  if and only if there exist  $b \in \mathbb{C}$  and  $\theta \in [0, 2\pi)$  such that  $f(z) = \tan^{-1}(e^{i\theta} z) + b$ .*

(b) *For every  $w \in f(\mathbb{D}) \setminus \{f(0)\}$ ,*

$$(1.10) \quad \tanh |w - f(0)| \leq \prod_j |z_j(w)|.$$

*Equality holds for some  $w_o \in f(\mathbb{D}) \setminus \{f(0)\}$  if and only if there exists an inner function  $h$  with  $h(0) = 0$  such that*

$$f(z) = \tan^{-1} \left( \frac{h(z) + iu}{1 - iu h(z)} \right) + t, \quad z \in \mathbb{D},$$

*with  $u = \tanh(\operatorname{Im} w_o) \in (-1, 1)$  and  $t = \operatorname{Re} w_o \in \mathbb{R}$ . If  $f$  has this form, then it maps  $\mathbb{D}$  onto the vertical strip*

$$S_t = \{w \in \mathbb{C} : t - \frac{\pi}{4} < \operatorname{Re} w < t + \frac{\pi}{4}\}$$

*and the point  $f(0)$  lies on the mid-line  $\{w \in \mathbb{C} : \operatorname{Re} w = t\}$  of this strip. Moreover, equality holds in (1.10) for all  $w \neq f(0)$  on the mid-line of  $S_t$ .*

(c) *For every  $z \in \mathbb{D}$ ,*

$$(1.11) \quad |f(z) - f(0)| \leq \tanh^{-1} |z|.$$

*Equality holds for some  $z_o \in \mathbb{D} \setminus \{0\}$  if and only if there exist  $t \in \mathbb{R}$  and  $u \in (-1, 1)$  such that*

$$f(z) = \pm \tan^{-1} \left( \frac{ie^{-i\theta_o} z + iu}{1 + ue^{-i\theta_o} z} \right) + t, \quad z \in \mathbb{D},$$

*where  $\theta_o = \arg z_o$ . If  $f$  has this form, then equality holds in (1.11) for all  $z$  on the diameter  $\{re^{i\theta_o} : -1 < r < 1\}$  of  $\mathbb{D}$ .*

*Remark.* In view of Theorem 2 and of the monotonicity results in [6], one could conjecture that the function

$$r \mapsto \frac{W(f(r\mathbb{D}))}{\tan^{-1} r}, \quad 0 < r < 1,$$

is increasing. However, the referee of the paper showed that this conjecture is false by using the function  $f(z) = \tan^{-1}(z/\rho)$ , where  $\rho$  is a fixed small positive number.

We will prove Theorems 1 and 2 in sections 3 and 4 after the preparatory next section.

## 2. STEINER SYMMETRIZATION AND THE GREEN FUNCTION

**2.1. Steiner symmetrization.** The Steiner symmetrization of an open set  $A \subset \mathbb{C}$  with respect to a line  $\ell$  is an open set  $S_\ell A$ , symmetric with respect to  $\ell$ . We define it by determining the intersection of  $S_\ell A$  with every line perpendicular to  $\ell$ . Let  $\gamma$  be such a line. Then  $\gamma \cap S_\ell A$  is an open linear segment on  $\gamma$ , symmetric with respect to  $\ell$  and having length equal to  $m_1(\gamma \cap A)$ ; here  $m_1$  denotes the one-dimensional Lebesgue measure. If  $\gamma \cap A = \emptyset$ , then  $\gamma \cap S_\ell A = \emptyset$ .

The Steiner symmetrization  $S_\ell K$  of a compact set  $K$  is defined similarly with the difference that  $\gamma \cap S_\ell K$  is a *closed* segment. Also, if  $m_1(\gamma \cap K) = 0$  but  $\gamma \cap K \neq \emptyset$ , then, by definition,  $\gamma \cap S_\ell K$  is the singleton  $\ell \cap \gamma$ . The set  $S_\ell K$  is a compact set, symmetric with respect to the line  $\ell$ .

