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# A NOTE ON THE TOPOLOGICAL STABLE RANK OF AN IDEAL

YOU QING JI AND YUAN HANG ZHANG

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Abstract. We show that if $\mathfrak{J}$ is an ideal of a Banach algebra $\mathfrak{A}$, then the left (right) topological stable rank of $\mathfrak{J}$ is no greater than the left (right) topological stable rank of $\mathfrak{A}$.

By a Banach algebra $\mathfrak{A}$ we mean a complex Banach algebra which may be not unital. When we speak of ideals we will always mean closed two-sided ideals. The topological stable rank for Banach algebras was defined in Rieffel [6] as a non-commutative analogue of the covering dimension for compact spaces that was modelled on the Bass stable rank of rings [1]. Since it was introduced, the concept has been very useful in studying non-stable K-theory and some spectral or structure properties of Banach algebras. While investigating the topological stable rank, Rieffel [6] also defined two other stable ranks: the connected stable rank and the general stable rank.

In classical covering dimension theory, it is known that the dimension of an open subset $U$ of a locally compact Hausdorff space $X$ is no greater than the dimension of the whole space $X$. Comparing with this, we are interested in whether an analog and generalization of the inequality can be obtained in topological stable rank. So it is natural to ask: is it always true that the left (right) topological stable rank of an ideal $\mathfrak{J}$ is no greater than the left (right) topological stable rank of the whole algebra $\mathfrak{A}$? In the seminal paper Rieffel asserted it is true under the assumption that $\mathfrak{J}$ has a bounded approximate identity [6, Theorem 4.4], and assuming that the quotient map splits, Davidson and the first author showed it is also true [3, Theorem 3.1]. As stated in the abstract, the purpose of this note is to show that this always holds for any ideal in any Banach algebra.

We begin by recalling some standard definitions and facts. Given a unital Banach algebra $\mathfrak{A}$, we denote by $Lg_n(\mathfrak{A})$ (resp. $Rg_n(\mathfrak{A})$) the set of $n$-tuples of elements of $\mathfrak{A}$ which generate $\mathfrak{A}$ as a left ideal (resp. as a right ideal), that is,

$$Lg_n(\mathfrak{A}) = \{(a_1, \dots, a_n) \in \mathfrak{A}^n : \exists (b_1, \dots, b_n) \in \mathfrak{A}^n \text{ with } \sum_{i=1}^{n} b_i a_i = 1\}.$$

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The elements of $Lg_n(\mathfrak{A})$ are called left unimodular. For typographical reasons, we will consider them as columns but write them as rows. The *left topological stable rank* of $\mathfrak{A}$, denoted by $\mathrm{ltsr}(\mathfrak{A})$, is the least positive integer $n$ for which $Lg_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$.

When no such integer exists, we set $\mathrm{ltsr}(\mathfrak{A}) = \infty$. The *right topological stable rank* of $\mathfrak{A}$, denoted by $\mathrm{rtsr}(\mathfrak{A})$, is defined analogously. If $\mathrm{ltsr}(\mathfrak{A}) = \mathrm{rtsr}(\mathfrak{A})$, we refer to their common value simply as the topological stable rank of $\mathfrak{A}$, written $\mathrm{tsr}(\mathfrak{A})$. When $\mathfrak{A}$ is not unital, we define the left (right) topological stable rank of $\mathfrak{A}$ to be that of its unitization $\mathfrak{A}^{\tilde{}}$.

The key to our methods is the following lemma, which was inspired by a trick due to Vaserstein [7, Lemma 2.0]. It helps us recognize what $Lg_n(\mathfrak{J}^{\tilde{}})(Rg_n(\mathfrak{J}^{\tilde{}}))$ really are.

**Lemma 1.** *Let $\mathfrak{A}$ be a unital Banach algebra and let $\mathfrak{J}$ be an ideal in $\mathfrak{A}$. Then for each positive integer $n$, $Lg_n(\mathfrak{J}^{\tilde{}}) = Lg_n(\mathfrak{A}) \cap (\mathfrak{J}^{\tilde{}})^n; Rg_n(\mathfrak{J}^{\tilde{}}) = Rg_n(\mathfrak{A}) \cap (\mathfrak{J}^{\tilde{}})^n$.*

*Proof.* We prove only the left version.

