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HYPERBOLIZING METRIC SPACES

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Dedicated to Fred Gehring on the occasion of his 85th birthday

ABSTRACT. It was proved by M. Bonk, J. Heinonen and P. Koskela that the quasihyperbolic metric hyperbolizes (in the sense of Gromov) uniform metric spaces. In this paper we introduce a new metric that hyperbolizes all locally compact noncomplete metric spaces. The metric is generic in the sense that (1) it can be defined on any metric space; (2) it preserves the quasiconformal geometry of the space; (3) it generalizes the j-metric, the hyperbolic cone metric and the hyperbolic metric of hyperspaces; and (4) it is quasi-isometric to the quasihyperbolic metric of uniform metric spaces. In particular, the Gromov hyperbolicity of these metrics also follows from that of our metric.

1. INTRODUCTION

Suppose that (Z, d) is a locally compact noncomplete metric space. Let $\overline{Z}$ be its metric completion and let $\partial Z = \overline{Z} \setminus Z$. The quantities

$$\frac{1}{\operatorname{dist}(z, \partial Z)} \quad \text{and} \quad \frac{d(x, y)}{\operatorname{dist}(x, \partial Z) \operatorname{dist}(y, \partial Z)}$$

are ubiquitous in geometric function theory. They are used in the definitions of various metrics such as the hyperbolic cone metric (see [2]), the j-metric (see [3]), the quasihyperbolic metric (see [4]) and the hyperbolic metric of hyperspaces (see [7]). The purpose of this paper is to show that the metric u_Z, defined by

$$u_Z(x, y) = 2 \log \frac{d(x, y) + \max\{\operatorname{dist}(x, \partial Z), \operatorname{dist}(y, \partial Z)\}}{\sqrt{\operatorname{dist}(x, \partial Z) \operatorname{dist}(y, \partial Z)}}, \tag{1.1}$$

hyperbolizes the space Z without changing its quasiconformal geometry (see Theorem 2.1) and to show that the metrics mentioned above are quasi-isometric to u_Z which, in particular, implies their Gromov hyperbolicity (see Section 3). The metric u_Z and some of the results of this paper were first announced in [6].

2. HYPERBOLIZATION

Let (Z, d) be an arbitrary metric space. The distance from a point $z \in Z$ to a set $A \subset Z$ is denoted by $\operatorname{dist}(z, A)$. The diameter of a set $A \subset Z$ is denoted by

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diam(A). The *Hausdorff distance* $d_H(A,B)$ between A and B is given by

$$d_H(A,B) = \big[\sup_{a\in A} \operatorname{dist}(a,B)\big] \vee \big[\sup_{b\in B} \operatorname{dist}(b,A)\big].$$

Here and in what follows, we set $r \vee s = \max\{r,s\}$ and $r \wedge s = \min\{r,s\}$.

Given $x,y,z \in Z$, the quantity $(x|y)_z = [d(x,z)+d(y,z)-d(x,y)]/2$ is called the *Gromov product* of x and y with respect to z. The space Z is called *Gromov hyperbolic* if there exists $\delta \ge 0$ such that

$$(x|y)_w \ge (x|z)_w \wedge (z|y)_w - \delta \tag{2.1}$$

for all $x,y,z,w \in Z$. We also say that Z is δ-*hyperbolic* and refer to (2.1) as the δ-*hyperbolicity* condition.

Let (Z',d') be another metric space. A homeomorphism $f\colon Z \to Z'$ is called K-*quasiconformal*, $K \ge 1$, if for each $x \in Z$ we have

$$\limsup_{r\to 0} \frac{\sup\{d'(f(x),f(y))\colon d(x,y) \le r\}}{\inf\{d'(f(x),f(y))\colon d(x,y) \ge r\}} \le K.$$

A map $g\colon Z \to Z'$ is called *quasi-isometric* if there exist $k \ge 0$ and $\lambda \ge 1$ such that $\operatorname{dist}(x', g(Z)) \le k$ for each $x' \in Z'$ and for all $x,y \in Z$ we have

$$\frac{1}{\lambda} d(x,y) - k \le d'(g(x),g(y)) \le \lambda d(x,y) + k.$$

Let M be a nonempty closed proper subset of Z. For convenience we put $d_M(z) = \operatorname{dist}(z,M)$. For $x,y \in Z\setminus M$ we define

$$u_Z(x,y) = 2\log \frac{d(x,y) + d_M(x) \vee d_M(y)}{\sqrt{d_M(x)d_M(y)}}. \tag{2.2}$$