For the basic properties and applications of Steiner symmetrization in complex analysis, we refer to [7], [8]. Here we mention only two elementary properties. Suppose that  $D$  is a planar domain and  $\ell$  is a line. Then  $S_\ell D$  is a simply connected domain. If  $S_\ell D = D$ , then we say that  $D$  is Steiner symmetric with respect to  $\ell$ . The second property is that

$$\text{Diam} S_\ell D \leq \text{Diam} D.$$

**2.2. The Green function.** The basic theory of the Green function is presented e.g. in [16]. The Green function for the unit disk has an explicit formula:

$$(2.1) \quad g(z_1, z_2, \mathbb{D}) = \log \left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right|.$$

It follows easily from this formula that

$$(2.2) \quad g(z_1, z_2, \mathbb{D}) \geq g(-|z_1|, |z_2|, \mathbb{D}),$$

with equality if and only if the points  $z_1, z_2$  lie on the same diameter but on different radii of  $\mathbb{D}$ .

Suppose that  $D$  is a domain in  $\mathbb{C}$  possessing a Green function. Let  $z_1, z_2 \in D$ ,  $z_1 \neq z_2$ . Let  $\ell$  be a line. Then we have (see [2], [9, Chapter 9])

$$(2.3) \quad g(z_1, z_2, D) \leq g(p_\ell(z_1), p_\ell(z_2), S_\ell D),$$

where  $p_\ell(z_j)$  is the vertical projection of  $z_j$  on  $\ell$ ,  $j = 1, 2$ . Moreover (see [3, p. 416]), equality holds in (2.3) if and only if there exists a line  $l$  parallel to  $\ell$  such that  $z_1, z_2 \in l$  and  $S_l D \stackrel{n.e.}{=} D$ . The last relation means that the two sets differ on a set of zero logarithmic capacity.

We will need another result on the behavior of the Green function under symmetrization.

**Lemma 1.** *Let  $D$  be a domain in  $\mathbb{C}$  which is Steiner symmetric with respect to a line  $\ell$ . Let  $z, \zeta \in \ell \cap D$ . Let  $\gamma$  be the line perpendicular to  $\ell$  at the point  $(z + \zeta)/2$ . Then*

$$(2.4) \quad g(z, \zeta, D) \leq g(z, \zeta, S_\gamma D).$$

*Equality holds if and only if  $S_\gamma D \stackrel{n.e.}{=} D$ .*

To prove Lemma 1 we need some facts about condensers and the modulus metric.

**2.3. Condensers.** A condenser is a pair  $(D, K)$ , where  $D$  is a domain in  $\mathbb{C}$  and  $K$  is a compact subset of  $D$ . The capacity  $\text{cap}(D, K)$  of the condenser  $(D, K)$  is defined via the Dirichlet integral; see [7], [8]. If  $\ell$  is a line, then  $(S_\ell D, S_\ell K)$  is a condenser and

$$(2.5) \quad \text{cap}(S_\ell D, S_\ell K) \leq \text{cap}(D, K).$$

Moreover (see [19] or [15]), equality holds if and only if  $S_\ell D \stackrel{n.e.}{=} D$  and  $S_\ell K \stackrel{n.e.}{=} K$ .

**2.4. The modulus metric.** Let  $D$  be a domain in  $\mathbb{C}$  and let  $z, \zeta \in D$ ,  $z \neq \zeta$ . We define

$$(2.6) \quad \mu(z, \zeta, D) = \inf_{z, \zeta \in K} \text{cap}(D, K),$$

where the infimum is taken over all curves  $K$  in  $D$  with  $z, \zeta \in K$ . This definition provides  $D$  with a conformally invariant metric, called the modulus metric. We refer to [20] for the basic properties of the modulus metric. Here we are interested in its connection with the Green function.

