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# ANALYTICAL SOLUTIONS TO THE NAVIER-STOKES-POISSON EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY AND WITH PRESSURE

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Abstract. We study some particular solutions to the Navier-Stokes-Poisson equations with density-dependent viscosity and with pressure, in radial symmetry. With an extension of the previous known blow-up solutions for the Euler-Poisson equations with pressureless Navier-Stokes-Poisson density-dependent viscosity, we constructed the corresponding self-similar blow-up solutions for the Navier-Stokes-Poisson equations with density-dependent viscosity and with pressure. Our solutions can provide concrete examples for testing the validation and stabilities of numerical methods for the systems.

## 1. Introduction

The evolution of Newtonian self-gravitating fluids can be formulated by the isentropic Euler-Poisson equations of the following form:

$$(1)\qquad \begin{cases} \rho_t+\nabla\cdot(\rho\vec{u}) &= 0,\\ \rho[\vec{u}_t+(\vec{u}\cdot\nabla)\,\vec{u})]+\nabla P &= -\delta\rho\nabla\Phi+vis(\rho,\vec{u}),\\ \Delta\Phi(t,x) &= \alpha(N)\rho-\Lambda, \end{cases}$$

where $\alpha(N)$ is a constant related to the unit ball in $R^N$: $\alpha(1)=2$, $\alpha(2)=2\pi$ and for $N\geq 3$,

$$(2)\qquad \alpha(N)=N(N-2)\mathrm{Vol}(N)=N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2+1)},$$

where $\mathrm{Vol}(N)$ is the volume of the unit ball in $R^N$ and $\Gamma$ is a Gamma function. As usual, $\rho=\rho(t,\vec{x})$ and $\vec{u}=\vec{u}(t,\vec{x})\in\mathbf{R}^N$ are the density and the velocity respectively. $P=P(\rho)$ is the pressure, and $\Lambda$ is the background constant.

When $\delta=1$, the system can model fluids that are self-gravitating, such as gaseous stars. In addition, the evolution of the simple cosmology can be modelled by the dust distribution without the pressure term. This describes the stellar systems of collisionless and gravitational $n$-body systems [8]. The pressureless Euler-Poisson equations can be derived from the Vlasov-Poisson-Boltzmann model

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with the zero mean free path [10]. For $N = 3$, the equations (1) are the classical (non-relativistic) descriptions of a galaxy in astrophysics. See [2] and [3] for details about the systems. When $\delta = -1$, the system is the compressible Euler-Poisson equations with repulsive forces. The equation $(1)_3$ is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons. In this case, the system can be viewed as a semiconductor model. See [5] and [6] for a detailed analysis of the system. When $\delta = 0$, the potential forces are ignored. The system is called Euler/Navier-Stokes equations. See [4], [16] and [17] for a detailed analysis of the system.

In the above system, the self-gravitational potential field $\Phi = \Phi(t, x)$ is determined by the density $\rho$ through the Poisson equation.

Also, $vis(\rho, \vec{u})$ is the viscosity function:

$$vis(\rho, \vec{u}) := \triangledown(\mu(\rho) \triangledown \cdot \vec{u}). \tag{3}$$

In this article, we seek the radial solutions

$$\rho(t, \vec{x}) = \rho(t, r) \text{ and } \vec{u} = \frac{\vec{x}}{r} V(t, r) =: \frac{\vec{x}}{r} V \tag{4}$$

with $r = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$. By the standard computation, the Euler-Poisson equations in radial symmetry can be written in the following form:

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{N-1}{r}\rho V = 0, \\ \rho\left(V_t + VV_r\right) + P_r(\rho) + \delta\rho\Phi_r\left(\rho\right) = \mu_r\left(\rho\right)\left(\frac{N-1}{r}V + V_r\right) \\ \qquad + \mu(\rho)\left[-\left(\frac{N-1}{r^2}\right)V + \frac{N-1}{r}V_r + V_{rr}\right]. \end{cases} \tag{5}$$