Note that $d_M(x) > 0$ for each $x \in Z\setminus M$. Clearly, $u_Z(x,y) \ge 0$, $u_Z(x,y) = u_Z(y,x)$ and $u_Z(x,y) = 0$ if and only if $x = y$. We have the following two lower bounds for u_Z valid for all $x,y \in Z\setminus M$ (compare to [1, (2.3) and (2.4)]).

$$\Big|\log \frac{d_M(x)}{d_M(y)}\Big| = 2\log \frac{d_M(x)\vee d_M(y)}{\sqrt{d_M(x)d_M(y)}} \le u_Z(x,y) \tag{2.3}$$

and

$$\log\Big(1 + \frac{d(x,y)}{d_M(x)}\Big) \le \log\Big(1 + \frac{d(x,y)}{d_M(x)}\Big)\Big(1 + \frac{d(x,y)}{d_M(y)}\Big) \le u_Z(x,y). \tag{2.4}$$

Observe also that the function $d_M\colon Z \to [0,\infty)$ is continuous. In fact,

$$|d_M(x) - d_M(y)| \le d(x,y) \qquad \text{for all} \qquad x,y \in Z. \tag{2.5}$$

In particular, given $x \in Z\setminus M$, we have

$$u_Z(x,y) \le \log \frac{[2d(x,y) + d_M(x)]^2}{d_M(x)[d_M(x) - d(x,y)]} \tag{2.6}$$

for all $y \in Z\setminus M$ with $d(x,y) < d_M(x)$.

Theorem 2.1. *Let (Z,d) be an arbitrary metric space and let M be a nonempty closed proper subset of Z. Then*

(1) *u_Z is a metric on $Z\setminus M$;*
(2) *the space $(Z\setminus M, u_Z)$ is δ-hyperbolic with $\delta \le \log 4$;*
(3) *the identity map between $(Z\setminus M, d)$ and $(Z\setminus M, u_Z)$ is 5-quasiconformal;*
(4) *if the space (Z,d) is complete, then so is $(Z\setminus M, u_Z)$.*

Proof. To prove (1), we only need to verify the triangle inequality. Given $x, y, z \in Z \setminus M$, it is easy to see that

$$[d_M(x) \vee d_M(z)]d(y,z) + [d_M(y) \vee d_M(z)]d(x,z) \geq d_M(z)d(x,y)$$

and

$$[d_M(x) \vee d_M(z)][d_M(y) \vee d_M(z)] \geq [d_M(x) \vee d_M(y)]d_M(z).$$

Hence

$$[d(x,z)+d_M(x)\vee d_M(z)][d(y,z)+d_M(y)\vee d_M(z)] \geq [d(x,y)+d_M(x)\vee d_M(y)]d_M(z),$$

which implies $u_Z(x,y) \leq u_Z(x,z) + u_Z(y,z)$. Thus, u_Z is a metric.

To prove (2), we show that u_Z satisfies (2.1) with $\delta = \log 4$. Put $\mu(x,y) = d(x,y) + d_M(x) \vee d_M(y)$ and observe that $\mu(x,y) \geq 0$, $\mu(x,y) = \mu(y,x)$ and that μ satisfies the triangle inequality. Let x, y, z, w be arbitrary points in $Z \setminus M$. By [7, Lemma 3.7] we have $\mu(x,y)\mu(z,w) \leq 4\big[\mu(x,z)\mu(y,w) \vee \mu(x,w)\mu(y,z)\big]$. Then

$$\frac{1}{\mu(x,y)\mu(z,w)} \geq \frac{1}{4}\Big[\frac{1}{\mu(x,z)\mu(y,w)} \wedge \frac{1}{\mu(x,w)\mu(y,z)}\Big]$$

or, equivalently,

$$\frac{\mu(x,w)\mu(y,w)}{\mu(x,y)} \geq \frac{1}{4}\Big[\frac{\mu(x,w)\mu(z,w)}{\mu(x,z)} \wedge \frac{\mu(y,w)\mu(z,w)}{\mu(y,z)}\Big].$$

Hence

$$\begin{aligned}(x|y)_w = \log\frac{\mu(x,w)\mu(y,w)}{\mu(x,y)d_M(w)} &\geq \log\frac{\mu(x,w)\mu(z,w)}{\mu(x,z)d_M(w)} \wedge \log\frac{\mu(y,w)\mu(z,w)}{\mu(y,z)d_M(w)} - \log 4\\ &= (x|z)_w \wedge (y|z)_w - \log 4,\end{aligned}$$

as required.

