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ASYMPTOTIC EXPANSIONS OF CERTAIN PARTIAL THETA FUNCTIONS

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Abstract. We establish an asymptotic expansion for a class of partial theta functions generalizing a result found in Ramanujan's second notebook. Properties of the coefficients in this more general asymptotic expansion are studied, with connections made to combinatorics and a certain Dirichlet series.

1. Introduction

In his second notebook [4, p. 324], S. Ramanujan recorded an asymptotic expansion for the partial theta function

$$2\sum_{n=0}^{\infty}(-1)^n q^{n^2+n} = 2\sum_{n=0}^{\infty}(-1)^n \left(\frac{1-t}{1+t}\right)^{n^2+n} \sim 1+t+t^2+2t^3+5t^4+\cdots, \tag{1.1}$$

where $q = \frac{1-t}{1+t} \to 1^-$, or $t \to 0^+$. This asymptotic series is very interesting since there is no *a priori* reason to believe that the coefficients (in the variable t) are positive integers. After the positive integrality of the coefficients was established by W. Galway [5], using relations between alternating permutations and Euler numbers, R. Stanley [8, Section 5] provided a combinatorial interpretation for these asymptotic series coefficients as the number of fixed-point-free alternating involutions in the symmetric group $\mathcal{S}_{2n}$. It therefore follows trivially that the coefficients on the right-hand side of (1.1) are positive integers.

In this article, we find asymptotic expansions for more general partial theta functions and false theta functions. Our first goal is to find an asymptotic expansion for

$$2\sum_{n=0}^{\infty}(-1)^n q^{an^2+bn}, \tag{1.2}$$

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as $q \to 1^-$, where $a > 0$ and b is real. Replacing q by $q^{\frac{1}{a}}$, we may assume that $a = 1$. Using an argument similar to that given in [4, pp. 544–547], we prove the following theorem.

Theorem 1.1. *Let E_{2n}, $n \geq 0$, denote the $2n$-th Euler number, and let $H_n(x)$, $n \geq 0$, be the n-th Hermite polynomial. Then, as $t \to 0^+$,*

$$2\sum_{n=0}^{\infty}(-1)^n\left(\frac{1-t}{1+t}\right)^{n^2+bn} \sim \left(\frac{1-t}{1+t}\right)^{\frac{2b-1}{4}}\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!2^{2n}}\log^n\left(\frac{1+t}{1-t}\right) \times H_{2n}\left(\frac{b-1}{2}\log^{1/2}\left(\frac{1+t}{1-t}\right)\right). \tag{1.3}$$

Our second task is to prove that the coefficients (of powers of t) in the asymptotic expansion above are integers. To prove the integrality of the coefficients, we need an identity established by S. O. Warnaar [9, p. 380] relating certain false theta functions and partial theta functions; namely,

$$\sum_{n=0}^{\infty}(-1)^n a^n q^{n^2+n} = \sum_{n=0}^{\infty}\frac{(q;q^2)_n(aq;q^2)_n(aq)^n}{(-aq;q)_{2n+1}}, \tag{1.4}$$

where

$$(a)_n := (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j).$$

Here we use an argument given by Galway [5] in the case $b = 1$.

Our third task is to provide an asymptotic expansion for false theta functions of the form

$$2\sum_{n=-\infty}^{\infty}(\text{sgn } n)q^{n^2+bn} \qquad (\text{sgn } 0 = +).$$

Our final task is to study the connection of L-functions associated with certain polynomials and the aforementioned asymptotic series coefficients. After this, we conclude with a conjecture and a question.

2. Asymptotic expansions for partial theta functions

In this section, we find an asymptotic expansion for partial theta functions, i.e., Theorem 1.1. Before proving Theorem 1.1, we need three lemmas.