Under a common assumption, the viscosity function can be defined:

$$\mu(\rho) := \kappa\rho^{\theta}, \tag{6}$$

where $\kappa$ and $\theta \geq 0$ are the constants. For the study of the above system, the reader may refer to [13], [16], [18] and [21]. In particular, when $\theta = 0$, that returns the expression for the only $V$-dependent viscosity function:

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{N-1}{r}\rho V = 0, \\ \rho\left(V_t + VV_r\right) + P_r(\rho) + \delta\rho\Phi_r\left(\rho\right) = \kappa\left[V_{rr} + \frac{N-1}{r}V_r - \left(\frac{N-1}{r^2}\right)V\right], \end{cases} \tag{7}$$

which are the usual form of the Navier-Stokes-Poisson equations. The equations $(1)_1$ and $(1)_2$ $(vis(\rho, u) \neq 0)$ are the compressible Navier-Stokes equations with forcing term. The equation $(1)_3$ is the Poisson equation through which the gravitational potential is determined by the density distribution of the density itself. Thus, the system (1) is called the Navier-Stokes-Poisson equations.

Here, if the $vis(\rho, u) = 0$, the system is called the Euler-Poisson equations. In this case, the equations can be viewed as a perfect gas model. For $N = 3$, (1) is a classical (nonrelativistic) description of a galaxy, in astrophysics. See [3], [15] for details about the system.

$P = P(\rho)$ is the pressure. The $\gamma$-law can be applied on the pressure $P(\rho)$, i.e.

$$P\left(\rho\right) = K\rho^{\gamma} := \frac{\rho^{\gamma}}{\gamma}, \tag{8}$$

which is a common hypothesis. The constant $\gamma = c_P/c_v \geq 1$, where $c_P$, $c_v$ are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in (8). With $K = 0$, we say that the system is pressureless.

For the local existence of the Euler-Poisson equations, the interested reader may refer to the results of Makino [14], Gamblin [9] and Bezard [1].

In the following, we always seek solutions in radial symmetry. Thus, the Poisson equation $(1)_3$ is transformed to

$$r^{N-1}\Phi_{rr}\left(t,x\right)+\left(N-1\right)r^{N-2}\Phi_{r}=\alpha\left(N\right)\rho r^{N-1}, \tag{9}$$

$$\Phi_{r}=\frac{\alpha\left(N\right)}{r^{N-1}}\int_{0}^{r}\left(\rho(t,s)-\Lambda\right)s^{N-1}ds. \tag{10}$$

In this paper, we are concerned with blow-up solutions for the $N$-dimensional pressureless Navier-Stokes-Poisson equations with the density-dependent viscosity. Our aim is to construct a family of blow-up solutions.

Historically in astrophysics, Goldreich and Weber [11] constructed the analytical blow-up (collapsing) solutions of the 3-dimensional Euler-Poisson equations for $\gamma = 4/3$ and for the nonrotating gas spheres. After that, Makino [15] obtained the rigorously mathematical proof of the existence of such kinds of blow-up solutions. In [7], Deng, Xiang and Yang extended the above blow-up solutions in $R^N$ $(N \geq 3)$. Recently, Yuen obtained the blow-up solutions in $R^2$ with $\gamma = 1$ in [21]. The class of self-similar solutions is rewritten as: for $N \geq 3$ and $\gamma = (2N-2)/N$, in [7]:

$$\left\{\begin{array}{c}\rho(t,r)=\left\{\begin{array}{c}\dfrac{1}{a(t)^{N}}y(\frac{r}{a(t)})^{N/(N-2)},\text{ for }r<a(t)Z_{\mu}\\ 0,\text{ for }a(t)Z_{\mu}\leq r\end{array}\right. ,\ V(t,r)=\dfrac{\dot{a}(t)}{a(t)}r,\\ \ddot{a}(t)=\dfrac{-\lambda}{a(t)^{N-1}},\ a(0)=a_{0}>0,\ \dot{a}(0)=a_{1},\\ \ddot{y}(z)+\dfrac{N-1}{z}\dot{y}(z)+\dfrac{\alpha(N)}{(2N-2)K}y(z)^{N/(N-2)}=\mu,\ y(0)=\alpha>0,\ \dot{y}(0)=0,\end{array}\right. \tag{11}$$