To show (3), we observe that $u_Z(x,y) \to 0$ if and only if $d(x,y) \to 0$, which follows from (2.4) and (2.6). Then both the identity map and its inverse are continuous. Hence the identity map is a homeomorphism. Now fix $x \in Z \setminus M$ and let $r < d_M(x)$. Using (2.4) we obtain

$$\inf_{d(x,y)\geq r} u_Z(x,y) \geq \inf_{d(x,y)\geq r} \log\frac{d(x,y)+d_M(x)}{d_M(x)} \geq \log\frac{r+d_M(x)}{d_M(x)}.$$

Similarly, using (2.6) we obtain

$$\sup_{d(x,y)\leq r} u_Z(x,y) \leq \sup_{d(x,y)\leq r} \log\frac{[2d(x,y)+d_M(x)]^2}{d_M(x)[d_M(x)-d(x,y)]} \leq \log\frac{[2r+d_M(x)]^2}{d_M(x)[d_M(x)-r]}.$$

Using, for instance, L'Hôpital's Rule one can easily show that the quotient of the second logarithmic function to the first tends to 5 as r tends to 0. Thus,

$$\limsup_{r\to 0}\frac{\sup\{u_Z(x,y)\colon d(x,y)\leq r\}}{\inf\{u_Z(x,y)\colon d(x,y)\geq r\}} \leq 5.$$

The proof of (4) is similar to that of [1, Proposition 2.8]. Let $\{x_n\}$ be a Cauchy sequence in $(Z \setminus M, u_Z)$. It follows from (2.3) and (2.4) that

$$0 < r = \inf_n d_M(x_n) \leq \sup_n d_M(x_n) = R < \infty$$

and that

$$d(x_n,x_m) \leq d_M(x_n)(e^{u_Z(x_n,x_m)}-1) \leq R(e^{u_Z(x_n,x_m)}-1).$$

Hence $\{x_n\}$ is a Cauchy sequence in (Z,d). Since (Z,d) is complete, $\{x_n\}$ converges to some point x in Z. Since $d_M(x_n) \geq r > 0$, it follows from (2.5) that $d_M(x) > 0$, i.e., $x \in Z \setminus M$. Since the identity map between $(Z \setminus M, d)$ and $(Z \setminus M, u_Z)$ is continuous, $\{x_n\}$ converges to x in $(Z \setminus M, u_Z)$, as required. □

Observe that the identity map is not, in general, conformal; i.e., given $x \in Z \setminus M$, the limit

$$\lim_{y \to x} \frac{u_Z(x,y)}{d(x,y)} \qquad \text{does not always exist.}$$

Indeed, if y approaches x so that $d_M(y) = d_M(x)$, then

$$\lim_{y \to x} \frac{u_Z(x,y)}{d(x,y)} = \lim_{y \to x} \frac{2}{d(x,y)} \log \frac{d(x,y) + d_M(x)}{d_M(x)} = \frac{2}{d_M(x)};$$

while if y approaches x so that $d_M(y) = d_M(x) + d(x,y)$, then

$$\lim_{y \to x} \frac{u_Z(x,y)}{d(x,y)} = \lim_{y \to x} \frac{2}{d(x,y)} \log \frac{2d(x,y) + d_M(x)}{\sqrt{d_M(x)[d_M(x) + d(x,y)]}} = \frac{3}{d_M(x)}.$$

3. Relations to other δ-hyperbolic metrics

In this section we obtain bounds for the metric u_Z in terms of the j-metric, the quasihyperbolic metric, the hyperbolic cone metric and the hyperbolic metric of hyperspaces, respectively. As a consequence of these bounds we obtain alternative proofs of the δ-hyperbolicity of these metrics.

For proper subdomains of Euclidean spaces, the quasihyperbolic metric and the j-metric were introduced by F. Gehring and B. Palka (see [4]) and F. Gehring and B. Osgood (see [3]), respectively. The δ-hyperbolicity of the j-metric in such domains was proved by P. Hästö (see [5, Theorem 1]). The δ-hyperbolicity of the quasihyperbolic metric in uniform domains in Euclidean spaces was proved by M. Bonk, J. Heinonen and P. Koskela (see [1, Theorem 3.6]), which was extended to uniform domains in Banach spaces by J. Väisälä (see [8, Theorem 2.12]).

