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DEPTH OF EDGE RINGS ARISING FROM FINITE GRAPHS

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Abstract. Let G be a finite graph and $K[G]$ the edge ring of G. Based on the technique of Gröbner bases and initial ideals, it will be proved that, given integers f and d with $7 \leq f \leq d$, there exists a finite graph G on $[d] = \{1, \ldots, d\}$ with depth $K[G] = f$ and with Krull-dim $K[G] = d$.

Introduction

The edge ring [3] and its toric ideal [4] arising from a finite graph have been studied from the viewpoints of both commutative algebra and combinatorics. Especially, the normality of the edge ring as well as the Gröbner bases of its toric ideal is extensively investigated. However, the fundamental question when an edge ring is Cohen–Macaulay is presumably open.

Let G be a finite simple graph, i.e., a finite graph with no loop and no multiple edge, on the vertex set $[d] = \{1, \ldots, d\}$ and $E(G) = \{e_1, \ldots, e_r\}$ its edge set. Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ be the polynomial ring in d variables over a field K and write $K[G]$ for the subring of $K[\mathbf{t}]$ generated by those squarefree quadratic monomials $\mathbf{t}^e = t_i t_j$ with $e = \{i, j\} \in E(G)$. The semigroup ring $K[G]$ is called the *edge ring* of G. Let Krull-dim $K[G]$ denote the Krull dimension of $K[G]$ and depth $K[G]$ the depth of $K[G]$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_r]$ be the polynomial ring in r variables over a field K. The kernel I_G of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[G]$ defined by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, r$ is called the *toric ideal* of G. One has $K[G] \cong K[\mathbf{x}]/I_G$. If G is connected and is nonbipartite (resp. bipartite), then Krull-dim $K[G] = d$ (resp. Krull-dim $K[G] = d - 1$).

The criterion of normality [3, Corollary 2.3] of edge rings guarantees that $K[G]$ is normal if either G is bipartite or $d \leq 6$. If $d = 7$, then there exists a finite graph G for which $K[G]$ is nonnormal. However, it follows easily that $K[G]$ is Cohen–Macaulay whenever $d \leq 7$. Computing the depth of the edge rings of all connected nonbipartite graphs G with 7 vertices shows that the depth of $K[G]$ is at least 7. Moreover, our computational experiment would naturally lead the authors into the temptation to give the following conjecture.

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Conjecture 0.1. *Let G be a finite connected nonbipartite graph on $[d]$ with $d \geq 7$. Then* $\operatorname{depth} K[G] \geq 7$.

Now, even though Conjecture 0.1 is completely open, by taking Conjecture 0.1 into consideration, this paper will be devoted to proving the following.

Theorem 0.2. *Given integers f and d with $7 \leq f \leq d$, there exists a finite graph G on $[d]$ with* $\operatorname{depth} K[G] = f$ *and with* Krull-dim $K[G] = d$.

Let $k \geq 1$ be an arbitrary integer and G_{k+6} the finite graph on $[k+6]$ of Figure 0.1. The essential part of a proof of Theorem 0.2 is to show that

$$\operatorname{depth} K[G_{k+6}] = \operatorname{depth} K[\mathbf{x}]/I_{G_{k+6}} = 7. \tag{0.1}$$

In Section 1, by virtue of the formula [1, Theorem 2.1], the inequality $\operatorname{depth} K[G_{k+6}] \leq 7$ will be proved. In Section 2, we compute a Gröbner basis of $I_{G_{k+6}}$ and an initial ideal $\operatorname{in}(I_{G_{k+6}})$ of $I_{G_{k+6}}$, and show the inequality $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_{G_{k+6}}) \geq 7$. In general, one has $\operatorname{depth} K[\mathbf{x}]/I_{G_{k+6}} \geq \operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_{G_{k+6}})$ (e.g., [2, Theorem 3.3.4 (d)]). Thus the desired equality (0.1) follows.