where $\mu = [N(N-2)\lambda]/((2N-2)K)$ and the finite $Z_\mu$ is the first zero of $y(z)$; for $N = 2$ and $\gamma = 1$, in [21],

$$\left\{\begin{array}{c}\rho(t,r)=\dfrac{1}{a(t)^{2}}e^{y(r/a(t))},\ V(t,r)=\dfrac{\dot{a}(t)}{a(t)}r,\\ \ddot{a}(t)=\dfrac{-\lambda}{a(t)},\ a(0)=a_{0}>0,\ \dot{a}(0)=a_{1},\\ \ddot{y}(z)+\dfrac{1}{z}\dot{y}(z)+\dfrac{2\pi}{K}e^{y(z)}=\mu,\ y(0)=\alpha,\ \dot{y}(0)=0,\end{array}\right. \tag{12}$$

where $K > 0$, $\mu = 2\lambda/K$ with a sufficiently small $\lambda$ and $\alpha$ are constants. For the construction of self-similar solutions to the Euler, Navier-Stokes and pressureless Navier-Stokes-Poisson equations, readers may refer to the recent results in [20], [19], [23] and [24].

In this article, it is natural to extend the more general results to cover the full system (5) in mathematical theory: the Navier-Stokes-Poisson equations with density-dependent viscosity and *with pressure*. In short, we successfully deduce the nonlinear partial differential equations (5) into the much simpler ordinary differential equations.

**Theorem 1.** *For the Navier-Stokes-Poisson equations in* $R^N$ $(N \geq 2)$ *with* $\gamma = \frac{2N-2}{N}$, (5) *there exists a family of solutions, namely:*

$$\rho(t,r) = \begin{cases} \frac{f(\frac{r}{a(t)})}{a(t)^N}, \ \text{for } r < a(t)Z_\mu \\ 0, \ \text{for } a(t)Z_\mu \leq r \end{cases}, \ V(t,r) = \frac{\dot{a}(t)}{a(t)}r, \tag{13}$$

*where* $f(z)$ *and* $a(t)$ *are the following functions and the finite* $Z_\mu$ *is the first zero of* $f(z)$*: (1) With* $\Lambda = 0$*, and (1a)* $\theta = \frac{2N-3}{N}$*,*

$$\begin{cases} a(t) = mt + n, \\ \gamma K f(z)^{\gamma-2}\dot{f}(z) - m\kappa\theta N f(z)^{\theta-2}\dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}}\int_0^z f(s)s^{N-1}ds, \ f(0) = \alpha > 0, \end{cases} \tag{14}$$

*where* $m$*,* $n > 0$ *and* $\alpha$ *are constant; (1b)* $\theta = \frac{3N-4}{2N}$*,*

$$\begin{cases} a(t) = (mt+n)^{2/N}, \\ \gamma K f(z)^{\gamma-2}\dot{f}(z) - 2m\kappa\theta f(z)^{\theta-2}\dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}}\int_0^z f(s)s^{N-1}ds = \frac{2(N-2)m^2}{N^2}z, \\ f(0) = \alpha > 0. \end{cases} \tag{15}$$

*In particular, for* $m < 0$*, the solutions* (14) *and* (15) *blow up at the finite time* $T = -m/n$*. (2) With* $\delta\Lambda > 0$ *and* $\theta = \frac{2N-3}{N}$*, we have*

$$\begin{cases} a(t) = e^{\sqrt{\frac{\delta\alpha(N)\Lambda}{N}}t}, \\ (K\gamma - N\delta\alpha(N)\Lambda)\, f(z)^{\theta-2}\dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}}\int_0^z f(s)s^{N-1}ds, \ f(0) = \alpha, \ f(0) = \alpha > 0. \end{cases} \tag{16}$$

Our analytical solutions can provide the concrete examples for testing the validation and stabilities of numerical methods for the systems.