**Lemma 2.** *There exists a strictly decreasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  such that for every simply connected domain  $D \subsetneq \mathbb{C}$  and every pair  $z, \zeta \in D$ ,*

$$(2.7) \quad \mu(z, \zeta, D) = \Phi(g(z, \zeta, D)).$$

*Proof.* By conformal invariance, we may assume that  $D = \mathbb{D}$ . In this case we have explicit expressions for both the modulus metric and the Green function. Indeed by [20, p. 104]

$$\mu(z, \zeta, \mathbb{D}) = \Phi_1(\rho(z, \zeta, \mathbb{D})),$$

where  $\rho(z, \zeta, \mathbb{D})$  is the hyperbolic distance between  $z$  and  $\zeta$  in  $\mathbb{D}$  and  $\Phi_1$  is a function given explicitly in [20] via elliptic integrals. Here we need only the fact that  $\Phi_1$  is strictly increasing.

Also, by the explicit expressions of the hyperbolic distance and the Green function in  $\mathbb{D}$ ,

$$\rho(z, \zeta, \mathbb{D}) = \Phi_2(g(z, \zeta, \mathbb{D})),$$

where  $\Phi_2$  is the strictly decreasing function given by

$$\Phi_2(x) = \frac{1}{2} \log \frac{1 + e^{-x}}{1 - e^{-x}}, \quad x > 0.$$

We conclude that  $\mu(z, \zeta, D) = \Phi(g(z, \zeta, D))$ , where  $\Phi = \Phi_1 \circ \Phi_2$ . □

**2.5. Proof of Lemma 1.** Suppose that  $D, \ell, z, \zeta, \gamma$  are as in Lemma 1. Since Steiner symmetrization decreases the capacity of condensers, the extremal curve in the definition of  $\mu(z, \zeta, D)$  is the closed segment  $[z, \zeta]$ , namely

$$(2.8) \quad \mu(z, \zeta, D) = \text{cap}(D, [z, \zeta]).$$

Now we apply Steiner symmetrization with respect to  $\gamma$ . Note that  $S_\gamma[z, \zeta] = [z, \zeta]$ . By (2.8) and the fact that  $S_\gamma$  decreases the capacity of condensers, we conclude that

$$(2.9) \quad \mu(z, \zeta, D) = \text{cap}(D, [z, \zeta]) \geq \text{cap}(S_\gamma D, [z, \zeta]) = \mu(z, \zeta, S_\gamma D).$$

The inequality (2.4) follows from (2.9) and Lemma 2.

Now suppose that we have equality in (2.4). Since  $\Phi$  is strictly decreasing, it follows from (2.9) that

$$(2.10) \quad \text{cap}(D, [z, \zeta]) = \text{cap}(S_\gamma D, [z, \zeta]).$$

By the result of Pouliasis [15] mentioned above, we conclude that  $S_\gamma D \stackrel{n.e.}{=} D$ . The converse is obvious.  $\square$

**2.6. Lindelöf's principle.** An inequality stronger than that of Littlewood mentioned in the introduction is (a special case of) Lindelöf's principle; see [12, Ch. III, §4]: If  $f : \mathbb{D} \rightarrow \mathbb{C}$  is holomorphic and  $f(\mathbb{D})$  is a domain possessing Green function, then for  $w \in f(\mathbb{D}) \setminus \{f(0)\}$ ,

$$(2.11) \quad \sum_j \log \frac{1}{|z_j(w)|} = \sum_j g(z_j(w), 0, \mathbb{D}) \leq g(w, f(0), f(\mathbb{D})).$$

This inequality implies the well-known inequality

$$(2.12) \quad g(z_1, z_2, \mathbb{D}) \leq g(f(z_1), f(z_2), f(\mathbb{D})), \quad z_1, z_2 \in \mathbb{D}.$$

If equality holds in (2.12) for a pair  $z_1, z_2 \in \mathbb{D}$ , then  $f$  is a conformal mapping; see [16, p. 112].