Lemma 2.1. *If n is a nonnegative integer, $|\arg\beta| < \pi/4$, and $a > 0$, then*

$$\int_0^{\infty} x^{2n}e^{-\beta^2x^2}\cos(ax)dx = (-1)^n\frac{\sqrt{\pi}}{(2\beta)^{2n+1}}e^{-a^2/(4\beta^2)}H_{2n}\left(\frac{a}{2\beta}\right),$$

where $H_n(x)$ denotes the n-th Hermite polynomial.

This integral evaluation is found in [6, p. 529, formula 3.952, no. 9]. It should be noted that the formula in [6] is incorrectly given; the editors have used the font **H** for Struve functions instead of the font H for Hermite polynomials.

Lemma 2.2. *Let $a > 0$, b real, and $\theta > 0$. If $H_n(x)$, $n \geq 0$, denotes the n-th Hermite polynomial, then*

$$I_n := I_n(b) := \frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} z^{2n}e^{biz}e^{-z^2/\theta}dz = \frac{(-1)^n\theta^n}{2^{2n}}e^{-b^2\theta/4}H_{2n}\left(\frac{b\sqrt{\theta}}{2}\right). \tag{2.1}$$

Corollary 2.3. *For $n \geq 0$,*

$$I_n(0) = \frac{\theta^n (2n)!}{2^{2n} n!}. \tag{2.2}$$

Proof. This result follows from Lemma 2.2 upon using the fact [6, p. 1058, formula 8.953, no. 1]

$$H_{2n}(0) = \frac{(-1)^n (2n)!}{n!}, \qquad n \geq 0. \qquad \square$$

Note that this result agrees with Lemma 1.6.1 in [4, p. 544].

Proof of Lemma 2.2. Let $u = z/\sqrt{\theta}$. Using Cauchy's theorem below, we find that

$$\begin{aligned} I_n &= \frac{\theta^n}{\sqrt{\pi}} \int_{-\infty+\frac{a}{\sqrt{\theta}}i}^{\infty+\frac{a}{\sqrt{\theta}}i} u^{2n} e^{ib\sqrt{\theta}u} e^{-u^2} du \\ &= \frac{\theta^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^{2n} e^{ib\sqrt{\theta}u} e^{-u^2} du \\ &= \frac{\theta^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^{2n} \cos(b\sqrt{\theta}u) e^{-u^2} du \\ &= \frac{2\theta^n}{\sqrt{\pi}} \int_{0}^{\infty} u^{2n} \cos(b\sqrt{\theta}u) e^{-u^2} du \\ &= \frac{2\theta^n}{\sqrt{\pi}} (-1)^n \frac{\sqrt{\pi}}{2^{2n+1}} e^{-b^2\theta/4} H_{2n}\left(\frac{b\sqrt{\theta}}{2}\right) \\ &= \frac{(-1)^n \theta^n}{2^{2n}} e^{-b^2\theta/4} H_{2n}\left(\frac{b\sqrt{\theta}}{2}\right), \end{aligned}$$

where we used Lemma 2.1. $\square$

Lemma 2.4. *If a and θ are positive, b is real, and n is a nonnegative integer,*

$$J_n := \frac{1}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(2n+b)iz} dz = e^{-(n+b/2)^2\theta}. \tag{2.3}$$

The case $b = 1$ was proved in [4, pp. 544–545].

Proof of Lemma 2.4. Let $z = u\sqrt{\theta}$. Then

$$\begin{aligned} J_n &= \frac{1}{\sqrt{\pi}} \int_{-\infty+\frac{a}{\sqrt{\theta}}i}^{\infty+\frac{a}{\sqrt{\theta}}i} e^{-u^2+(2n+b)iu\sqrt{\theta}} du \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty+\frac{a}{\sqrt{\theta}}i}^{\infty+\frac{a}{\sqrt{\theta}}i} e^{-(u-(n+b/2)i\sqrt{\theta})^2-(n+b/2)^2\theta} du. \end{aligned}$$