## 2. Separable solutions

Before we present the proof of Theorem 1, Lemmas 3 and 12 of [22] are quoted here.

**Lemma 2** (Lemma 3 of [22]). *For the equation of conservation of mass in radial symmetry:*

$$\rho_t + V\rho_r + \rho V_r + \frac{N-1}{r}\rho V = 0, \tag{17}$$

*there exist solutions*

$$\rho(t,r) = \frac{f(r/a(t))}{a(t)^N}, \ V(t,r) = \frac{\dot{a}(t)}{a(t)}r \tag{18}$$

*with the form* $f \geq 0 \in C^1$ *and* $a(t) > 0 \in C^1$*.*

It is clear to check our solutions to satisfy the Navier-Stokes-Poisson equations.

*Proof.* As we use the functional structure of the above lemma, our solutions (13) fit well to the mass equation $(5)_1$. (1a) For the momentum equation $(5)_2$, we have:

$$\rho(V_t + V\cdot V_r) + K\left(\rho^\gamma\right)_r - \kappa(\rho^\theta)_r\left(\frac{N-1}{r}V + V_r\right) \tag{19}$$

$$-\kappa\rho_r(V_{rr} + \frac{N-1}{r}V_r - \frac{N-1}{r^2}V) + \rho\frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \rho s^{N-1}ds$$

$$= \rho\frac{\ddot{a}(t)}{a(t)}r + K\left(\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^\gamma\right)_r - \kappa\left(\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^\theta\right)_r\frac{N\dot{a}(t)}{a(t)} \tag{20}$$

$$+ \rho\frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds$$

$$= \gamma K\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\gamma-1}\frac{\partial}{\partial r}\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right] - \kappa\theta\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\theta-1}\frac{\partial}{\partial r}\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]\frac{N\dot{a}(t)}{a(t)} \tag{21}$$

$$+ \rho\frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds$$

with $a(t) = mt + n$. Then, we have:

$$= \gamma K\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\gamma-1}\frac{\dot{f}(\frac{r}{a(t)})}{a(t)^{N+1}} - \kappa\theta\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\theta-1}\frac{\dot{f}(\frac{r}{a(t)})}{a(t)^{N+2}}N\dot{a}(t) \tag{22}$$

$$+ \rho\frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds$$

$$= \rho\left\{\frac{\gamma K f(\frac{r}{a(t)})^{\gamma-2}\dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+N+1}} - \frac{\kappa\theta N f(\frac{r}{a(t)})^{\theta-2}\dot{f}(\frac{r}{a(t)})}{a(t)^{N(\theta-2)+N+2}}\dot{a}(t) + \frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds\right\} \tag{23}$$

$$= \frac{\rho}{a(t)^{N-1}}\left\{\frac{\gamma K f(\frac{r}{a(t)})^{\gamma-2}\dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+2}} - \frac{\kappa\theta N f(\frac{r}{a(t)})^{\theta-2}\dot{f}(\frac{r}{a(t)})}{a(t)^{N(\theta-2)+3}}\dot{a}(t) + \frac{\delta\alpha(N)}{\frac{r^{N-1}}{a(t)^{N-1}}}\int_0^{r/a(t)} \frac{f(s)}{a(t)^N}s^{N-1}ds\right\}. \tag{24}$$

We plug in the condition for $\gamma = \frac{2N-2}{N}$ and $\theta = \frac{2N-3}{N}$ and with the new variable $z := r/a(t)$:

$$= \frac{\rho}{a(t)^{N-1}}\left\{\gamma K f(z)^{\gamma-2}\dot{f}(z) - m\kappa\theta N f(z)^{\theta-2}\dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}}\int_0^z f(s)s^{N-1}ds\right\}. \tag{25}$$

In the theorem, we require that the ordinary differential equation for $y(z)$ is

$$\begin{cases} \gamma K f(z)^{\gamma-2}\dot{f}(z) - m\kappa\theta N f(z)^{\theta-2}\dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}}\int_0^z f(s)s^{N-1}ds = 0, \\ y(0) = \alpha > 0. \end{cases} \tag{26}$$

Therefore, our solutions satisfy the momentum equation.