3.1. **The j-metric.** Let (Z,d) be an arbitrary metric space and let M be a nonempty closed proper subset of Z. For $x, y \in Z \setminus M$ we define

$$j_Z(x,y) = \frac{1}{2} \log \Big(1 + \frac{d(x,y)}{d_M(x)}\Big)\Big(1 + \frac{d(x,y)}{d_M(y)}\Big). \tag{3.1}$$

Clearly, $j_Z(x,y) = j_Z(y,x)$ and $j_Z(x,y) = 0$ if and only if $x = y$. Since the j-metric has not been considered in this setting before, we will show that j_Z is indeed a metric. That is, we verify that j_Z satisfies the triangle inequality. Let $x, y, z \in Z \setminus M$ be arbitrary points. It follows from (2.5) that

$$d_M(z)\big(d(x,z) + d(y,z)\big) \leq d_M(x)d(y,z) + d_M(z)d(x,z) + d(x,z)d(y,z).$$

In particular,

$$d_M(z)(d_M(x) + d(x,y)) \leq \big(d_M(x) + d(x,z)\big)\big(d_M(z) + d(y,z)\big).$$

Dividing both sides by $d_M(x)d_M(z)$ we obtain

$$\Big(1 + \frac{d(x,y)}{d_M(x)}\Big) \leq \Big(1 + \frac{d(x,z)}{d_M(x)}\Big)\Big(1 + \frac{d(y,z)}{d_M(z)}\Big).$$

Similarly, we obtain

$$\Big(1+\frac{d(x,y)}{d_M(y)}\Big) \le \Big(1+\frac{d(x,z)}{d_M(z)}\Big)\Big(1+\frac{d(y,z)}{d_M(y)}\Big).$$

Combining the last two inequalities we obtain $j_Z(x,y) \le j_Z(x,z) + j_Z(z,y)$.

Theorem 3.1. *For all $x, y \in Z \setminus M$ we have*

$$2j_Z(x,y) \le u_Z(x,y) \le 2j_Z(x,y) + 2\log 2. \tag{3.2}$$

In particular, the space $(Z \setminus M, j_Z)$ is δ-hyperbolic with $\delta \le \frac{5}{2}\log 2$.

Proof. It follows from (2.5) that $d_M(x) \vee d_M(y) \le d_M(x) \wedge d_M(y) + d(x,y)$ for all $x, y \in Z \setminus M$. Hence

$$[d(x,y) + d_M(x) \vee d_M(y)]^2 \le 4[d_M(x) + d(x,y)][d_M(y) + d(x,y)].$$

In particular,

$$u_Z(x,y) \le \log\Big(1+\frac{d(x,y)}{d_M(x)}\Big)\Big(1+\frac{d(x,y)}{d_M(y)}\Big) + 2\log 2. \tag{3.3}$$

Combining (2.4) and (3.3), we obtain (3.2). The latter in combination with Theorem 2.1 implies the second part. □

3.2. **The quasihyperbolic metric.** Let (Z,d) be a locally compact rectifiably connected noncomplete metric space and let $\partial Z = \overline{Z} \setminus Z$, where $\overline{Z}$ is the metric completion of Z. The quasihyperbolic metric is defined by

$$k_Z(x,y) = \inf_\gamma \int_\gamma \frac{|dz|}{\operatorname{dist}(z,\partial Z)},$$

where the infimum is taken over all rectifiable curves γ joining the points x and y in Z. In this setting the quasihyperbolic metric was studied by M. Bonk, J. Heinonen and P. Koskela (see [1]). Recall that the space Z is called *A-uniform* ($A \ge 1$) if every pair of points in Z can be joined by a curve $\gamma\colon [0,1] \to Z$ such that $\operatorname{length}(\gamma) \le Ad(\gamma(0),\gamma(1))$ and

$$\operatorname{length}(\gamma|[0,t]) \wedge \operatorname{length}(\gamma|[t,1]) \le A \operatorname{dist}(\gamma(t), \partial Z)$$

for all $t \in [0,1]$ (see [1, Definition 1.9]). It follows from [1, (2.4) and (2.16)] that

$$j_Z(x,y) \le k_Z(x,y) \le 4A^2 j_Z(x,y) \tag{3.4}$$

for all $x, y \in Z$. Note that the second inequality holds under the assumption that Z is A-uniform. In particular, the space (Z, k_Z) is proper, geodesic and δ-hyperbolic provided Z is uniform (see [1, Theorem 3.6]).

