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SCHUR-AGLER CLASS RATIONAL INNER FUNCTIONS ON THE TRIDISK

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Abstract. We prove two results with regard to rational inner functions in the Schur-Agler class of the tridisk. Every rational inner function of degree $(n, 1, 1)$ is in the Schur-Agler class, and every rational inner function of degree $(n, m, 1)$ is in the Schur-Agler class after multiplication by a monomial of sufficiently high degree.

1. Prologue

In this article, we continue the study of the Schur-Agler class of the polydisk by focusing on rational inner functions. The *Schur-Agler class* appears naturally in operator theory as the class of holomorphic functions $f : \mathbb{D}^n \to \mathbb{D}$ which satisfy the von Neumann inequality; i.e. for all commuting n-tuples of strict contractions $(T_1, \dots, T_n)$ on some separable Hilbert space, we have

$$||f(T_1, \dots, T_n)|| \leq 1.$$

The *Schur class* simply refers to the holomorphic functions $f : \mathbb{D}^n \to \mathbb{D}$. Our general motivating question is this:

How does the Schur-Agler class fit inside the Schur class?

For $n = 1, 2$ these two classes coincide, but they differ for $n \geq 3$, and this is not well understood. More recent efforts in this area have focused on generalizations and properties of the Schur-Agler class. See [Anderson et al., 2008], [Ball and Bolotnikov, 2002], [Ball and Bolotnikov, 2010]. For more specific progress on this question, one probably has to go back to work of the 1970's on counterexamples to von Neumann's inequality. See [Varopoulos, 1974], [Crabb and Davie, 1975], [Lotto, 1994], [Holbrook, 2001].

Motivated by the recent major strides in the study of two variable rational inner functions from [Cole and Wermer, 1999], [Geronimo and Woerdeman, 2004], [Ball et al., 2005], along with our own efforts [Knese, 2008], [Knese, 2010a], the approach of this article is to make progress on this question by studying rational inner functions in the Schur-Agler class on $\mathbb{D}^3$. For further motivation and background to this approach we refer the reader to [Knese, 2010b] and [Knese, 2010c]. We now

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introduce our topic purely in terms of polynomials, as our main results serve to establish a close connection between sums of squares decompositions for positive trigonometric polynomials and the Schur-Agler class on the tridisk $\mathbb{D}^3$.

2. Rational inner functions in the Schur-Agler class

Let $\mathbb{D}, \mathbb{D}^n, \mathbb{T}, \mathbb{T}^n$ denote the unit disk, polydisk, torus, and n-torus. We say $p \in \mathbb{C}[z_1, \dots, z_n]$ has multi-degree at most $\mathbf{d} = (d_1, \dots, d_n)$ if it has degree at most d_j in the variable z_j.

If p has multi-degree at most $\mathbf{d}$ we may form a type of reflection (depending on the degree)

$$\tilde{p}(z) := z^{\mathbf{d}} \overline{p(1/\bar{z}_1, \dots, 1/\bar{z}_n)} \in \mathbb{C}[z_1, \dots, z_n],$$

and if in addition p has no zeros on $\mathbb{D}^n$, then the rational function

$$\phi(z) = \frac{\tilde{p}(z)}{p(z)} \tag{2.1}$$

is a rational inner function; i.e. has modulus one a.e. on $\mathbb{T}^n$ and modulus at most one on $\mathbb{D}^n$, by the maximum principle. Theorem 5.2.5 of [Rudin, 1969] proves that every rational inner function on $\mathbb{D}^n$ arises as in (2.1).

In particular,

$$\begin{aligned} &|p(z)|^2 - |\tilde{p}(z)|^2 = 0 \text{ on } \mathbb{T}^n, \\ &|p(z)|^2 - |\tilde{p}(z)|^2 \geq 0 \text{ on } \overline{\mathbb{D}}^n. \end{aligned} \tag{2.2}$$

On the other hand, any expression of the form

$$\sum_{j=1}^{n} (1 - |z_j|^2) SOS_j(z), \tag{2.3}$$

where each SOS_j is a sum of squared moduli of polynomials, also satisfies this inequality. It turns out that $\tilde{p}/p$ is in the Schur-Agler class exactly when the left side of (2.2) is of the form (2.3).