(1a) Similarly, for $\Lambda = 0$, $\gamma = \frac{2N-2}{N}$ and $\theta = \frac{3N-4}{2N}$, we have:

(27)

$$\rho(V_t + V \cdot V_r) + K\left(\rho^{\gamma}\right)_r - \kappa(\rho^{\theta})_r\left(\frac{N-1}{r}V + V_r\right)$$
$$- \kappa\rho_r(V_{rr} + \frac{N-1}{r}V_r - \frac{N-1}{r^2}V) + \rho\frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \rho s^{N-1}ds$$

(28)

$$= \rho\frac{\ddot{a}(t)}{a(t)}r + K\left(\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\gamma}\right)_r - \kappa\left(\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\theta}\right)_r\frac{N\dot{a}(t)}{a(t)}$$
$$+ \rho\frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds$$

(29)

$$= \rho\frac{\frac{2}{N}(\frac{2}{N}-1)m^2}{(mt+n)^2}r + \gamma K\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\gamma-1}\frac{\partial}{\partial r}\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]$$
$$- \kappa\theta\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\theta-1}\frac{\partial}{\partial r}\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]2m(mt+n)^{-1} + \rho\frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds$$

with $a(t) = (mt+n)^{2/N}$. Then, we have

(30)

$$= \rho\frac{2\,(2-N)\,m^2r}{N^2(mt+n)^2} + \gamma K\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\gamma-1}\frac{\dot{f}(\frac{r}{a(t)})}{a(t)^{N+1}}$$
$$- 2m\kappa\theta\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\theta-1}\frac{\dot{f}(\frac{r}{a(t)})}{a(t)^{N+2}}(mt+n)^{(2-N)/N} + \rho\frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds$$

(31)

$$= \rho\left\{\begin{array}{c} \frac{2(2-N)m^2r}{N^2(mt+n)^2} + \frac{K\gamma f(\frac{r}{a(t)})^{\gamma-2}\dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+N+1}} \\ -\frac{2m\kappa\theta f(\frac{r}{a(t)})^{\theta-2}\dot{f}(\frac{r}{a(t)})}{a(t)^{N(\theta-2)+N+2}}(mt+n)^{(2-N)/N} + \frac{\delta\alpha(N)}{r^{N-1}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds \end{array}\right\}$$

(32)

$$= \frac{\rho}{(mt+n)^{2(N-1)/N}}\left\{\begin{array}{r} \frac{2(2-N)m^2}{N^2}\frac{(mt+n)^{2(N-1)/N}}{(mt+n)^2}r + \frac{\gamma K f(\frac{r}{a(t)})^{\gamma-2}\dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+2}} \\ -\frac{2m\kappa\theta f(\frac{r}{a(t)})^{\theta-2}\dot{f}(\frac{r}{a(t)})}{a(t)^{N(\theta-2)+3}}(mt+n)^{s-1} \\ + \frac{\delta a(N)}{\frac{r^{N-1}}{a(t)^{N-1}}}\int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N}s^{N-1}ds \end{array}\right\}$$

(33)

$$= \frac{\rho}{a(t)^{N-1}} \left\{ \begin{array}{c} \frac{2(2-N)m^2}{N^2} \frac{r}{(mt+n)^{N/2}} + \frac{\gamma K f(\frac{r}{a(t)})^{\gamma-2} \dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+2}} \\ -\frac{2m\kappa\theta f(\frac{r}{a(t)})^{\theta-2}\dot{f}(\frac{r}{a(t)})}{(mt+n)^{s(N(\theta-2)+3)}}(mt+n)^{s-1} + \frac{\delta\alpha(N)}{\frac{r^{N-1}}{a(t)^{N-1}}} \int_0^{r/a(t)} \frac{f(s)}{a(t)^N} s^{N-1} ds \end{array} \right\}$$