Now if Z is A-uniform, then it follows from (3.2) and (3.4) that

$$\frac{1}{2A^2} k_Z(x,y) \le u_Z(x,y) \le 2k_Z(x,y) + 2\log 2 \tag{3.5}$$

for all $x, y \in Z$. (Here the metric u_Z is as in (1.1).) Hence both the identity map $\operatorname{id}_Z\colon (Z,k_Z) \to (Z,u_Z)$ and its inverse are quasi-isometric. In particular, as (Z,k_Z) is geodesic and quasi-isometric to (Z,u_Z), an argument due to M. Bonk gives an alternative proof of the δ-hyperbolicity of k_Z (see [5, Lemma 4]).

3.3. **The hyperbolic cone metric.** Let (X,d) be a bounded metric space and let $\mathrm{Con}(X) = X \times (0, \mathrm{diam}(X)]$. The hyperbolic cone metric is defined by

$$\rho\big((x,r),(y,s)\big) = 2\log\frac{d(x,y)+r\vee s}{\sqrt{rs}}.$$

The space $(\mathrm{Con}(X),\rho)$ is δ-hyperbolic for some δ (see [2, Theorem 7.2]). Our computations imply that $\delta \le 5\log 2$ (see Theorem 3.2).

We consider the space $Z = X \times [0, \mathrm{diam}(X)]$ equipped with the metric d',

$$d'\big((x,r),(y,s)\big) = d(x,y) + |r-s|,$$

and let $M = X \times \{0\}$ so that $\mathrm{Con}(X) = Z \setminus M$. Observe that for each $(x,r) \in \mathrm{Con}(X)$, we have $\inf\{d'\big((x,r),(y,0)\big)\colon (y,0)\in M\} = \inf_{y\in X}[d(x,y)+r] = r$. Applied to the space (Z,d') and the subset $M \subset Z$, the metric u_Z takes the form

$$u_Z\big((x,r),(y,s)\big) = 2\log\frac{d(x,y)+|r-s|+r\vee s}{\sqrt{rs}}.$$

Since $|r-s| \le r\vee s$, we obtain the following result.

Theorem 3.2. *For all $x,y \in Z\setminus M$ we have*

$$\rho(x,y) \le u_Z(x,y) \le \rho(x,y) + 2\log 2.$$

In particular, the space $(\mathrm{Con}(X),\rho)$ is δ-hyperbolic with $\delta \le 5\log 2$.

3.4. **The hyperbolic metric of hyperspaces.** We recall the hyperbolization of hyperspaces from [7]. Let (X,d) be an arbitrary metric space and let $\mathcal{H}(X)$ be the hyperspace of all nondegenerate closed bounded subsets of X equipped with the metric $d_{\mathcal{H}}$,

$$d_{\mathcal{H}}(A,B) = 2\log\frac{d_H(A,B)+\mathrm{diam}(A)\vee\mathrm{diam}(B)}{\sqrt{\mathrm{diam}(A)\,\mathrm{diam}(B)}}.$$

The space $(\mathcal{H}(X), d_{\mathcal{H}})$ is δ-hyperbolic with $\delta \le 2\log 2$ (see [7, Theorem 4.7]).

Now let Z be the set of all nonempty closed bounded subsets of X endowed with the Hausdorff metric d_H. Let $M = Z \setminus \mathcal{H}(X)$ so that $\mathcal{H}(X) = Z\setminus M$. Observe that for any $A \in \mathcal{H}(X)$ we have $d_M(A) = \inf\{d_H(A,\{x\})\colon x\in X\}$ and $\mathrm{diam}(A)/2 \le d_M(A) \le \mathrm{diam}(A)$. Applied to the space (Z,d_H) and the subset $M\subset Z$, the metric u_Z takes the form

$$u_Z(A,B) = 2\log\frac{d_H(A,B)+d_M(A)\vee d_M(B)}{\sqrt{d_M(A)d_M(B)}}.$$

Consequently, we obtain the following result.

Theorem 3.3. *For all $A,B\in\mathcal{H}(X)$ we have*

$$d_{\mathcal{H}}(A,B) - 2\log 2 \le u_Z(A,B) \le d_{\mathcal{H}}(A,B) + 2\log 2.$$

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