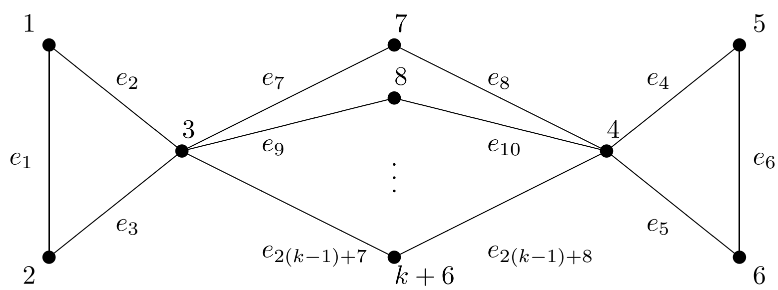

FIGURE 0.1. (finite graph G_{k+6})

Once we know that $\operatorname{depth} K[G_{k+6}] = 7$, to prove Theorem 0.2 is straightforward. In fact, given integers f and d with $7 \leq f \leq d$, let Γ denote the finite graph G_{d-f+7} on $[d-f+7]$ and write G for the finite graph on $[d]$ obtained from Γ by adding $f-7$ edges

$$\{1, d-f+8\}, \{1, d-f+9\}, \ldots, \{1, d\}$$

to Γ. It then follows that $\operatorname{depth} K[G] = \operatorname{depth} K[\Gamma] + f - 7$. Since $\operatorname{depth} K[\Gamma] = 7$, one has $\operatorname{depth} K[G] = f$, as required.

1. PROOF OF $\operatorname{depth} K[G_{k+6}] \leq 7$

Let $G = G_{k+6}$ of Figure 0.1. In this section, we prove that $\operatorname{depth} K[G] \leq 7$. Since the number of edges of G is $r = 2(k-1) + 8$, the Auslander–Buchsbaum formula implies that we may prove $\operatorname{pd} K[G] \geq r - 7 = 2k - 1$.

Let S_G be the semigroup arising from G. Let $\mathcal{A}_G = \{\underline{a}_1, \ldots, \underline{a}_r\}$ be the set of columns of the incidence matrix of G where $\underline{a}_l$ corresponds to the edge e_l (which corresponds to the variable x_l). Therefore, $S_G = \mathbb{N}\mathcal{A}_G$.

To prove $\operatorname{pd} K[G] \geq 2k - 1$, we use the following theorem due to Briales, Campillo, Marijuán, and Pisón [1]. For $\underline{s} \in S_G$, we define the simplicial complex

$$\Delta_{\underline{s}} = \{F \subset [r] \ : \ \underline{s} - \underline{n}_F \in S_G\},$$

where $\underline{n}_F = \sum_{l\in F} \underline{a}_l$. We denote by $\beta_{i,\underline{s}}(K[G])$, the ith multigraded Betti number of $K[G]$ in degree $\underline{s}$.

Lemma 1.1 ([1, Theorem 2.1])**.** *Let G be a finite simple graph. Then*

$$\beta_{j+1,\underline{s}}(K[G]) = \dim_K \tilde{H}_j(\Delta_{\underline{s}}; K).$$

We consider the case where

$$\underline{s} = (1, 1, k+1, k+1, 1, 1, 2, 2, \dots, 2).$$

By Lemma 1.1, it is sufficient to prove the following lemma:

Lemma 1.2. *Set $\underline{s} = (1, 1, k+1, k+1, 1, 1, 2, 2, \dots, 2)$. Then*

$$\dim_K \tilde{H}_{2k-2}(\Delta_{\underline{s}}; K) \neq 0.$$

We set $\Delta = \Delta_{\underline{s}}$. Before proving Lemma 1.2, we compute the simplicial complex Δ.

Lemma 1.3. *Set $\underline{s} = (1, 1, k+1, k+1, 1, 1, 2, 2, \dots, 2)$. Then the facets of $\Delta_{\underline{s}}$ are the following subsets of $[r]$:*

$$F_{1,i} = \{1, 4, 5, 7, 8, \dots, 2(k-1)+8\} \setminus \{2(i-1)+8\}, \qquad i = 1, \dots, k;$$
$$F_{2,j} = \{2, 3, 6, 7, 8, \dots, 2(k-1)+8\} \setminus \{2(j-1)+7\}, \qquad j = 1, \dots, k.$$

Proof. Since $\underline{s} - \underline{n}_{F_{1,i}} = \underline{a}_{2(i-1)+7} \in S_G$, we have $F_{1,i} \in \Delta_{\underline{s}} = \Delta$. (It follows that $\underline{s} \in S_G$.) Similarly, we have $F_{2,j} \in \Delta$.

To prove that there are no facets other than $F_{1,i}, F_{2,j}$, it is enough to show that

- $\{1, 2\}, \{1, 3\}, \{4, 6\}, \{5, 6\} \notin \Delta$;
- $\{1, 6\} \notin \Delta$;
- $\{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\} \notin \Delta$;
- $F_0 = \{7, 8, \dots, 2(k-1)+8\} \notin \Delta$.