**2.7. The Green function for a strip.** We will need the following estimate for the Green function of the strip

$$S_o = \{w \in \mathbb{C} : -\frac{\pi}{4} < \text{Re } w < \frac{\pi}{4}\}.$$

**Lemma 3.** *Let  $w_o, w_1$  be two points in the strip  $S_o$  with  $w_o \neq w_1$ . Then*

$$(2.13) \quad g(w_o, w_1, S_o) \leq g(-i|w_o - w_1|/2, i|w_o - w_1|/2, S_o)$$

*with equality if and only if  $\text{Re } w_o = \text{Re } w_1 = 0$ .*

*Proof.* We will use the hyperbolic distance  $\rho(z, w, S_o)$  for  $S_o$ . We denote its density by  $\sigma(w, S_o)$ . Set  $s = |w_o - w_1|/2$ . To prove (2.13), it suffices to show that

$$(2.14) \quad \rho(w_o, w_1, S_o) \geq \rho(-is, is, S_o).$$

The proof of (2.14) is based on the fact that the mid-line of  $S_o$  is a hyperbolic geodesic for  $S_o$ . By the explicit expression for the hyperbolic density  $\sigma(w, S_o)$  (see e.g. [9, p. 684]),

$$(2.15) \quad \sigma(w, S_o) \geq \sigma(i|w|, S_o), \quad w \in S_o.$$

Because of symmetry, we may assume that  $|\text{Re } w_o| \leq |\text{Re } w_1|$  and that  $\text{Re } w_o \geq 0$ .

Case 1.  $\operatorname{Re} w_1 > 0$ .

We may assume that the points  $w_o, w_1$  lie on a line through the origin. Let  $q$  be the hyperbolic geodesic (in the hyperbolic geometry of  $S_o$ ) joining  $w_o$  to  $w_1$ . This is an analytic curve lying in the right mid-strip. By (2.15),

$$\begin{aligned} (2.16) \quad \rho(w_o, w_1, S_o) &= \int_q \sigma(w, S_o) |dw| \geq \int_{|w_o|}^{|w_1|} \sigma(ir, S_o) dr \\ &= \rho(i|w_o|, i|w_1|, S_o) = \rho(-is, is, S_o), \end{aligned}$$

and (2.14) is proved.

Case 2.  $\operatorname{Re} w_1 < 0$ .

In this case, the hyperbolic geodesic joining  $w_o, w_1$  intersects the imaginary axis. We may assume that the point of intersection is the origin. By Case 1,

$$\begin{aligned} (2.17) \quad \rho(w_o, w_1, S_o) &= \rho(w_o, 0, S_o) + \rho(0, w_1, S_o) \\ &\geq \rho(-i|w_o|, 0, S_o) + \rho(0, i|w_1|, S_o) \\ &= \rho(-i|w_o|, i|w_1|, S_o) \\ &= \rho(-is, is, S_o), \end{aligned}$$

and (2.14) is proved in Case 2 as well. □

### 3. PROOF OF THEOREM 1

Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a holomorphic function with  $\operatorname{Diam} f(\mathbb{D}) = 2$ . Let  $w \in f(\mathbb{D}) \setminus \{f(0)\}$ . By Lindelöf's principle,

$$(3.1) \quad \sum_j \log \frac{1}{|z_j(w)|} \leq g(w, f(0), f(\mathbb{D})).$$

Let  $\ell$  be the line passing through  $f(0)$  and  $w$ . By the symmetrization inequality (2.3),

$$(3.2) \quad g(w, f(0), f(\mathbb{D})) \leq g(w, f(0), S_\ell f(\mathbb{D})).$$

By Lemma 1,

$$(3.3) \quad g(w, f(0), S_\ell f(\mathbb{D})) \leq g(w, f(0), S_\gamma S_\ell f(\mathbb{D})),$$

where  $\gamma$  is the line perpendicular to  $\ell$  at the point  $\frac{w+f(0)}{2}$ .