(34)

$$= \frac{\rho}{a(t)^{N-1}} \left\{ \begin{array}{c} \frac{2(2-N)m^2}{N^2} \frac{r}{a(t)} + \frac{\gamma K f(\frac{r}{a(t)})^{\gamma-2} \dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+2}} \\ -2m\kappa\theta f(\frac{r}{a(t)})^{\theta-2}\dot{f}(\frac{r}{a(t)}) + \frac{\delta\alpha(N)}{\frac{r^{N-1}}{a(t)^{N-1}}} \int_0^{r/a(t)} \frac{f(s)}{a(t)^N} s^{N-1} ds \end{array} \right\}$$

with $\gamma = \frac{2N-2}{N}$ and $\theta = \frac{3N-4}{2N}$ for $N \geq 2$. and with the new variable $z := r/a(t)$:

$$(35) \qquad = \frac{\rho}{a(t)^{N-1}} \left\{ \frac{2(2-N)m^2}{N^2} z + \gamma K f(z)^{\gamma-2} \dot{f}(z) - 2m\kappa\theta f(z)^{\theta-2} \dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds \right\}.$$

In the theorem, we require the ordinary differential equation for $f(z)$:

(36)

$$\begin{cases} \gamma K f(z)^{\gamma-2} \dot{f}(z) - 2m\kappa\theta f(z)^{\theta-2} \dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds = \frac{2(N-2)m^2}{N^2} z, \\ y(0) = \alpha > 0. \end{cases}$$

Therefore, our solutions satisfy the momentum equation.

(2) For $\delta\Lambda > 0$ and $\gamma = \theta = \frac{2N-2}{N}$, we have

$$(37) \qquad \rho(V_t + V \cdot V_r) + K\left(\rho^{\gamma}\right)_r - \kappa(\rho^{\theta})_r \left(\frac{N-1}{r} V + V_r\right) - \kappa\rho_r(V_{rr} + \frac{N-1}{r} V_r - \frac{N-1}{r^2} V) + \frac{\delta\alpha(N)\rho}{r^{N-1}} \int_0^r (\rho s^{N-1} - \Lambda) ds$$

$$(38) \qquad = \rho \frac{\delta\alpha(N)\Lambda}{N} r + K\left(\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\gamma}\right)_r - \left(\kappa\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\theta}\right)_r \frac{N\dot{a}(t)}{a(t)} + \rho\frac{\delta\alpha(N)}{r^{N-1}} \int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N} s^{N-1} ds - \rho\frac{\delta\alpha(N)\Lambda}{r^{N-1}} \int_0^r s^{N-1} ds.$$

With $a(t) = e^{\sqrt{\frac{\delta\alpha(N)\Lambda}{N}} t}$, then we get:

$$(39) \qquad = K\gamma\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\gamma-1} \frac{\dot{f}(\frac{r}{a(t)})}{a(t)^{N+1}} - \kappa\theta\left[\frac{f(\frac{r}{a(t)})}{a(t)^N}\right]^{\theta-1} \frac{\dot{f}(\frac{r}{a(t)})}{a(t)^{N+1}} \sqrt{N\delta\alpha(N)\Lambda} + \rho\frac{\delta\alpha(N)}{r^{N-1}} \int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N} s^{N-1} ds$$

$$= \rho \left\{ \gamma K \left[ \frac{f(\frac{r}{a(t)})}{a(t)^N} \right]^{\gamma-2} \frac{\dot{f}(\frac{r}{a(t)})}{a(t)^{N+1}} - \kappa\theta\sqrt{N\delta\alpha(N)\Lambda} \left[ \frac{f(\frac{r}{a(t)})}{a(t)^N} \right]^{\theta-2} \frac{\dot{f}(\frac{r}{a(t)})}{a(t)^{N+1}} + \frac{\delta\alpha(N)}{r^{N-1}} \int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N} s^{N-1} ds \right\} \tag{40}$$