The inclusion "$\subseteq$" is obvious; we show the other inclusion. Let $u = (\lambda_i + j_i) \in Lg_n(\mathfrak{A}) \cap (\mathfrak{J}^{\tilde{}})^n$, where $j_i \in \mathfrak{J}$ and $\lambda_i$ scalar, $i = 1, 2, \cdots, n$. Then we can find a $b = (b_i) \in Rg_n(\mathfrak{A})$ such that $b_1(\lambda_1 + j_1) + b_2(\lambda_2 + j_2) + \cdots + b_n(\lambda_n + j_n) = 1$. Observe that at least one $\lambda_i \neq 0$. Then there exists an invertible scalar matrix $T$ so that $Tu = (1 + \tilde{j}_1, \tilde{j}_2, \cdots, \tilde{j}_n)$; here $(\tilde{j}_i) \in (\mathfrak{J})^n$. Now set $a = (a_i) = bT^{-1}$. It follows that $a_1(1 + \tilde{j}_1) + a_2\tilde{j}_2 + \cdots + a_n\tilde{j}_n = 1$. Then if we multiply this equation on the left by $-\tilde{j}_1$, we obtain

$$(-\tilde{j}_1 a_1)(1 + \tilde{j}_1) + (-\tilde{j}_1 a_2)\tilde{j}_2 + \cdots + (-\tilde{j}_1 a_n)\tilde{j}_n = -\tilde{j}_1 - 1 + 1.$$

Thus, we can conclude that $(1 - \tilde{j}_1 a_1)(1 + \tilde{j}_1) + (-\tilde{j}_1 a_2)\tilde{j}_2 + \cdots + (-\tilde{j}_1 a_n)\tilde{j}_n = 1$, that is,

$$(1 - \tilde{j}_1 a_1, -\tilde{j}_1 a_2, \cdots, -\tilde{j}_1 a_n)TT^{-1}(1 + \tilde{j}_1, \tilde{j}_2, \cdots, \tilde{j}_n) = 1.$$

Hence $(1 - \tilde{j}_1 a_1, -\tilde{j}_1 a_2, \cdots, -\tilde{j}_1 a_n)T(\lambda_i + j_i) = 1$. Observe that

$$(1 - \tilde{j}_1 a_1, -\tilde{j}_1 a_2, \cdots, -\tilde{j}_1 a_n)T \in (\mathfrak{J}^{\tilde{}})^n.$$

Then $u = (\lambda_i + j_i)$ is in $Lg_n(\mathfrak{J}^{\tilde{}})$, and the proof is complete. □

Since the left and right topological stable ranks of a Banach algebra may not agree [4], we state our main theorem in two versions.

**Theorem 2.** *Let $\mathfrak{A}$ be a Banach algebra and let $\mathfrak{J}$ be an ideal in $\mathfrak{A}$. Then* $\mathrm{ltsr}(\mathfrak{J}) \leq \mathrm{ltsr}(\mathfrak{A})$ *and* $\mathrm{rtsr}(\mathfrak{J}) \leq \mathrm{rtsr}(\mathfrak{A})$.

*Proof.* We need to prove only the left version. While the right version is almost the same, we leave it to the reader. Without loss of generality, it can be assumed that $\mathfrak{A}$ has an identity element, for otherwise we can adjoin one without changing the situation. Write $I$ for the identity element. For the sake of completeness, we would like to review some arguments which are essentially contained in [3, Theorem 3.1].

Suppose that $\mathrm{ltsr}(\mathfrak{A}) = n$, that $T_i = a_i I + J_i$ are given with $J_i \in \mathfrak{J}$, and that $0 < \varepsilon < 1/4$. By changing each $a_i$ if necessary by at most $\varepsilon/2$, we may suppose that $|a_i| \geq \varepsilon/2$. We assume that this has been done. Let $M = \max\{1, \|T_i\| : 1 \leq i \leq n\}$. Use $\mathrm{ltsr}(\mathfrak{A}) = n$ to find $A_i \in \mathfrak{A}$ with $\|A_i\| < \varepsilon^2/6M$ and $X_i \in \mathfrak{A}$ so that $\sum_{i=1}^n X_i(T_i + A_i) = I$. Observe that

$$T_i + A_i = (I + a_i^{-1}A_i)(I + (I + a_i^{-1}A_i)^{-1}J_i a_i{}^{-1})a_i.$$

Set $T_i' = (I + (I + a_i^{-1}A_i)^{-1}J_i a_i^{-1})a_i = (I + a_i^{-1}A_i)^{-1}(T_i + A_i)$, and note that

$$\begin{aligned}\|T_i' - T_i\| &\leq \|(I + a_i^{-1}A_i)^{-1} - I\| \, \|T_i\| + \|(I + a_i^{-1}A_i)^{-1}\| \, \|A_i\| \\ &\leq \frac{M\|a_i^{-1}A_i\| + \|A_i\|}{1 - \|a_i^{-1}A_i\|} \leq \frac{(M\frac{2}{\varepsilon} + 1)\frac{\varepsilon^2}{6M}}{1 - \frac{2}{\varepsilon}\frac{\varepsilon^2}{6M}} \\ &\leq \frac{\varepsilon}{3}\big(1 + \frac{\varepsilon}{2M}\big)\big(1 + \frac{\varepsilon}{2M}\big) < \frac{27\varepsilon}{64} < \frac{\varepsilon}{2}.\end{aligned}$$

Then $\sum_{i=1}^n X_i(I + a_i^{-1}A_i)T_i' = I$.