The domain  $S_\gamma S_\ell f(\mathbb{D})$  is a Steiner symmetric with respect to each of the perpendicular lines  $\ell$  and  $\gamma$ , and also

$$\operatorname{Diam} S_\gamma S_\ell f(\mathbb{D}) \leq \operatorname{Diam} f(\mathbb{D}) = 2.$$

It easily follows (see [4, Lemma 2]) that  $S_\gamma S_\ell f(\mathbb{D})$  is contained in the disk  $\Delta$  of radius 1, centered at  $\frac{w+f(0)}{2}$ . By the domain monotonicity and the conformal invariance of the Green function and (2.1),

$$\begin{aligned} (3.4) \quad g(w, f(0), S_\gamma S_\ell f(\mathbb{D})) &\leq g(w, f(0), \Delta) \\ &= g\left(-\frac{|f(0) - w|}{2}, \frac{|f(0) - w|}{2}, \mathbb{D}\right) \\ &= \log \frac{4 + |f(0) - w|^2}{4 |f(0) - w|}. \end{aligned}$$

Now (1.6) follows from (3.1)-(3.4).

Suppose that equality holds in (1.6) for some  $w_o \in f(\mathbb{D}) \setminus \{f(0)\}$ , namely,

$$(3.5) \quad \frac{4|w_o - f(0)|}{4 + |w_o - f(0)|^2} = \prod_j |z_j(w_o)|.$$

Then equality holds (for  $w = w_o$ ) in (3.1)-(3.4). By the equality statements for the inequalities (2.3) and (2.4) and for the domain monotonicity of the Green function,

$$(3.6) \quad f(\mathbb{D}) \stackrel{n.e.}{=} \Delta = \left\{ w \in \mathbb{C} : \left| w - \frac{f(0) + w_o}{2} \right| < 1 \right\}.$$

We consider the linear fractional transformations

$$\phi_1(w) = w - \frac{f(0) + w_o}{2} : \Delta \rightarrow \mathbb{D}$$

and

$$\phi_2(w) = \frac{w - (f(0) - w_o)/2}{1 - w(f(0) - w_o)/2} : \mathbb{D} \rightarrow \mathbb{D}.$$

The function  $h := \phi_2 \circ \phi_1 \circ f$  maps  $\mathbb{D}$  to  $\mathbb{D}$  with

$$h(0) = \phi_2 \circ \phi_1(f(0)) = \phi_2\left(\frac{f(0) - w_o}{2}\right) = 0.$$

Moreover, for every  $j$ ,

$$(3.7) \quad \begin{aligned} h(z_j(w_o)) &= \phi_2 \circ \phi_1(f(z_j(w_o))) = \phi_2 \circ \phi_1(w_o) \\ &= \frac{w_o - f(0)}{1 + |w_o - f(0)|^2/4}. \end{aligned}$$

By (3.5) and (3.7),

$$(3.8) \quad |h(z_j(w_o))| = \prod_j |z_j(w_o)|.$$

Hence we have equality in Littlewood's inequality (1.1) for the function  $h$ . By Lehto's result mentioned in the introduction, the function  $h$  is inner. It follows that

$$(3.9) \quad \begin{aligned} f(z) &= \phi_1^{-1} \circ \phi_2^{-1} \circ h(z) = \phi_2^{-1}(h(z)) + \frac{f(0) + w_o}{2} \\ &= \frac{h(z) + (f(0) - w_o)/2}{1 + h(z)(f(0) - w_o)/2} + \frac{f(0) + w_o}{2} \\ &= \frac{h(z) + a}{1 + \bar{a} h(z)} + a + w_o, \quad z \in \mathbb{D}, \end{aligned}$$

with  $a = \frac{f(0) - w_o}{2}$ . Note that

$$|a| = \frac{|f(0) - w_o|}{2} < 1,$$

because  $f(0)$  and  $w_o$  belong to  $f(\mathbb{D})$  and  $\text{Diam} f(\mathbb{D}) = 2$ .