Since the first entry of $\underline{s} - \underline{n}_{\{1,2\}}$ is $-1 < 0$, it follows that $\underline{s} - \underline{n}_{\{1,2\}} \notin S_G$. Therefore $\{1, 2\} \notin \Delta$. By symmetry, we also have $\{1, 3\}, \{4, 6\}, \{5, 6\} \notin \Delta$.

Second we show that $\{1, 6\} \notin \Delta$. Suppose, on the contrary, that $\{1, 6\} \in \Delta$, i.e.,

$$\underline{s} - \underline{n}_{\{1,6\}} = (0, 0, k+1, k+1, 0, 0, 2, 2, \dots, 2) \in S_G.$$

Then we can write $\underline{s} - \underline{n}_{\{1,6\}} = \sum_{l=1}^{r} c_l \underline{a}_l$, where $c_l \in \mathbb{N}$. Since $(\underline{s} - \underline{n}_{\{1,6\}})_1 = (\underline{s} - \underline{n}_{\{1,6\}})_2 = 0$ and $(\underline{s} - \underline{n}_{\{1,6\}})_3 = k+1$, we have $c_1 = c_2 = c_3 = 0$ and $\sum_{i=1}^{k} c_{2(i-1)+7} = k+1$. Similarly, we have $c_4 = c_5 = c_6 = 0$ and $\sum_{j=1}^{k} c_{2(j-1)+8} = k+1$. Then $\sum_{i=1}^{k} c_{2(i-1)+7} + \sum_{j=1}^{k} c_{2(j-1)+8} = 2(k+1)$, but it must be $2k$. This is a contradiction.

Next we show that $\{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\} \notin \Delta$. Suppose that $\{2, 4\} \in \Delta$, i.e.,

$$\underline{s} - \underline{n}_{\{2,4\}} = (0, 1, k, k, 0, 1, 2, 2, \dots, 2) \in S_G.$$

Then we can write $\underline{s} - \underline{n}_{\{2,4\}} = \sum_{l=1}^{r} c_l \underline{a}_l$, where $c_l \in \mathbb{N}$. Since $(\underline{s} - \underline{n}_{\{2,4\}})_1 = 0$ and $(\underline{s} - \underline{n}_{\{2,4\}})_2 = 1$, we have $c_3 = 1$. Similarly, we have $c_5 = 1$. Thus

$$(0, 0, k-1, k-1, 0, 0, 2, 2, \dots, 2) \in S_G.$$

Then a similar argument to the proof of $\{1, 6\} \notin \Delta$ yields a contradiction. Therefore $\{2, 4\} \notin \Delta$. By symmetry, we also have $\{2, 5\}, \{3, 4\}, \{3, 5\} \notin \Delta$.

Last, we show $F_0 \notin \Delta$. It follows from

$$\underline{s} - \underline{n}_{F_0} = (1, 1, 1, 1, 1, 1, 0, 0, \dots, 0) \notin S_G. \qquad \square$$

Now we prove Lemma 1.2.

Proof of Lemma 1.2. Let Δ_1 be the subcomplex of Δ whose facets are $F_{1,i}$, $i = 1, \ldots, k$, and let Δ_2 be the subcomplex of Δ whose facets are $F_{2,j}$, $j = 1, \ldots, k$. Then $\Delta = \Delta_1 \cup \Delta_2$. Also, the facets of the simplicial complex $\Delta_1 \cap \Delta_2$ are

$$\{7, 8, \ldots, 2(k-1)+8\} \setminus \{2(j-1)+7, 2(i-1)+8\}, \qquad i, j = 1, \ldots, k.$$

In particular, $\dim(\Delta_1 \cap \Delta_2) = 2k-3$. Note that both of Δ_1 and Δ_2 are cones over some simplicial complexes, and so the reduced homologies of all of these vanish. Therefore the Mayer–Vietoris sequence

$$\cdots \longrightarrow \tilde{H}_i(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \longrightarrow \tilde{H}_i(\Delta; K)$$
$$\longrightarrow \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_{i-1}(\Delta_1; K) \oplus \tilde{H}_{i-1}(\Delta_2; K) \longrightarrow \cdots$$

yields

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \qquad \text{for all } i.$$

We can see that $\tilde{H}_{2k-3}(\Delta_1 \cap \Delta_2; K) \neq 0$ by considering the alternating sum of all facets of $\Delta_1 \cap \Delta_2$:

$$\sum_{1 \leq i,j \leq k} (-1)^{i+j} \{7, 8, \ldots, 2(k-1)+8\} \setminus \{2(j-1)+7, 2(i-1)+8\}.$$

Therefore we have $\tilde{H}_{2k-2}(\Delta; K) \neq 0$. □

2. Proof of $\operatorname{depth} K[G_{k+6}] \geq 7$

As before, let $G = G_{k+6}$ be as in Figure 0.1. In this section we prove that $\operatorname{depth} K[G] \geq 7$.