Theorem 2.1. *Given a polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ with no zeros in $\mathbb{D}^n$ and degree at most $\mathbf{d}$, $\tilde{p}/p$ is in the Schur-Agler class exactly when there exists a decomposition*

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{n} (1 - |z_j|^2) SOS_j(z),$$

where each SOS_j is a sum of squared moduli of polynomials.

This theorem is implicit in [Cole and Wermer, 1999]. We refined the above theorem as follows.

Theorem 2.2 ([Knese, 2010b])**.** *If $p \in \mathbb{C}[z_1, \dots, z_n]$ has multi-degree at most $\mathbf{d} = (d_1, \dots, d_n)$ and $\tilde{p}/p$ is in the Schur-Agler class, then given a decomposition*

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{n} (1 - |z_j|^2) K_j(z, z),$$

where each K_j is a positive semi-definite function, it must be the case that K_j is a sum of squares of polynomials of degree at most

$$\begin{cases} d_j - 1 \text{ in } z_j, & \\ d_k \text{ in } z_k & \text{for } k \neq j. \end{cases}$$

In particular, K_j can be written as a sum of at most $d_j \prod_{k\neq j}(d_k+1)$ polynomials (by dimensionality).

Recall that a function $K(z,\zeta)$ is positive semi-definite if for every finite set F the matrix

$$(K(z,\zeta))_{z,\zeta\in F}$$

is positive semi-definite. (We would need an ordering to form an actual matrix, but this is unimportant.) For more information on positive semi-definite kernels, refer to [Agler and McCarthy, 2002], Section 2.7.

The main results of this paper relate to rational inner functions in the Schur-Agler class on $\mathbb{D}^3$. The first interesting result in this area is due to Kummert.

Theorem 2.3 ([Kummert, 1989a])**.** *If $p \in \mathbb{C}[z_1,z_2,z_3]$ has degree $(1,1,1)$ and has no zeros on $\overline{\mathbb{D}}^3$, then $\tilde{p}/p$ is in the Schur-Agler class.*

We gave the following minor improvement to the details of the sums of squares decomposition of $\tilde{p}/p$ in [Knese, 2010c].

Theorem 2.4. *If $p \in \mathbb{C}[z_1,z_2,z_3]$ has degree $(1,1,1)$ and no zeros on $\overline{\mathbb{D}}^3$, then there exist sums of squares terms such that*

$$|p|^2 - |\tilde{p}|^2 = \sum_{j=1}^{3}(1-|z_j|^2)SOS_j(z),$$

where SOS_3 is a sum of two squares, while SOS_1, SOS_2 are sums of four squares.

Our two main results are the following. We extend the above results to the case of polynomials of degree $(n,m,1)$ and exhibit a new phenomenon in the study of the Schur-Agler class.

Theorem 2.5. *If $p \in \mathbb{C}[z_1,z_2,z_3]$ has degree $(n,1,1)$ and no zeros on $\overline{\mathbb{D}}^3$, then $\tilde{p}/p$ is in the Schur-Agler class. Furthermore, we have a decomposition*

$$|p|^2 - |\tilde{p}|^2 = \sum_{j=1}^{3}(1-|z_j|^2)SOS_j(z),$$

where SOS_3 is a sum of two squares, while SOS_1, SOS_2 are sums of $4(n-1)$, $2(n+1)$ squares respectively.

We are not claiming that the bounds of $4(n-1)$ and $2(n+1)$ are best possible.

Theorem 2.6. *If $p \in \mathbb{C}[z_1,z_2,z_3]$ has no zeros on $\overline{\mathbb{D}}^3$ and degree at most $(n,m,1)$, then there exist integers $r,s \geq 0$ such that*

$$\frac{z_1^r z_2^s \tilde{p}(z_1,z_2,z_3)}{p(z_1,z_2,z_3)}$$

is in the Schur-Agler class.