Conversely, assume that

$$(3.10) \quad f(z) = \frac{h(z) + a}{1 + \bar{a} h(z)} + a + w_o,$$



where  $a \in \mathbb{D} \setminus \{0\}$  and  $h$  is an inner function with  $h(0) = 0$ . Then  $f$  maps  $\mathbb{D}$  onto a disk  $\Delta$  of radius 1, centered at  $a + w_o$ . Since  $a \in \mathbb{D} \setminus \{0\}$ , we have  $w_o \in f(\mathbb{D}) \setminus \{f(0)\}$  and  $f(0) = 2a + w_o \neq w_o$ . Moreover, solving for  $h$  in (3.10), we obtain

$$(3.11) \quad \begin{aligned} h(z_j(w_o)) &= \frac{f(z_j(w_o)) - a - w_o - a}{1 - \bar{a}(f(z_j(w_o)) - a - w_o)} \\ &= \frac{-2a}{1 + |a|^2} = \frac{w_o - f(0)}{1 + |w_o - f(0)|^2/4}. \end{aligned}$$

Since  $h$  is inner, Lehto's equality statement (applied to  $h$ ) and (3.11) give

$$(3.12) \quad \prod_j |z_j(w_o)| = \frac{|w_o - f(0)|}{1 + |w_o - f(0)|^2/4}.$$

Therefore, we have equality in (1.6) for  $w = w_o$ . □

#### 4. PROOF OF THEOREM 2

(a) Let  $0 < r < 1$ . There exist points  $\tilde{w}_r, w_r$  on  $\partial f(r\mathbb{D})$  such that

$$(4.1) \quad W(f(r\mathbb{D})) = \operatorname{Re} w_r - \operatorname{Re} \tilde{w}_r.$$

Let  $\tilde{z}_r, z_r$  be points with  $|\tilde{z}_r| = |z_r| = r$  and  $\tilde{w}_r = f(\tilde{z}_r)$ ,  $w_r = f(z_r)$ . By (2.2),

$$(4.2) \quad \log \frac{1 + r^2}{2r} = g(-r, r, \mathbb{D}) \leq g(\tilde{z}_r, z_r, \mathbb{D}).$$

By Lindelöf's principle,

$$(4.3) \quad g(\tilde{z}_r, z_r, \mathbb{D}) \leq g(\tilde{w}_r, w_r, f(\mathbb{D})).$$

Let  $S_{\mathbb{R}}$  denote Steiner symmetrization with respect to the real axis and  $S_{\gamma}$  denote Steiner symmetrization with respect to the vertical line intersecting  $\mathbb{R}$  at the point  $t := \operatorname{Re}(\tilde{w}_r + w_r)/2$ . By the symmetrization inequalities for the Green function ((2.3) and Lemma 1),

$$(4.4) \quad g(\tilde{w}_r, w_r, f(\mathbb{D})) \leq g(\operatorname{Re} \tilde{w}_r, \operatorname{Re} w_r, S_{\gamma} S_{\mathbb{R}} f(\mathbb{D})).$$

The simply connected domain  $S_{\gamma} S_{\mathbb{R}} f(\mathbb{D})$  is contained in the strip

$$S_t = \{w \in \mathbb{C} : -\frac{\pi}{4} + t < \operatorname{Re} w < t + \frac{\pi}{4}\}.$$

The domain monotonicity of the Green function yields

$$(4.5) \quad g(\operatorname{Re} \tilde{w}_r, \operatorname{Re} w_r, S_{\gamma} S_{\mathbb{R}} f(\mathbb{D})) \leq g(\operatorname{Re} \tilde{w}_r, \operatorname{Re} w_r, S_t).$$

We now use the conformal mapping  $\tan : S_o \rightarrow \mathbb{D}$  and the conformal invariance of the Green function to obtain

$$(4.6) \quad \begin{aligned} g(\operatorname{Re} \tilde{w}_r, \operatorname{Re} w_r, S_t) &= g\left(-\frac{W(f(r\mathbb{D}))}{2}, \frac{W(f(r\mathbb{D}))}{2}, S_o\right) \\ &= g\left(-\tan \frac{W(f(r\mathbb{D}))}{2}, \tan \frac{W(f(r\mathbb{D}))}{2}, \mathbb{D}\right) \\ &= \log \frac{1 + \tan^2(W(f(r\mathbb{D}))/2)}{2 \tan(W(f(r\mathbb{D}))/2)}. \end{aligned}$$

It follows from (4.2)-(4.6) that

$$(4.7) \quad \frac{1 + r^2}{2r} \leq \frac{1 + \tan^2(W(f(r\mathbb{D}))/2)}{2 \tan(W(f(r\mathbb{D}))/2)}.$$

The function  $x \mapsto (1 + x^2)/(2x)$ ,  $0 < x < 1$ , is strictly decreasing. Therefore (4.7) implies

$$(4.8) \quad \tan \frac{W(f(r\mathbb{D}))}{2} \leq r,$$

which is equivalent to (1.9).

