

ON MAXIMAL RANGES OF VECTOR MEASURES FOR SUBSETS AND PURIFICATION OF TRANSITION PROBABILITIES

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ABSTRACT. Consider a measurable space with an atomless finite vector measure. This measure defines a mapping of the σ -field into a Euclidean space. According to the Lyapunov convexity theorem, the range of this mapping is a convex compactum. Similar ranges are also defined for measurable subsets of the space. Two subsets with the same vector measure may have different ranges. We investigate the question whether, among all the subsets having the same given vector measure, there always exists a set with the maximal range of the vector measure. The answer to this question is positive for two-dimensional vector measures and negative for higher dimensions. We use the existence of maximal ranges to strengthen the Dvoretzky-Wald-Wolfowitz purification theorem for the case of two measures.

1. INTRODUCTION

Let (X, \mathcal{F}) be a measurable space and $\mu = (\mu_1, \dots, \mu_m)$, $m = 1, 2, \dots$, be a finite atomless vector measure on it. We recall that a measure ν is called atomless if for each $Z \in \mathcal{F}$, such that $\nu(Z) > 0$, there exists $Z' \in \mathcal{F}$ such that $Z' \subset Z$ and $0 < \nu(Z') < \nu(Z)$. A vector measure $\mu = (\mu_1, \dots, \mu_m)$, is called finite and atomless if each measure μ_i , $i = 1, \dots, m$, is finite and atomless. For each $Y \in \mathcal{F}$ consider the range $R_\mu(Y) = \{\mu(Z) : Z \in \mathcal{F}, Z \subseteq Y\}$ of the vector measures of all its measurable subsets Z . According to the Lyapunov convexity theorem [8], the sets $R_\mu(Y)$ are convex compactums in \mathbb{R}^m . For a review on this theorem and its applications, see [9].

In this paper we study whether for any $p \in R_\mu(X)$, the set of all ranges $\{R_\mu(Y) : \mu(Y) = p, Y \in \mathcal{F}\}$ contains a maximal element. In other words, is it always true that for any $p \in R_\mu(X)$ there exists a subset $Y^* \in \mathcal{F}$ such that $\mu(Y^*) = p$ and $R_\mu(Y^*) \supseteq R_\mu(Y)$ for any $Y \in \mathcal{F}$ with $\mu(Y) = p$? We show that the answer is positive when $m = 2$ and negative when $m > 2$. Furthermore, for $m = 2$, this maximal range can be constructed by a simple geometric transformation of $R_\mu(X)$.

In addition to the maximal range, it is possible to consider a minimal range. For $q \in R_\mu(X)$, the set $M^* \in \mathcal{F}$ with $\mu(M^*) = q$ has minimal range corresponding to

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q if $R_\mu(M) \supseteq R_\mu(M^*)$ for any $M \in \mathcal{F}$ with $\mu(M) = q$. We show that a set has a maximal range corresponding to p if and only if its complement has a minimal range corresponding to $\mu(X) - p$. Therefore, minimal ranges also exist for dimension two and they may not exist for higher dimensions.

Lyapunov's theorem is relevant to the purification of transition probabilities discovered by Dvoretzky, Wald and Wolfowitz [2, 3] for a finite image set. Let (A, \mathcal{A}) be a measurable space and π be a transition probability from X to A ; that is, $\pi(B|x)$ is a measurable function on (X, \mathcal{F}) for any $B \in \mathcal{A}$ and $\pi(\cdot|x)$ is a probability measure on (A, \mathcal{A}) for any $x \in X$. According to Dvoretzky, Wald and Wolfowitz [2, 3], two transition probabilities π_1 and π_2 are called strongly equivalent if

$$(1.1) \quad \int_X \pi_1(B|x) \mu_i(dx) = \int_X \pi_2(B|x) \mu_i(dx), \quad i = 1, \dots, m, \quad B \in \mathcal{A}.$$

A transition probability π is called pure if each measure $\pi(\cdot|x)$ is concentrated at one point. A pure transition probability π is defined by a measurable mapping $\varphi : X \rightarrow A$ such that $\pi(B|x) = I\{\varphi(x) \in B\}$ for all $B \in \mathcal{A}$. According to the contemporary terminology, a transition probability can be purified if for it there exists a strongly equivalent pure transition probability.