This phenomenon has not been observed in the study of the Schur-Agler class (although it is analogous to results in "sums of squares" such as Quillen's theorem [Quillen, 1968]). We do not yet have an example of p such that $\tilde{p}/p$ is not Schur-Agler while $z_1^r z_2^s \tilde{p}/p$ is Schur-Agler. However, in the last section we explain how a construction might proceed.

We will first discuss two necessary preliminary results and then proceed to the proof of Theorems 2.5 and 2.6.

3. Preliminary results

The following result is proven in [Megretski, 2003], [Dritschel, 2004], [Geronimo and Lai, 2006], and [Dumitrescu, 2007].

Theorem 3.1. *Suppose t is a d variable, positive trigonometric polynomial:*

$$t(z) = \sum_{-N \leq |\alpha| \leq N} t_\alpha z^\alpha > 0 \text{ for all } z = (z_1, \dots, z_d) \in \mathbb{T}^d,$$

where we use multi-index notation with $\alpha \in \mathbb{Z}^d$. Then t can be written as a sum of squares of polynomials; i.e. there exist $A_j \in \mathbb{C}[z_1, \dots, z_d]$ such that

$$t(z) = \sum_{j=1}^{M} |A_j(z)|^2 \qquad (z \in \mathbb{T}^n).$$

Known proofs of this result require both strict positivity and can only control the numbers of polynomials (and their degrees) in the sums of squares decomposition in terms of a bound below on t. See [Geronimo and Lai, 2006] for more details. It is this subtlety that creates the need to multiply by a sufficiently high degree monomial in Theorem 2.6. We get around this in Theorem 2.5 via the following lemma.

Lemma 3.2. *Let t be a non-negative, two variable trigonometric polynomial of degree $(n, 1)$, i.e.*

$$t(z_1, z_2) = t_0(z_1) + z_2 t_1(z_1) + \overline{z_2 t_1(z_1)} \geq 0, \tag{3.1}$$

where t_0, t_1 are one variable trigonometric polynomials of degree at most n. Then, there exist $A_1, A_2 \in \mathbb{C}[z_1, z_2]$ of degree at most $(n, 1)$ such that

$$t(z) = |A_1(z)|^2 + |A_2(z)|^2 \qquad (z \in \mathbb{T}^2).$$

This lemma is implicitly known, but in a different language and/or context (and with more complicated proofs and less detail). In [Gabardo, 1998] and [Bakonyi and Naevdal, 1998], it is proven and phrased in the language of trigonometric moment problems. The connection to sums of squares is because of the main result of [Rudin, 1963], which, loosely speaking, says that given a subset Λ of $\mathbb{Z}^d_+$, one can solve (truncated) trigonometric moment problems on $\Lambda - \Lambda$ if and only if non-negative trigonometric polynomials with Fourier support in $\Lambda - \Lambda$ are sums of squares of polynomials with coefficient support in Λ.

Proof. The proof is really the same as the degree $(1, 1)$ case, which we gave in [Knese, 2010c]. By minimizing (3.1) over z_2, we see that $t_0(z_1) \geq 2|t_1(z_1)|$. This implies that the 2×2 matrix trigonometric polynomial

$$T(z_1) = \begin{bmatrix} \frac{1}{2} t_0(z_1) & t_1(z_1) \\ \overline{t_1(z_1)} & \frac{1}{2} t_0(z_1) \end{bmatrix}$$

is positive semi-definite. By the matrix Fejér-Riesz theorem (due to M. Rosenblum; see [Dritschel, 2004] for a recent proof and references), T can be factored as

$$T(z_1) = A(z_1)^* A(z_1),$$

where $A \in \mathbb{C}^{2\times 2}[z_1]$ is a matrix polynomial of degree at most n. Then,

$$t(z_1, z_2) = \begin{bmatrix} 1 & \bar{z}_2 \end{bmatrix} T(z_1) \begin{bmatrix} 1 \\ z_2 \end{bmatrix} = \left\| A(z_1) \begin{bmatrix} 1 \\ z_2 \end{bmatrix} \right\|^2$$

is a sum of two squares of the desired type. □

4. Proof of Theorems 2.5 and 2.6

To prove Theorems 2.5 and 2.6 simultaneously we merely need to keep track of whether we are using Lemma 3.2 or Theorem 3.1. A brief notational warning: if E is a column vector of polynomials in the variables z_1, z_2 (as will occur below), we shall write $\|E(z_1, z_2)\|^2$ for the sum of squares of the entries of E, and often to save space we write $\|E\|^2$ for the same expression. These are all pointwise euclidean norms and do not represent any kind of function space norm.