Suppose that for some  $r \in (0, 1)$ , we have equality in (1.9). Then for this  $r$ , we have equality in each of the inequalities (4.2)-(4.5). We will use the equality statements from section 2. Equality in (4.3) implies that  $f$  is a conformal mapping and therefore  $f(\mathbb{D})$  is simply connected. Equality in (4.4) implies that  $f(\mathbb{D})$  is Steiner symmetric with respect to a horizontal line and with respect to  $\gamma$  and that  $\text{Im } \tilde{w}_r = \text{Im } w_r$ . Equality in (4.5), together with the above facts, implies that  $f(\mathbb{D}) = S_t$ . We set  $b := f(0)$ . The function  $g(z) := \tan(f(z) - b)$  maps  $\mathbb{D}$  conformally onto itself and  $g(0) = 0$ . Hence there exist a  $\theta \in [0, 2\pi)$  such that  $g(z) = e^{i\theta}z$ . It follows that

$$f(z) = \tan^{-1}(e^{i\theta}z) + b.$$

Conversely, if  $f$  has this form for some  $\theta \in [0, 2\pi)$  and some  $b \in \mathbb{C}$ , then it maps  $\mathbb{D}$  conformally onto the strip  $S_t$  with  $t = \text{Re } b$ . Moreover, for  $0 < r < 1$ , the points  $e^{-i\theta}r$ ,  $-e^{-i\theta}r$  are mapped to the points  $b - \tan^{-1}r$ ,  $b + \tan^{-1}r$ , which implies that

$$W(f(r\mathbb{D})) = 2 \tan^{-1} r,$$

and hence we have equality in (1.9).

(b) Let  $w \in f(\mathbb{D}) \setminus \{f(0)\}$ . Set  $s = \frac{1}{2}|f(0) - w_o|$ . By Lindelöf's principle, Lemma 3, the conformal invariance and the domain monotonicity of Green function,

$$(4.9) \quad \begin{aligned} \sum_j \log \frac{1}{|z_j|} &= \sum_j g(z_j, 0, \mathbb{D}) \leq g(w, f(0), f(\mathbb{D})) \\ &\leq g(-is, is, S_o) = g(-\tanh s, \tanh s, \mathbb{D}) \\ &= \log \frac{1 + \tanh^2 s}{2 \tanh s} = \log \frac{1}{\tanh(2s)}. \end{aligned}$$

It follows that

$$\tanh(2s) \leq \prod_j |z_j|,$$

which is equivalent to (1.10).

Suppose that we have equality in (1.10) for  $w = w_o \in f(\mathbb{D}) \setminus \{f(0)\}$ . Then all the inequalities in (4.9) become equalities for  $w = w_o$ . Hence  $f(\mathbb{D}) \stackrel{n.e.}{=} S_{\text{Re } w_o}$ , and  $f(0)$  lies on the mid-line of  $S_{\text{Re } w_o}$ . Consider the function

$$(4.10) \quad h(z) = \frac{\tan(f(z) - \text{Re } w_o) - i \tanh(\text{Im } f(0))}{1 + i \tanh(\text{Im } f(0)) \tan(f(z) - \text{Re } w_o)}$$

which maps  $\mathbb{D}$  to  $\mathbb{D}$  with  $h(0) = 0$ . Moreover,

$$(4.11) \quad \begin{aligned} h(z_j(w_o)) &= \frac{i \tanh(\text{Im } w_o) - i \tanh(\text{Im } f(0))}{1 - \tanh(\text{Im } f(0)) \tanh(\text{Im } w_o)} \\ &= i \tanh(\text{Im } w_o - \text{Im } f(0)). \end{aligned}$$