For a finite set A , Dvoretzky, Wald and Wolfowitz [2, 3] proved that any transition probability can be purified (we recall that μ is atomless). Edwards [4, Theorem 4.5] generalized this result to the case of a countable set A . Khan and Rath [6, Theorem 2] gave another proof of this generalization. Loeb and Sun [7, Example 2.7] constructed an elegant example when a transition probability cannot be purified for $m = 2$, $X = [0, 1]$, and $A = [-1, 1]$. However, purification holds for a countable set of nonatomic, finite, signed Loeb measures when A is a complete separable metric space [7, Corollary 2.6].

Note that for a countable (finite or infinite) set A , a transition probability π can be purified if and only if there exists a partition $\{Z^a \in \mathcal{F} : a \in A\}$ of X such that

$$(1.2) \quad \int_X \pi(a|x) \mu(dx) = \mu(Z^a), \quad a \in A,$$

where μ is an m -dimensional finite atomless vector measure. Since $\int_X \pi(a|x) \mu(dx)$ are vectors in \mathbb{R}^m , a natural question is: under what conditions for an arbitrary set of vectors $\{p^a : a \in A\}$ does there exist a partition $\{Z^a \in \mathcal{F} : a \in A\}$ of X such that $p^a = \mu(Z^a)$ for each $a \in A$? We use the theorem on maximal ranges proved in this paper to show that for $m = 2$, such a partition exists if and only if (i) $\sum_{a \in A} p^a = \mu(X)$, and (ii) $\sum_{a \in B} p^a \in R_\mu(X)$ for any finite subset B of A . For $m = 2$, the Dvoretzky-Wald-Wolfowitz theorem for a countable set A [4, 6] follows from the sufficient part of this statement.

We formulate the main results in the following section, prove the existence of maximal and minimal subsets for $m = 2$ in Section 3, provide counterexamples when $m > 2$ in Section 4, and describe geometric constructions of maximal ranges in Section 5. Section 6 is devoted to the proof of the theorem on the existence of a partition.

2. MAIN RESULTS

Definition 2.1. Given a measurable subset Y of the measurable space (X, \mathcal{F}) with a vector measure μ and a vector $p \in R_\mu(Y)$, we define

(a) the set of all subsets of Y with vector measure p ,

$$\mathcal{S}_\mu^p(Y) = \{Z \in \mathcal{F}_Y : \mu(Z) = p\},$$

where $\mathcal{F}_Y = \{Z \subseteq Y : Z \in \mathcal{F}\}$;

(b) the union of all the ranges of all subsets of Y with the vector measure p ,

$$R_\mu^p(Y) = \bigcup_{Z \in \mathcal{S}_\mu^p(Y)} R_\mu(Z);$$

(c) the intersection of the $R_\mu(Y)$ with its shift by a vector $-(\mu(Y) - p)$,

$$Q_\mu^p(Y) = (R_\mu(Y) - \{\mu(Y) - p\}) \cap R_\mu(Y),$$

where $S_1 - S_2 = \{q - r : q \in S_1, r \in S_2\}$ for $S_1, S_2 \in \mathbb{R}^m$. In particular, $R_\mu(Y) - \{r\}$ is a parallel shift of $R_\mu(Y)$ by $-r$.

Definition 2.2. For a measurable subset $Y \in \mathcal{F}$, the set $Z^* \in \mathcal{S}_\mu^p(Y)$ such that $R_\mu(Z^*) = R_\mu^p(Y)$ is called the maximal subset of Y with the measure p . The set $M^* \in \mathcal{S}_\mu^q(Y)$ such that $R_\mu(M^*) \subseteq R_\mu(M)$ for any $M \in \mathcal{S}_\mu^q(Y)$ is called the minimal subset of Y with the measure q .

Our first result is the following theorem.

Theorem 2.3. For a two-dimensional finite atomless vector measure $\mu = (\mu_1, \mu_2)$ and for a vector $p \in R_\mu(X)$, there exists a maximal set $Z^* \in \mathcal{S}_\mu^p(X)$ and, in addition, $R_\mu^p(X) = Q_\mu^p(X)$.