Write $p(z) = a(z_1, z_2) + b(z_1, z_2)z_3$, where $a, b \in \mathbb{C}[z_1, z_2]$ have degree at most (n, m). For $z_1, z_2 \in \mathbb{T}$, by direct computation

$$|p|^2 - |\tilde{p}|^2 = (1 - |z_3|^2)(|a(z_1, z_2)|^2 - |b(z_1, z_2)|^2). \tag{4.1}$$

Then, for $(z_1, z_2) \in \mathbb{T}^2$, $|a(z_1, z_2)|^2 - |b(z_1, z_2)|^2$ is a non-negative two variable trigonometric polynomial of degree at most (n, m). As p has no zeros on $\overline{\mathbb{D}}^3$, $|a|^2 - |b|^2$ is in fact strictly positive on $\mathbb{T}^2$, since a zero (z_1', z_2') would imply $|p(z_1', z_2', \cdot)| = |\tilde{p}(z_1', z_2', \cdot)|$, and this would mean $z_3 \mapsto p(z_1', z_2', z_3)$ has a zero on $\mathbb{T}$.

By Lemma 3.2 or by Theorem 3.1, we may write

$$|a(z_1, z_2)|^2 - |b(z_1, z_2)|^2 = \|E(z_1, z_2)\|^2 \text{ on } \mathbb{T}^2,$$

where E is a vector polynomial with values in $\mathbb{C}^N$ (this provides a convenient way to represent sums of squares). In the degree $(n, 1, 1)$ case we may take $N = 2$, and in the $(n, m, 1)$ case we do not know what N is. Set $\tilde{E}(z_1, z_2) = z_1^{n+r} z_2^{m+s} \overline{E(1/\bar{z}_1, 1/\bar{z}_2)}$, where we assume E has degree $(n + r, m + s)$. Again, in the case $m = 1$, we may take $r = s = 0$.

We also remark that since p has no zeros on $\overline{\mathbb{D}}^3$, a has no zeros on $\overline{\mathbb{D}}^2$. By the maximum principle,

$$\frac{\tilde{b}(z_1, z_2)}{a(z_1, z_2)}$$

is analytic and has modulus strictly less than one since $|b| = |\tilde{b}|$ on $\mathbb{T}^2$ and since $|a| > |b|$ on $\mathbb{T}^2$. In particular, $a + z_1^r z_2^s \tilde{b}$ has no zeros on $\overline{\mathbb{D}}^2$.

We may polarize formula (4.1) and get for $z_1, z_2 \in \mathbb{T}$,

$$p(z_1, z_2, z_3)\overline{p(z_1, z_2, \zeta_3)} - \tilde{p}(z_1, z_2, z_3)\overline{\tilde{p}(z_1, z_2, \zeta_3)} = (1 - z_3\bar{\zeta}_3)\|E(z_1, z_2)\|^2, \tag{4.2}$$

for $z_3, \zeta_3 \in \mathbb{C}$, which we rearrange into

$$\begin{aligned} &p(z_1, z_2, z_3)\overline{p(z_1, z_2, \zeta_3)} + z_3\bar{\zeta}_3\|E(z_1, z_2)\|^2 \\ &= \tilde{p}(z_1, z_2, z_3)\overline{\tilde{p}(z_1, z_2, \zeta_3)} + \|E(z_1, z_2)\|^2. \end{aligned}$$