Setting $v = u - (n + b/2)i\sqrt{\theta}$, we find that

$$\begin{aligned} J_n &= \frac{1}{\sqrt{\pi}} e^{-(n+b/2)^2\theta} \int_{-\infty+(\frac{a}{\sqrt{\pi}}-(n+b/2)\sqrt{\theta})i}^{\infty+(\frac{a}{\sqrt{\theta}}-(n+b/2)\sqrt{\theta})i} e^{-v^2} dv \\ &= \frac{1}{\sqrt{\pi}} e^{-(n+b/2)^2\theta} \int_{-\infty}^{\infty} e^{-v^2} dv \\ &= e^{-(n+b/2)^2\theta}, \end{aligned}$$

where in the penultimate line we used Cauchy's theorem. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Write

$$F_1(q) := 2\sum_{n=0}^{\infty}(-1)^n q^{n^2+bn} = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+b/2)^2-b^2/4}. \tag{2.4}$$

Let

$$q = e^{-\theta} = \frac{1-t}{1+t}, \qquad t>0, \qquad \text{or} \qquad \theta = \log\frac{1+t}{1-t}. \tag{2.5}$$

Thus,

$$F_1(q) = e^{b^2\theta/4} 2\sum_{n=0}^{\infty}(-1)^n e^{-(n+b/2)^2\theta} =: e^{b^2\theta/4}G_1(\theta). \tag{2.6}$$

From Lemma 2.4,

$$e^{-(n+b/2)^2\theta} = \frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(2n+b)iz}dz. \tag{2.7}$$

Multiply both sides of (2.7) by $2(-1)^n$ and sum on n, $0 \le n < \infty$, to obtain

$$\begin{aligned} G_1(\theta) &= \frac{2}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta}\sum_{n=0}^{\infty}(-1)^n e^{(2n+b)iz}dz \\ &= \frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(b-1)iz}\frac{dz}{\cos z}, \end{aligned}$$

where we interchanged the order of summation and integration by using the absolute and uniform convergence of the series on the path of integration, as $a>0$. Using the generating function

$$\sec x = \sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}x^{2n}, \qquad |x| < \pi/2,$$

for the Euler numbers E_{2n} and proceeding as in [4, p. 546], we arrive at

$$G_1(\theta) = \frac{1}{\sqrt{\pi\theta}}\sum_{n=0}^{N}\frac{(-1)^n E_{2n}}{(2n)!}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(b-1)iz}z^{2n}dz + R_N, \tag{2.8}$$

where, by an argument completely analogous to that in [4, pp. 546–547],

$$\begin{aligned} |R_N| &= \left|\frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(b-1)iz}\left(\sec z - \sum_{n=0}^{N}\frac{(-1)^n E_{2n}}{(2n)!}z^{2n}\right)dz\right| \\ &= O(\theta^{N+1/2}) \end{aligned}$$

if we choose $a = \sqrt{\theta}$.

In summary, by Lemma 2.2,

$$G_1(\theta) = e^{-(b-1)^2\theta/4}\sum_{n=0}^{N}\frac{E_{2n}\theta^n}{(2n)!2^{2n}}H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) + O(\theta^{N+1/2}).$$

Thus, by (2.6), as $t \to 0^+$,

$$F_1(q) = e^{(2b-1)\theta/4}\sum_{n=0}^{N}\frac{E_{2n}\theta^n}{(2n)!2^{2n}}H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) + O(\theta^{N+1/2}). \tag{2.9}$$

Recalling that $\theta = \log \frac{1+t}{1-t}$, we finish the proof. $\square$

We have proved that $F_1(q)$ has an asymptotic expansion of the form

$$F_1(q) = 2\sum_{n=0}^{\infty}(-1)^n \left(\frac{1-t}{1+t}\right)^{n^2+bn} \sim \sum_{n=0}^{\infty} a_n t^n, \tag{2.10}$$

as t tends to 0^+.