Hence

$$(4.12) \quad |h(z_j(w_o))| = \tanh |w_o - f(0)| = \prod_j |z_j(w_o)|.$$

This means that we have equality in Littlewood’s inequality for the function  $h$ . By Lehto’s equality statement (see the introduction),  $h$  is inner. Solving for  $f$  in (4.10), we obtain

$$(4.13) \quad f(z) = \tan^{-1} \left( \frac{h(z) + iu}{1 - iu h(z)} \right) + t, \quad z \in \mathbb{D},$$

with  $u = \tanh(\operatorname{Im} w_o) \in (-1, 1)$  and  $t = \operatorname{Re} w_o \in \mathbb{R}$ .

Conversely, assume that  $f$  has the above form for some  $u \in (-1, 1)$ , some  $t \in \mathbb{R}$ , and some inner function  $h$  with  $h(0) = 0$ . Then  $f$  maps  $\mathbb{D}$  onto  $S_t$  with

$$(4.14) \quad \operatorname{Re} f(0) = t, \quad \operatorname{Im} f(0) = \tanh^{-1} u.$$

Solving for  $h$  in (4.13) we find

$$(4.15) \quad h(z) = \frac{\tan(f(z) - t) - iu}{1 + iu \tan(f(z) - t)}, \quad z \in \mathbb{D}.$$

Using this expression for  $h$  and (4.14), we infer that for every  $w$  in the mid-line of  $S_t$ , namely for  $\operatorname{Re} w = t$ , we have

$$(4.16) \quad h(z_j(w)) = i \tanh(\operatorname{Im} w - \operatorname{Im} f(0)).$$

Since  $h$  is inner, Lehto’s result (applied to  $h$ ) and (4.16) imply that

$$(4.17) \quad \prod_j |z_j(w)| = \tanh |w - f(0)|.$$

Therefore for such a  $w$ , (1.10) holds with equality.

(c) The inequality (1.11) follows at once from (1.10). However, in order to treat the equality case, we give a straightforward proof. Let  $z \in \mathbb{D}$ . By the symmetrization argument (as in the proof of (b)),

$$(4.18) \quad g(z, 0, \mathbb{D}) \leq g(f(z), f(0), S_t),$$

where  $t = \operatorname{Re} f(0)$ . This is equivalent to (1.11).

Suppose that equality holds in (1.11) for some  $z_o \in \mathbb{D} \setminus \{0\}$ . This means that

$$(4.19) \quad g(z_o, 0, \mathbb{D}) = g(f(z_o), f(0), S_t),$$

where  $t = \operatorname{Re} f(z_o) = \operatorname{Re} f(0)$ . By arguments similar to those in (b),  $f$  is a conformal mapping of  $\mathbb{D}$  onto  $S_t$  and the points  $0, z_o$  are mapped on the mid-line of  $S_t$ . Simple calculations show that all such mappings have the form

$$f(z) = \pm \tan^{-1} \left( \frac{ie^{-i\theta_o} z + iu}{1 + ue^{-i\theta_o} z} \right) + t, \quad z \in \mathbb{D},$$

where  $\theta_o = \arg z_o$  and  $u = \tanh(\operatorname{Im} f(z_o))$ .

Conversely, if  $f$  has this form for some  $u \in (-1, 1)$  and some  $t, \theta_o \in \mathbb{R}$ , then it maps  $\mathbb{D}$  conformally onto  $S_t$ . Also, every  $z$  lying on the diameter  $\{re^{i\theta_o} : -1 < r < 1\}$  of  $\mathbb{D}$  is mapped on the mid-line of the strip  $S_t$ . By the conformal invariance of the Green function, for such a  $z \neq 0$ ,

$$g(0, z, \mathbb{D}) = g(f(0), f(z), S_t),$$

which is equivalent to (1.11) with equality. □

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