Observe that $(T_i') \in Lg_n(\mathfrak{A}) \cap (\mathfrak{J}^\sim)^n$. Applying Lemma 1 shows that there are $Y_i \in \mathfrak{J}^\sim$ such that $\sum_{i=1}^n Y_i T_i' = I$. This shows that $\mathrm{ltsr}(\mathfrak{J}) \leq \mathrm{ltsr}(\mathfrak{A})$. $\square$

Let $\mathfrak{A}$ be a unital Banach algebra. Denote by $\mathrm{bsr}(\mathfrak{A}), \mathrm{csr}(\mathfrak{A})$ and $\mathrm{gsr}(\mathfrak{A})$ the Bass stable rank, the connected stable rank and the general stable rank respectively. Also, we define the connected (resp. general) stable rank of an non-unital $\mathfrak{A}$ to be the connected (resp. general) stable rank of its unitization $\mathfrak{A}^\sim$. The reader is referred to [6] for concrete definitions and more details.

By the virtue of Lemma 1, we can also estimate the connected (resp. general) stable rank of a Banach algebra $\mathfrak{A}$ in terms of the connected (resp. general) stable rank of $\mathfrak{J}$ and the connected (resp. connected) stable rank of $\mathfrak{A}/\mathfrak{J}$. So we get a similar result of Nica [5, Theorem 12.5] in full generality.

**Proposition 3.** *Let $\mathfrak{A}$ be a Banach algebra and let $\mathfrak{J}$ be an ideal in $\mathfrak{A}$. Then* $\mathrm{csr}(\mathfrak{A}) \leq \max\{\mathrm{csr}(\mathfrak{J}), \mathrm{csr}(\mathfrak{A}/\mathfrak{J})\}$*,* $\mathrm{gsr}(\mathfrak{A}) \leq \max\{\mathrm{gsr}(\mathfrak{J}), \mathrm{csr}(\mathfrak{A}/\mathfrak{J})\}$.

*Proof.* We provide only a sketch. Since $(\mathfrak{A}/\mathfrak{J})^\sim \cong \mathfrak{A}^\sim/\mathfrak{J}$, it suffices to consider the case in which $\mathfrak{A}$ has an identity element. The arguments run just as in the one in Nica's paper [5, Theorem 12.5], except that we apply directly Lemma 1 rather than use the conclusion that an ideal with an approximate identity satisfies $Lg_n(\mathfrak{J}^\sim) = Lg_n(\mathfrak{A}) \cap (\mathfrak{J}^\sim)^n$ for each positive integer $n$ [5, Lemma 12.1]. $\square$

Finally, we conclude with an example.

For a long time, the following question has been posed by some people (maybe in different forms), i.e., Corach, Suárez [2], Nica [5], whether one can find a Banach algebra $\mathfrak{A}$ such that $\mathrm{bsr}(\mathfrak{A}) < \mathrm{tsr}(\mathfrak{A}) = \infty$. By using a striking result of [4], we can find one.

**Example 4.** In [4], Davidson et al. constructed a nest algebra $\mathfrak{A}$ such that $\mathrm{ltsr}(\mathfrak{A}) = \infty$, while $\mathrm{rtsr}(\mathfrak{A}) = 2$. Let $\mathfrak{A}^* = \{a^* : a \in \mathfrak{A}\}$. It is easy to see that $\mathrm{ltsr}(\mathfrak{A}^*) = 2, \mathrm{rtsr}(\mathfrak{A}^*) = \infty$. Let $\mathfrak{B} = \mathfrak{A} \bigoplus \mathfrak{A}^*$. Then by [6, Corollary 2.4], we have $\mathrm{bsr}(\mathfrak{A}) \leq \min\{\mathrm{ltsr}(\mathfrak{A}), \mathrm{rtsr}(\mathfrak{A})\}$=2 and $\mathrm{bsr}(\mathfrak{A}^*) \leq \min\{\mathrm{ltsr}(\mathfrak{A}^*), \mathrm{rtsr}(\mathfrak{A}^*)\}$=2. Thus, $\mathrm{bsr}(\mathfrak{B})$ =max$\{\mathrm{bsr}(\mathfrak{A}), \mathrm{bsr}(\mathfrak{A}^*)\} \leq 2$, but $\mathrm{ltsr}(\mathfrak{B}) = \mathrm{rtsr}(\mathfrak{B}) = \infty$.

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Department of Mathematics, Jilin University, Changchun 130012, People's Republic of China

*E-mail address*: jiyq@jlu.edu.cn

Institute of Mathematics, Jilin University, Changchun 130012, People's Republic of China

*E-mail address*: zhangyuanhang0229@126.com