Theorem 2.5. *If b is a positive integer, then each a_n above, $n \geq 0$, is an integer.*

Proof. After Galway [5], we set $s := t/(1+t)$, so that, by (2.5), $t = s/(1-s)$ and $q = 1-2s$. Assume that

$$F_1(q) \sim \sum_{n=0}^{\infty} b_n s^n.$$

Note that the integrality of b_n implies the integrality of a_n. We restate (1.4), i.e.,

$$\sum_{n=0}^{\infty}(-1)^n a^n q^{n^2+n} = \sum_{n=0}^{\infty} \frac{(q;q^2)_n (aq;q^2)_n (aq)^n}{(-aq;q)_{2n+1}}.$$

Replacing a by q^{b-1}, we see that

$$\sum_{n=0}^{\infty}(-1)^n q^{n^2+bn} = \sum_{n=0}^{\infty} \frac{(q;q^2)_n (q^b;q^2)_n (q)^{bn}}{(-q^b;q)_{2n+1}} =: \sum_{n=0}^{\infty} T_n. \tag{2.11}$$

Since, as $q \to 1^-$,

$$(q;q^2)_n = \prod_{k=0}^{n-1}(1-q^{2k+1}) = \prod_{k=0}^{n-1} O(s) = O(s^n),$$

$$(q^b;q^2)_n = \prod_{k=0}^{n-1}(1-q^{2k+b}) = \prod_{k=0}^{n-1} O(s) = O(s^n),$$

we conclude that $T_n = O(s^{2n})$.

For $s \in (0, \frac{1}{2})$, i.e., $q \in (0,1)$,

$$T_{n+1} = \frac{(1-q^{2n+1})(1-q^{b+2n})q^b}{(1+q^{b+2n+1})(1+q^{b+2n+2})} T_n \leq q^b T_n.$$

Thus, by induction, we easily conclude that $T_{n+k} \leq q^{kb} T_n$ for all $k \geq 0$. Therefore, for $b \geq 1$ and $0 < q < 1$,

$$\sum_{k=0}^{\infty} T_{n+k} \leq T_n \sum_{k=0}^{\infty} q^{bk} = \frac{T_n}{1-q^b} \leq \frac{T_n}{1-q},$$

which, since $1-q = 2s$ and $T_n = O(s^{2n})$, implies that $\sum_{k=0}^{\infty} T_{n+k} = O(s^{2n-1})$.

In summary, by (2.4) and (2.11), we conclude that

$$F_1(q) = 2\sum_{n=0}^{\infty} T_n = 2\sum_{n=0}^{N-1} T_n + 2\sum_{k=0}^{\infty} T_{N+k} = 2\sum_{n=0}^{N-1} T_n + O(s^{2N-1}).$$

Expanding each T_n as a power series in s for $0 \leq n \leq N-1$, we can obtain the first $2N$ coefficients of the asymptotic expansion for $F(q)$. Thus, to show the integrality of the coefficients, it is sufficient to show that each $2T_n$, $n \geq 0$, has only integral coefficients in its power series expansion.

Note that, for all $k \geq 1$, we have $q^k \in 1 + 2s\mathbb{Z}[s]$, and thus $1 - q^k \in 2s\mathbb{Z}[s]$ and $1 + q^k \in 2 + 2s\mathbb{Z}[s]$. Therefore,

$$2T_n = 2\frac{(q;q^2)_n(q^b;q^2)_n q^{bn}}{(-q^b;q)_{2n+1}} \in \mathbb{Z}[s]. \qquad \square$$

3. An asymptotic expansion for $2\sum_{n=0}^{\infty} q^{n^2+bn}$

In this section, we seek an asymptotic expansion for

$$2\sum_{n=0}^{\infty}\left(\frac{1-t}{1+t}\right)^{n^2+bn}.$$

As before, let us set $q = e^{-\theta}$ and put

$$F_2(q) = e^{b^2\theta/4} 2\sum_{n=0}^{\infty} e^{-(n+b/2)^2\theta} := e^{b^2\theta/4} G_2(\theta). \tag{3.1}$$