Then, for fixed $z_1, z_2 \in \mathbb{T}$ and for varying z_3, the map

$$\begin{bmatrix} p(z_1, z_2, z_3) \\ z_3 E(z_1, z_2) \end{bmatrix} \mapsto \begin{bmatrix} z_1^r z_2^s \tilde{p}(z_1, z_2, z_3) \\ E(z_1, z_2) \end{bmatrix} \tag{4.3}$$

gives a well-defined isometry $V(z_1, z_2)$ (which depends on z_1, z_2) from the span of the elements on the left to the span of the elements on the right (the span is taken over the above vectors as z_3 varies). More concretely, by examining coefficients of z_3, we map

$$\begin{bmatrix} a(z_1,z_2) \\ \vec{0} \end{bmatrix} \mapsto \begin{bmatrix} z_1^r z_2^s \tilde{b}(z_1,z_2) \\ E(z_1,z_2) \end{bmatrix}, \qquad \begin{bmatrix} b(z_1,z_2) \\ E(z_1,z_2) \end{bmatrix} \mapsto \begin{bmatrix} z_1^r z_2^s \tilde{a}(z_1,z_2) \\ \vec{0} \end{bmatrix}. \tag{4.4}$$

This is how the "lurking isometry argument" traditionally works. However, $V(z_1,z_2)$ does not extend uniquely to define a unitary on $\mathbb{C}^{N+1}$, and we would like to extend $V(z_1,z_2)$ so that V is rational in z_1,z_2.

The definition that Kummert gives in the $(1,1,1)$ case works here with a small modification.

Claim 1. Define

$$V = \frac{1}{a}\begin{bmatrix} z_1^r z_2^s \tilde{b} & \tilde{E}^t \\ E & \frac{E\tilde{E}^t - a(z_1^r z_2^s \tilde{a} + b)I}{a + z_1^r z_2^s \tilde{b}} \end{bmatrix}.$$

Then, V is holomorphic in $\mathbb{D}^2$ and unitary valued on $\mathbb{T}^2$, and V satisfies

$$V(z_1,z_2)\begin{bmatrix} p(z_1,z_2,z_3) \\ z_3 E(z_1,z_2) \end{bmatrix} = \begin{bmatrix} z_1^r z_2^s \tilde{p}(z_1,z_2,z_3) \\ E(z_1,z_2) \end{bmatrix} \tag{4.5}$$

for $(z_1,z_2) \in \mathbb{T}^2$ and hence for all $(z_1,z_2) \in \overline{\mathbb{D}}^2$ by analyticity. (This is just the content of (4.3).)

Proof of Claim 1. First, V is holomorphic since a and $a + z_1^r z_2^s \tilde{b}$ have no zeros on $\overline{\mathbb{D}}^2$.

Let

$$S(z_1,z_2) = \text{span}\left\{ \begin{bmatrix} p(z_1,z_2,z_3) \\ z_3 E(z_1,z_2) \end{bmatrix} : z_3 \in \mathbb{C} \right\}.$$

Notice that

$$S(z_1,z_2) = \text{span}\left\{ \begin{bmatrix} z_1^r z_2^s \tilde{p}(z_1,z_2,z_3) \\ E(z_1,z_2) \end{bmatrix} : z_3 \in \mathbb{C} \right\}.$$

This follows from looking at coefficients of z_3 as

$$S(z_1,z_2) = \text{span}\left\{ \begin{bmatrix} a(z_1,z_2) \\ \vec{0} \end{bmatrix}, \begin{bmatrix} b(z_1,z_2) \\ E(z_1,z_2) \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix}, \begin{bmatrix} 0 \\ E(z_1,z_2) \end{bmatrix} \right\},$$

and one can similarly show that

$$\text{span}\left\{ \begin{bmatrix} z_1^r z_2^s \tilde{p}(z_1,z_2,z_3) \\ E(z_1,z_2) \end{bmatrix} : z_3 \in \mathbb{C} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix}, \begin{bmatrix} 0 \\ E(z_1,z_2) \end{bmatrix} \right\}.$$

The goal is to show that $V(z_1,z_2)$ is a unitary by verifying (4.5), which shows that $V(z_1,z_2)$ is isometric on the subspace $S(z_1,z_2)$, and by showing that $V(z_1,z_2)$ maps $S(z_1,z_2)^\perp$ isometrically into itself.