$$= \rho \left\{ \frac{\gamma K f(\frac{r}{a(t)})^{\gamma-2} \dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+N+1}} - \frac{\kappa\theta\sqrt{N\delta\alpha(N)\Lambda} f(\frac{r}{a(t)})^{\theta-2} \dot{f}(\frac{r}{a(t)})}{a(t)^{N(\theta-2)+N+1}} + \frac{\delta\alpha(N)}{r^{N-1}} \int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N} s^{N-1} ds \right\} \tag{41}$$

$$= \frac{\rho}{a(t)^{N-1}} \left\{ \frac{\gamma K f(\frac{r}{a(t)})^{\gamma-2} \dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+2}} - \frac{\kappa\theta\sqrt{N\delta\alpha(N)\Lambda} f(\frac{r}{a(t)})^{\theta-2} \dot{f}(\frac{r}{a(t)})}{a(t)^{N(\theta-2)+2}} + \frac{\delta\alpha(N)}{\frac{r^{N-1}}{a(t)^{N-1}}} \int_0^r \frac{f(\frac{r}{a(t)})}{a(t)^N} s^{N-1} ds \right\} \tag{42}$$

$$= \frac{\rho}{a(t)^{N-1}} \left\{ \frac{\gamma K f(\frac{r}{a(t)})^{\gamma-2} \dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+2}} - \frac{\kappa\theta\sqrt{N\delta\alpha(N)\Lambda} f(\frac{r}{a(t)})^{\theta-2} \dot{f}(\frac{r}{a(t)})}{a(t)^{N(\gamma-2)+2}} + \frac{\delta\alpha(N)}{\frac{r^{N-1}}{a(t)^{N-1}}} \int_0^{r/a(t)} \frac{f(s)}{a(t)^N} s^{N-1} ds \right\} \tag{43}$$

$$= \frac{\rho}{a(t)^{N-1}} \left\{ \gamma K f(\frac{r}{a(t)})^{\gamma-2} \dot{f}(\frac{r}{a(t)}) - \kappa\theta\sqrt{N\delta\alpha(N)\Lambda} f(\frac{r}{a(t)})^{\theta-2} \dot{f}(\frac{r}{a(t)}) + \frac{\delta\alpha(N)}{\frac{r^{N-1}}{a(t)^{N-1}}} \int_0^{r/a(t)} \frac{f(s)}{a(t)^N} s^{N-1} ds \right\} \tag{44}$$

with $\gamma = \theta = \frac{2N-2}{N}$. We require the corresponding ordinary differential equations $f(z)$ with $z := r/a(t)$ :

$$\begin{cases} \left(\gamma K - \kappa\theta\sqrt{N\delta\alpha(N)\Lambda}\right) f(z)^{\gamma-2} \dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds = 0, \\ y(0) = \alpha > 0. \end{cases} \tag{45}$$

The proof is completed. □

*Remark* 3. If the complex number solutions $(\rho, u) \in C^{N+1}$ are considered, we have the corresponding solutions for $\delta\Lambda < 0$:

$$\begin{cases} \rho(t,r) = \frac{f(\frac{r}{a(t)})}{a(t)^N},\ V(t,r) = i\sqrt{\frac{-\delta\Lambda}{N}} r, \\ a(t) = e^{i\sqrt{\frac{-\delta\Lambda}{N}} t}, \\ \left(\gamma K - i\kappa\theta\sqrt{-N\delta\alpha(N)\Lambda}\right) f(z)^{\gamma-2} \dot{f}(z) + \frac{\delta\alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds = 0,\ f(0) = \alpha, \end{cases} \tag{46}$$

where $i$ is the complex constant.

*Remark* 4. Our method can be easily extended to the corresponding systems with linear damping:

$$\left\{\begin{array}{rl} \rho_t + \nabla\cdot(\rho\vec{u}) & = 0, \\ \rho[\vec{u}_t + (\vec{u}\cdot\nabla)\,\vec{u}] + \beta\rho\vec{u} + \nabla P & = -\delta\rho\nabla\Phi + vis(\rho,\vec{u}), \\ \Delta\Phi(t,x) & = \alpha(N)\rho - \Lambda, \end{array}\right. \tag{47}$$

where $\beta > 0$ is a constant.

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