We set $C_1 = (e_2, e_1, e_3)$ and $C_2 = (e_4, e_6, e_5)$, both of which are 3-cycles of G. By [4, Lemma 3.2], there are 3 kinds of primitive even closed walks Γ of G up to route:

(I) a 4-cycle: $\Gamma = (e_{2(i-1)+7}, e_{2(i-1)+8}, e_{2(j-1)+8}, e_{2(j-1)+7})$, where $i < j$;

(II) a walk on two 3-cycles C_1, C_2 and a single path connecting C_1 and C_2: $\Gamma = (C_1, e_{2(i-1)+7}, e_{2(i-1)+8}, C_2, e_{2(i-1)+8}, e_{2(i-1)+7})$, where $i = 1, \ldots, k$;

(III) a walk on two 3-cycles C_1, C_2 and two different paths combining C_1 and C_2: $\Gamma = (C_1, e_{2(i-1)+7}, e_{2(i-1)+8}, C_2, e_{2(j-1)+8}, e_{2(j-1)+7})$, where $i < j$.

It was proved in [4, Lemma 3.1] that binomials corresponding to these primitive even closed walks generate the toric ideal I_G. Let us consider the lexicographic order $< = <_{\text{lex}}$ with $x_1 > x_2 > x_3 > \cdots > x_{2(k-1)+8}$.

Lemma 2.1. *The set of binomials corresponding to primitive even closed walks (I), (II), (III) is a Gröbner basis of I_G with respect to $<_{\text{lex}}$.*

Proof. The result follows from a straightforward application of Buchberger's algorithm to the set of generators of I_G corresponding to the primitive even closed walks listed above. Let f and g be two such generators. We will prove that the S-polynomial, $S(f, g)$, arising from Buchberger's algorithm will reduce to 0 by generators of type (I), (II) and (III). For convenience of notation, we will assume that i, j, p, and q are all odd integers such that $7 \leq i < j, 7 \leq p < q$.

Case 1. Let $f = x_i x_{j+1} - x_{i+1} x_j$ and $g = x_p x_{q+1} - x_{p+1} x_q$ be generators of type (I). If $i \neq p$ and $j \neq q$, then the leading terms of f and g are relatively prime and thus the S-polynomial $S(f,g)$ will reduce to 0 (e.g., [2, Lemma 2.3.1]). Suppose $i = p$; then

$$\begin{aligned} S(f,g) &= \frac{\operatorname{lcm}(LM_{<_{\text{lex}}}(f), LM_{<_{\text{lex}}}(g))}{LT_{<_{\text{lex}}}(f)} f - \frac{\operatorname{lcm}(LM_{<_{\text{lex}}}(f), LM_{<_{\text{lex}}}(g))}{LT_{<_{\text{lex}}}(g)} g \\ &= x_{q+1}(x_i x_{j+1} - x_{i+1} x_j) - x_{j+1}(x_i x_{q+1} - x_{i+1} x_q) \\ &= x_{i+1} x_{j+1} x_q - x_{i+1} x_j x_{q+1} \\ &= x_{i+1}(x_{j+1} x_q - x_j x_{q+1}). \end{aligned}$$

Note that, up to sign, $x_{j+1} x_q - x_j x_{q+1}$ is a generator of I_G of type (I) and therefore $S(f,g)$ will reduce to 0. The case of $j = q$ is similar.

Case 2. Let f be the same as above and $g = x_1 x_4 x_5 x_p^2 - x_2 x_3 x_6 x_{p+1}^2$ be a generator of type (II). If $i \neq p$, then the leading terms of f and g are relatively prime and therefore negligible. If $i = p$, then

$$\begin{aligned} S(f,g) &= x_1 x_4 x_5 x_i (x_i x_{j+1} - x_{i+1} x_j) - x_{j+1}(x_1 x_4 x_5 x_i^2 - x_2 x_3 x_6 x_{i+1}^2) \\ &= x_2 x_3 x_6 x_{i+1}^2 x_{j+1} - x_1 x_4 x_5 x_i x_{i+1} x_j \\ &= -x_{i+1}(x_1 x_4 x_5 x_i x_j - x_2 x_3 x_6 x_{i+1} x_{j+1}), \end{aligned}$$

where $x_1 x_4 x_5 x_i x_j - x_2 x_3 x_6 x_{i+1} x_{j+1}$ is a generator of type (III).