To show (4.5) we first observe that

$$\begin{aligned} \tilde{E}(z_1,z_2)^t E(z_1,z_2) &= z_1^{n+r} z_2^{m+s} \|E(z_1,z_2)\|^2 \\ &= z_1^{n+r} z_2^{m+s} (|a(z_1,z_2)|^2 - |b(z_1,z_2)|^2) = z_1^r z_2^s (a\tilde{a} - b\tilde{b}), \end{aligned}$$

where a,b are reflected at degree (n,m) to give $\tilde{a},\tilde{b}$.

Here are the computations used to show (4.4) (which is equivalent to (4.5)):

$$V\begin{bmatrix} a \\ \vec{0} \end{bmatrix} = \begin{bmatrix} z_1^r z_2^r \tilde{b} \\ E \end{bmatrix}$$

and

$$V\begin{bmatrix} b \\ E \end{bmatrix} = \frac{1}{a}\begin{bmatrix} z_1^r z_2^s \tilde{b} b + z_1^r z_2^s (a\tilde{a} - b\tilde{b}) \\ \frac{b(a+z_1^r z_2^s \tilde{b}) + z_1^r z_2^s (a\tilde{a} - b\tilde{b}) - a(z_1^r z_2^s \tilde{a} + b)}{a + z_1^r z_2^s \tilde{b}} E \end{bmatrix} = \begin{bmatrix} z_1^r z_2^s \tilde{a} \\ \vec{0} \end{bmatrix}.$$

Now we show that $V(z_1, z_2)$, viewed as a linear map, is isometric on the orthogonal complement of $S(z_1, z_2)$. Set $X(z_1, z_2)$ equal to the orthogonal complement of $S(z_1, z_2)$ in $\mathbb{C}^{1+N}$, and observe that

$$X(z_1, z_2) = \{\begin{bmatrix} 0 \\ v \end{bmatrix} : v \perp E(z_1, z_2)\}.$$

Notice that $v \perp E(z_1, z_2)$ if $\tilde{E}^t v = 0$.

Let us observe what the definition of V does to elements of X. For $\vec{x} = \begin{bmatrix} 0 \\ v \end{bmatrix} \in X$, we have

$$V\vec{x} = \begin{bmatrix} 0 \\ -\frac{z_1^r z_2^s \tilde{a} + b}{a + z_1^r z_2^s \tilde{b}} v \end{bmatrix} = -\frac{z_1^r z_2^s \tilde{a} + b}{a + z_1^r z_2^s \tilde{b}} \vec{x}.$$

So, every element of X is an eigenvector with eigenvalue $-\frac{z_1^r z_2^s \tilde{a} + b}{a + z_1^r z_2^s \tilde{b}}$. This number is unimodular for $(z_1, z_2) \in \mathbb{T}^2$. This proves $V(z_1, z_2)$ is unitary valued, and the claim is proved. □

This means V is an $(N+1) \times (N+1)$ two-variable rational matrix-valued inner function. It was proved in [Kummert, 1989b] (see also [Ball et al., 2005]) that such functions have transfer function representations. Namely, there exists a $((N+1) + n_1 + n_2) \times ((N+1) + n_1 + n_2)$ block unitary

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix},$$

where A is an $(N+1) \times (N+1)$ matrix, B is an $(N+1) \times (n_1 + n_2)$, C is an $(n_1 + n_2) \times (N+1)$, and D is an $(n_1 + n_2) \times (n_1 + n_2)$ (all subdivided as indicated) such that

$$V(z_1, z_2) = A + Bd(z_1, z_2)(I - Dd(z_1, z_2))^{-1} C, \tag{4.6}$$

where

$$d(z_1, z_2) = \begin{bmatrix} z_1 I_1 & 0 \\ 0 & z_2 I_2 \end{bmatrix}.$$

Here I_1, I_2 are the n_1, n_2-dimensional identity matrices, respectively.