Then, by Lemma 2.4,

$$\begin{aligned} G_2(\theta) &= \frac{2}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta} \sum_{n=0}^{\infty} e^{(2n+b)iz} dz \\ &= \frac{i}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(b-1)iz} \frac{dz}{\sin z}. \end{aligned}$$

Recall that [6, p. 42, formula 1.411, no. 11]

$$\csc z = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^n 2\left(2^{2n+1}-1\right) B_{2n+2}}{(2n+2)!} z^{2n+1}, \quad |z| < \pi,$$

where B_n, $n \geq 0$, is the n-th Bernoulli number. Proceeding as before, we arrive at

$$\begin{aligned} G_2(\theta) &= \frac{i}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} \frac{1}{z} e^{(b-1)iz - z^2/\theta} dz \\ &\quad + \frac{i}{\sqrt{\pi\theta}} \sum_{n=0}^{N-1} \frac{(-1)^n 2\left(2^{2n+1}-1\right) B_{2n+2}}{(2n+2)!} \int_{-\infty+ai}^{\infty+ai} z^{2n+1} e^{(b-1)iz + -z^2/\theta} dz \\ &\quad + O(\theta^{N+1/2}) \\ &=: J(b-1) + \sum_{n=0}^{N-1} \frac{(-1)^n 2\left(2^{2n+1}-1\right) B_{2n+2}}{(2n+2)!} J_n(b-1) + O(\theta^{N+1/2}). \end{aligned} \tag{3.2}$$

To evaluate the integrals $J(b)$ and $J_n(b)$, we need the following three lemmas.

Lemma 3.1. *If n is a nonnegative integer, $|\arg \beta| < \pi/4$, and $a > 0$, then*

$$\int_0^{\infty} x^{2n+1} e^{-\beta^2 x^2} \sin(ax) dx = (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+2}} e^{-a^2/(4\beta^2)} H_{2n+1}\left(\frac{a}{2\beta}\right),$$

where $H_n(x)$ denotes the n-th Hermite polynomial.

This integral evaluation can be found in [6, p. 529, formula 3.952, no. 10]. (As before, this formula is misprinted with the wrong font for H in [6].)

Lemma 3.2. *Let $a > 0$, b real, and $\theta > 0$. If $H_n(x)$ denotes the n-th Hermite polynomial, then, for $n \geq 0$,*

$$J_n(b) = \frac{i}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} z^{2n+1} e^{biz-z^2/\theta} dz = \frac{(-1)^{n+1}(\sqrt{\theta})^{2n+1}}{2^{2n+1}} e^{-b^2\theta/4} H_{2n+1}\left(\frac{b\sqrt{\theta}}{2}\right).$$

Proof. Let $u = z/\sqrt{\theta}$. Then, by Cauchy's formula,

$$\begin{aligned} J_n(b) &= -\frac{2(\sqrt{\theta})^{2n+1}}{\sqrt{\pi}} \int_0^\infty u^{2n+1} \sin(b\sqrt{\theta}u) e^{-u^2} du \\ &= \frac{(-1)^{n+1}(\sqrt{\theta})^{2n+1}}{2^{2n+1}} e^{-b^2\theta/4} H_{2n+1}\left(\frac{b\sqrt{\theta}}{2}\right), \end{aligned}$$

where we have used Lemma 3.1. □

Lemma 3.3. *Let $a > 0$, b real, and $\theta > 0$. Then,*

$$J(b) = \frac{i}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} \frac{1}{z} e^{biz-z^2/\theta} dz = \sqrt{\frac{\pi}{\theta}} - \frac{1}{\sqrt{\theta}} e^{-b^2\theta/4} \sinh(\sqrt{\theta}b). \tag{3.3}$$