Case 3. Again, we assume that f is the same as above. Now assume g is of type (III), $g = x_1 x_4 x_5 x_p x_q - x_2 x_3 x_6 x_{p+1} x_{q+1}$. If $i \neq p, q$, then the leading terms of f and g will be relatively prime. Suppose $i = p$; then

$$\begin{aligned} S(f,g) &= x_1 x_4 x_5 x_q (x_i x_{j+1} - x_{i+1} x_j) - x_{j+1}(x_1 x_4 x_5 x_i x_q - x_2 x_3 x_6 x_{i+1} x_{q+1}) \\ &= -x_{i+1}(x_1 x_4 x_5 x_q x_j - x_2 x_3 x_6 x_{q+1} x_{j+1}) \end{aligned}$$

and again we have that $x_1 x_4 x_5 x_q x_j - x_2 x_3 x_6 x_{q+1} x_{j+1}$ is either a type (II) or type (III) generator of I_G. The case of $i = q$ is similar.

Case 4. Now let f and g both be generators of type (II), $f = x_1 x_4 x_5 x_i^2 - x_2 x_3 x_6 x_{i+1}^2$, $g = x_1 x_4 x_5 x_j^2 - x_2 x_3 x_6 x_{j+1}^2$. Then the S-polynomial

$$\begin{aligned} S(f,g) &= x_j^2 (x_1 x_4 x_5 x_i^2 - x_2 x_3 x_6 x_{i+1}^2) - x_i^2 (x_1 x_4 x_5 x_j^2 - x_2 x_3 x_6 x_{j+1}^2) \\ &= x_2 x_3 x_6 (x_i^2 x_{j+1}^2 - x_{i+1}^2 x_j^2) \\ &= x_2 x_3 x_6 (x_i x_{j+1} + x_{i+1} x_j)(x_i x_{j+1} - x_{i+1} x_j) \end{aligned}$$

is a multiple of a type (I) generator.

Case 5. Let f be the same as in Case 4 and $g = x_1 x_4 x_5 x_p x_q - x_2 x_3 x_6 x_{p+1} x_{q+1}$ be of type (III). First suppose that $i \neq p, q$. Let us consider the case of $i < p$. Then

$$\begin{aligned} S(f,g) &= x_p x_q (x_1 x_4 x_5 x_i^2 - x_2 x_3 x_6 x_{i+1}^2) - x_i^2 (x_1 x_4 x_5 x_p x_q - x_2 x_3 x_6 x_{p+1} x_{q+1}) \\ &= x_2 x_3 x_6 (x_i^2 x_{p+1} x_{q+1} - x_{i+1}^2 x_p x_q) \\ &= x_2 x_3 x_6 [x_i x_{q+1}(x_i x_{p+1} - x_{i+1} x_p) + x_i x_{i+1} x_p x_{q+1} - x_{i+1}^2 x_p x_q] \\ &= x_2 x_3 x_6 [x_i x_{q+1}(x_i x_{p+1} - x_{i+1} x_p) + x_{i+1} x_p (x_i x_{q+1} - x_{i+1} x_q)], \end{aligned}$$

and so $S(f,g)$ reduces to 0 by two type (I) generators. The cases of $p < i < q$ and $q < i$ are similar.

Now suppose $i = p$. Then the S-polynomial,

$$\begin{aligned} S(f,g) &= x_q(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) - x_i(x_1x_4x_5x_ix_q - x_2x_3x_6x_{i+1}x_{q+1}) \\ &= x_2x_3x_6x_{i+1}(x_ix_{q+1} - x_{i+1}x_q), \end{aligned}$$

is a multiple of a type (I) generator. The case of $i = q$ is similar.