Such a representation is equivalent to the formula

$$U \begin{bmatrix} I \\ z_1 G_1(z_1, z_2) \\ z_2 G_2(z_1, z_2) \end{bmatrix} = \begin{bmatrix} V(z_1, z_2) \\ G_1(z_1, z_2) \\ G_2(z_1, z_2) \end{bmatrix}, \tag{4.7}$$

where G_1, G_2 are $\mathbb{C}^{n_1 \times (N+1)}$-, $\mathbb{C}^{n_2 \times (N+1)}$-valued functions given by

$$\begin{bmatrix} G_1(z_1, z_2) \\ G_2(z_1, z_2) \end{bmatrix} = (I - Dd(z_1, z_2))^{-1} C.$$

Indeed, one can use formula (4.7) to explicitly solve for V as in (4.6).

Define

$$Y = \begin{bmatrix} p \\ z_3 E \end{bmatrix} \text{ and } H_j = G_j Y \text{ for } j = 1, 2.$$

Then, by these definitions

$$U \begin{bmatrix} I \\ z_1 G_1 \\ z_2 G_2 \end{bmatrix} Y = U \begin{bmatrix} Y \\ z_1 G_1 Y \\ z_2 G_2 Y \end{bmatrix} = U \begin{bmatrix} p \\ z_3 E \\ z_1 H_1 \\ z_2 H_2 \end{bmatrix}$$

and

$$\begin{bmatrix} V \\ G_1 \\ G_2 \end{bmatrix} Y = \begin{bmatrix} VY \\ H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} z_1^r z_2^s \tilde{p} \\ E \\ H_1 \\ H_2 \end{bmatrix},$$

where the last equality follows from (4.5).

By (4.7) we now have

$$U \begin{bmatrix} p \\ z_3 E \\ z_1 H_1 \\ z_2 H_2 \end{bmatrix} = \begin{bmatrix} z_1^r z_2^s \tilde{p} \\ E \\ H_1 \\ H_2 \end{bmatrix},$$

and since U is a unitary we have

$$\begin{aligned} &|p|^2 + |z_3|^2 \|E\|^2 + |z_1|^2 \|H_1\|^2 + |z_2|^2 \|H_2\|^2 \\ &= |z_1^r z_2^s \tilde{p}|^2 + \|E\|^2 + \|H_1\|^2 + \|H_2\|^2, \end{aligned}$$

which can be rearranged to give

$$|p|^2 - |z_1^r z_2^s \tilde{p}|^2 = \sum_{j=1,2} (1 - |z_j|^2) \|H_j\|^2 + (1 - |z_3|^2) \|E\|^2.$$

Even though we have not verified that H_1 and H_2 are polynomials, this is enough to prove that $z_1^r z_2^s \tilde{p}/p$ is in the Schur-Agler class. In fact, Theorem 2.2 forces $\|H_1\|^2$, $\|H_2\|^2$ to be sums of squares of polynomials of multi-degree $(n + r - 1, m + s, 1)$, $(n + r, m + s - 1, 1)$.

In the case $m = 1$, we have $r = s = 0$, E of degree $(n, 1, 0)$, and E has values in $\mathbb{C}^2$. So, $\|E\|^2$ is a sum of two squares, and by dimensional considerations $\|H_1\|^2$ is a sum of at most $4(n - 1)$ squares and $\|H_2\|^2$ is a sum of at most $2(n + 1)$ squares. This concludes the proof of Theorems 2.5 and 2.6.

5. How to construct examples

How might one construct an example of a rational inner function $\tilde{p}/p$ with p of degree $(n, m, 1)$ and the property that $\tilde{p}/p$ is not in the Schur-Agler class while $z_1^r z_2^s \tilde{p}/p$ is? Examining the above proof, it is a matter of choosing $a, b \in \mathbb{C}[z_1, z_2]$ such that (1) a has no zeros on $\mathbb{D}^2$, (2) $|a|^2 - |b|^2 \geq 0$ on $\mathbb{T}^2$, and (3) $|a|^2 - |b|^2$ is not a sum of squares of polynomials of degree (n, m). Positive trigonometric polynomials of degree (n, m) which cannot be written as a sum of squares of polynomials of degree (n, m) do exist (see [Dritschel, 2004] and [Dumitrescu, 2007]), so our problem reduces to finding such a trigonometric polynomial of the form $|a|^2 - |b|^2$. We leave this for future work (or future authors).

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