Proof. Setting $u = z/\sqrt{\theta}$, we arrive at

$$J(b) = \frac{i}{\sqrt{\pi\theta}} \int_{-\infty+\frac{a}{\sqrt{\theta}}i}^{\infty+\frac{a}{\sqrt{\theta}}i} \frac{1}{u} e^{b\sqrt{\theta}iu+-u^2} du.$$

Then, by the residue theorem,

$$\begin{aligned} J(b) &= \sqrt{\frac{\pi}{\theta}} - \frac{2}{\sqrt{\pi\theta}} \int_0^\infty \frac{1}{u} \sin(b\sqrt{\theta}u) e^{-u^2} du \\ &= \sqrt{\frac{\pi}{\theta}} - be^{-b^2\theta/4} {}_0F_1\left(-; \frac{3}{2}; \frac{\theta b^2}{4}\right), \end{aligned} \tag{3.4}$$

where, for $b \neq 0$, we used an integral evaluation from [6, p. 529, formula 3.952, no. 7]. Here ${}_0F_1(-; a; z)$ is the generalized hypergeometric function, usually so denoted. Furthermore, from [2, pp. 200, 202, eqs. (4.5.2), (4.6.3)],

$${}_0F_1\left(-; \frac{3}{2}; -\left(\frac{x}{2}\right)^2\right) = \frac{\sin x}{x}.$$

Hence,

$${}_0F_1\left(-; \frac{3}{2}; \frac{\theta b^2}{4}\right) = \frac{\sinh(\sqrt{\theta}b)}{\sqrt{\theta}b}. \tag{3.5}$$

Using (3.5) in (3.4), we complete the proof. □

Combining Lemmas 3.2 and 3.3, (3.5), and (3.2), we have proved the following theorem.

Theorem 3.4. *Let $G_2(\theta)$ be defined by (3.1). Then, as $\theta \to 0^+$,*

$$\begin{aligned} G_2(\theta) = \sqrt{\frac{\pi}{\theta}} &- e^{-(b-1)^2\theta/4} \frac{\sinh(\sqrt{\theta}(b-1))}{\sqrt{\theta}} \\ &- e^{-(b-1)^2\theta/4} \sum_{n=0}^{N-1} \frac{\left(2^{2n+1}-1\right) B_{2n+2}(\sqrt{\theta})^{2n+1}}{2^{2n}(2n+2)!} H_{2n+1}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) \\ &+ O(\theta^{N+1/2}). \end{aligned} \tag{3.6}$$

In particular, when $b=1$, $G_2(\theta) \sim \sqrt{\frac{\pi}{\theta}}$, since $H_{2n+1}(0)=0$, $n \geq 0$, which agrees with the familiar transformation formula for theta functions. Now we are ready to offer an asymptotic expansion for false theta functions of the form

$$F_3(q) := 2 \sum_{n=-\infty}^{\infty} (\text{sgn } n) q^{n^2+bn}, \tag{3.7}$$

where b is a positive real number. Setting $q = e^{-\theta}$ and using the asymptotic expansion of $F_2(q)$, given by (3.1) and (3.6), we can deduce the following theorem.

Theorem 3.5. *Let $F_3(q)$ be defined by (3.7), and put $q = e^{-\theta}$. Then, as $\theta \to 0$,*

$$\begin{aligned} e^{-b^2\theta/4} F_3(e^{-\theta}) &= 2 - e^{-(b-1)^2\theta/4} \frac{\sinh(\sqrt{\theta}(b-1))}{\sqrt{\theta}} - e^{-(b+1)^2\theta/4} \frac{\sinh(\sqrt{\theta}(b+1))}{\sqrt{\theta}} \\ &- e^{-(b-1)^2\theta/4} \sum_{n=0}^{N-1} \frac{2\left(2^{2n+1}-1\right) B_{2n+2} (\sqrt{\theta})^{2n+1}}{2^{2n+1}(2n+2)!} H_{2n+1}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) \\ &- e^{-(b+1)^2\theta/4} \sum_{n=0}^{N-1} \frac{2\left(2^{2n+1}-1\right) B_{2n+2} (\sqrt{\theta})^{2n+1}}{2^{2n+1}(2n+2)!} H_{2n+1}\left(\frac{(b+1)\sqrt{\theta}}{2}\right) \\ &+ O(\theta^{N+1/2}). \end{aligned}$$