Case 6. Finally, we consider the case that both f and g are of type (III): $f = x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}$, $g = x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}$. We may assume that $i \le p$. Let us first suppose that $i, j \ne p, q$. Then

$$\begin{aligned} S(f,g) &= x_px_q(x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}) \\ &\quad -x_ix_j(x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}) \\ &= x_2x_3x_6(x_ix_jx_{p+1}x_{q+1} - x_{i+1}x_{j+1}x_px_q) \\ &= x_2x_3x_6[x_jx_{q+1}(x_ix_{p+1} - x_{i+1}x_p) + x_{i+1}x_p(x_jx_{q+1} - x_{j+1}x_q)]. \end{aligned}$$

Now let $i = p$. We then have

$$\begin{aligned} S(f,g) &= x_qf - x_jg = -x_qx_2x_3x_6x_{i+1}x_{j+1} + x_jx_2x_3x_6x_{i+1}x_{q+1} \\ &= x_2x_3x_6x_{i+1}(x_jx_{q+1} - x_{j+1}x_q). \end{aligned}$$

The cases of $j = p$ and $j = q$ are similar. $\square$

Now we prove that $\operatorname{depth} K[G] \ge 7$. We denote by $\operatorname{in}(I_G)$ the initial ideal of I_G with respect to $<_{\text{lex}}$. Since

$$\operatorname{depth} K[G] = \operatorname{depth} K[\mathbf{x}]/I_G \ge \operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_G),$$

it is sufficient to prove that $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_G) \ge 7$. By the Auslander–Buchsbaum formula, it is enough to prove the following lemma:

Lemma 2.2.

$$\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/\operatorname{in}(I_G) \le 2k - 1.$$

Proof. First we compute $\operatorname{in}(I_G)$.

The binomials corresponding to type (I) are

$$x_{2(i-1)+7}x_{2(j-1)+8} - x_{2(i-1)+8}x_{2(j-1)+7}, \qquad \text{where } i < j.$$

The initial term of this binomial is $x_{2(i-1)+7}x_{2(j-1)+8}$ $(i < j)$. We denote by I' the ideal generated by these monomials. Note that x_8 and $x_{2(k-1)+7}$ do not appear in the minimal system of monomial generators of I'.

The binomials corresponding to types (II), (III) are

$$x_2x_3x_6x_{2(i-1)+8}x_{2(j-1)+8} - x_1x_4x_5x_{2(i-1)+7}x_{2(j-1)+7}, \qquad \text{where } i \le j.$$

The initial term of this binomial is $-x_1x_4x_5x_{2(i-1)+7}x_{2(j-1)+7}$ $(i \le j)$.

Therefore

$$\begin{aligned} \operatorname{in}(I_G) &= x_1x_4x_5(x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I' \\ &= ((x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I') \cap ((x_1x_4x_5) + I'). \end{aligned}$$

We set

$$\begin{aligned} I_1 &= (x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I', \\ I_2 &= (x_1x_4x_5) + I'. \end{aligned}$$

By the short exact sequence

$$0 \to K[\mathbf{x}]/I_1 \cap I_2 \to K[\mathbf{x}]/I_1 \oplus K[\mathbf{x}]/I_2 \to K[\mathbf{x}]/(I_1 + I_2) \to 0,$$

we have

$$\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/\operatorname{in}(I_G) \leq \max\{\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_1, \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_2, \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/(I_1+I_2)-1\}. \tag{2.1}$$

Now we investigate each of $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_1, \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_2, \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/(I_1+I_2)$. First we consider the ideal I_1. Note that $x_1, \dots, x_6$ and x_8 do not appear in the minimal system of monomial generators of I_1. Let $K[\mathbf{x}']$ be the polynomial ring over K with variables $x_7, x_9, x_{10}, \dots, x_{2(k-1)+8}$. Then $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_1 = \operatorname{pd}_{K[\mathbf{x}']} K[\mathbf{x}']/(I_1 \cap K[\mathbf{x}'])$. By Hilbert's syzygy theorem, we have $\operatorname{pd}_{K[\mathbf{x}']} K[\mathbf{x}']/(I_1 \cap K[\mathbf{x}']) \leq 2k-1$.

Next we consider the ideal $I_2 = (x_1x_4x_5) + I'$. Since the variables x_1, x_4, x_5 do not appear in the minimal system of generators of I', we have

$$\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_2 = \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I' + \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/(x_1x_4x_5) = \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I' + 1.$$

Then similarly to the case of I_1, we have $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I' \leq 2k-2$. Thus we have $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_2 \leq 2k-1$.

Last, we consider the ideal $I_1 + I_2 = (x_1x_4x_5) + I_1$. For the same reason as the case of I_2, we have $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/(I_1+I_2) = \operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/I_1 + 1 \leq 2k$.

Combining these results with (2.1), we have $\operatorname{pd}_{K[\mathbf{x}]} K[\mathbf{x}]/\operatorname{in}(I_G) \leq 2k-1$, as desired. □

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