As before, by setting $q = e^{-\theta} = \frac{1-t}{1+t}$, we can deduce an asymptotic series in powers of t.

4. Connection with certain L-series and partitions

It is now well known that we can possibly use partial theta function identities to obtain generating functions for certain L-functions; for example, see [3] and [10]. Here, we obtain such results for the L-series,

$$\mathcal{Z}(x^2+bx, -1, s) := \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+bn)^s}, \tag{4.1}$$

which was extensively studied by P. Cassou-Noguès [7] in a more general setting. In particular, $\mathcal{Z}(x^2+bx,-1,s)$ has an analytic continuation into the entire complex s-plane [7, Theorem 3].

In (2.11), we replace q by e^{-t} to obtain

$$\mathcal{F}(e^{-t}) := \sum_{n=1}^{\infty} (-1)^n e^{-(n^2+bn)t} = -1 + \sum_{n=0}^{\infty} \frac{(e^{-t};e^{-2t})_n (e^{-bt};e^{-2t})_n e^{-bnt}}{(-e^{-bt};e^{-t})_{2n+1}} := \mathcal{G}(e^{-t}). \tag{4.2}$$

Thus, for Re $s > \frac{1}{2}$,

$$\begin{aligned} &\int_0^\infty \mathcal{F}(e^{-t}) t^{s-1} dt \\ &= \int_0^\infty \sum_{n=1}^{\infty} (-1)^n e^{-(n^2+bn)t} t^{s-1} dt \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+bn)^s} \int_0^\infty e^{-t} t^{s-1} dt \\ &= \Gamma(s) \mathcal{Z}(x^2+bx, -1, s), \qquad (4.3) \end{aligned}$$

where $\mathcal{Z}(x^2+bx,-1,s)$ is defined in (4.1).

On the other hand, if we define c_n to be the n-th coefficient of the Taylor expansion for $\mathcal{G}(e^{-t})$ about $t=0$, we find that

$$\begin{aligned}\int_0^\infty &\mathcal{G}(e^{-t})t^{s-1}dt\\ &=\int_0^\infty\left(\sum_{n=0}^{N-1}c_nt^n+O(t^N)\right)t^{s-1}dt\\ &=\sum_{n=0}^{N-1}\frac{c_n}{s+n}+I(s),\end{aligned}\tag{4.4}$$

where $I(s)$ is an analytic function for $\operatorname{Re}s>-N$. Thus, c_n is the residue of $\Gamma(s)\mathcal{Z}(x^2+bx,-1,s)$ at $s=-n$, so that, by (4.2), (4.3), and (4.4), we have established the following theorem.

Theorem 4.1. *The Taylor series coefficients c_n about $t=0$ for $G(e^{-t})$ are given by*

$$c_n=\frac{(-1)^n}{n!}\mathcal{Z}(x^2+bx,-1,s),$$

for all $n\geq 0$, where $\mathcal{Z}(x^2+bx,-1,s)$ is defined by (4.1).

5. Conjecture and question

Numerical data suggest the following conjecture:

Conjecture 1. *For any positive integer b, for sufficiently large n, the coefficients a_n, which appear in the asymptotic expansion* (2.10)*, have the same sign.*

Question 1. In view of Stanley's work [8], it is natural to ask: Is there a combinatorial interpretation for the asymptotic series coefficients a_n in (2.10)?

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