Theorem 2.3 immediately implies that the set $R_\mu^p(X)$, which is the union of the ranges of μ on Z , for all $Z \in \mathcal{S}_\mu^p(X)$, is a convex compactum. Furthermore, if $R_\mu(X)$ and p are given, the set $R_\mu^p(X)$ is defined by two simple geometric operations, a shift and an intersection, since $Q_\mu^p(X)$ is defined by these operations.

The following theorem links the notions of maximal and minimal subsets.

Theorem 2.4. The set Z^* is the maximal subset of X with the measure p if and only if $M^* = X \setminus Z^*$ is the minimal subset of X with the measure $\mu(X) - p$.

We will use Theorem 2.3 to prove the following theorem that, as shown in Section 6, strengthens the Dvoretzky-Wald-Wolfowitz purification theorem [4, 6] for the case $m = 2$.

Theorem 2.5. Consider a measurable space (X, \mathcal{F}) with a two-dimensional finite atomless vector measure μ , a countable set A , and a set of two-dimensional vectors $\{p^a : a \in A\}$. A partition $\{Z^a \in \mathcal{F} : a \in A\}$ of X , with $p^a = \mu(Z^a)$ for all $a \in A$, exists if and only if (i) $\sum_{a \in A} p^a = \mu(X)$ and (ii) $\sum_{a \in B} p^a \in R_\mu(X)$ for any finite subset $B \subset A$.

3. MAXIMAL AND MINIMAL SUBSETS

In this section, we prove Theorems 2.3 and 2.4. Recall that for a set $S \subseteq \mathbb{R}^m$, its reflection across a point $c \in \mathbb{R}^m$ is $\text{Ref}(S, c) = \{2c\} - S$. If $S = \{s\}$ is a singleton, we shall write $\text{Ref}(s, c)$ instead of $\text{Ref}(\{s\}, c)$. A set $S \subseteq \mathbb{R}^m$ is called centrally symmetric if $\text{Ref}(S, c) = S$ for some point $c \in \mathbb{R}^m$ called the center of S . Any bounded centrally symmetric set has only one center.

In this section we let $Y \in \mathcal{F}$ be any measurable subset of X . Lemmas 3.1-3.3 hold for any finite atomless vector measure $\mu = (\mu_1, \dots, \mu_m)$ on (X, \mathcal{F}) , where $m = 1, 2, \dots$.

Lemma 3.1. *The set $R_\mu(Y)$ is centrally symmetric with the center $\frac{1}{2}\mu(Y)$.*

Proof. The proof is straightforward, and this fact was observed by Lyapunov [8, p. 476]. \square

Lemma 3.2. *The equality $R_\mu(Y) - \{\mu(Y) - p\} = \text{Ref}(R_\mu(Y), \frac{1}{2}p)$ holds for any $p \in R_\mu(Y)$.*

Proof. By Lemma 3.1, $R_\mu(Y) = \text{Ref}(R_\mu(Y), \frac{1}{2}\mu(Y)) = \{\mu(Y)\} - R_\mu(Y)$. Therefore, $R_\mu(Y) - \{\mu(Y) - p\} = (\{\mu(Y)\} - R_\mu(Y)) - \{\mu(Y) - p\} = \{p\} - R_\mu(Y) = \text{Ref}(R_\mu(Y), \frac{1}{2}p)$. \square

Lemma 3.3. *Each of the sets $R_\mu^p(Y)$ and $Q_\mu^p(Y)$ is centrally symmetric with the center $\frac{1}{2}p$.*

Proof. According to Lemma 3.1, each set $Z \in \mathcal{S}_\mu^p(Y)$ is centrally symmetric with the center $\frac{1}{2}p$. Therefore, $R_\mu^p(Y)$, which is the union of all the sets in $Z \in \mathcal{S}_\mu^p(Y)$, is also centrally symmetric with the center $\frac{1}{2}p$.

In addition,

$$\begin{aligned} \text{Ref}\left(Q_\mu^p(Y), \frac{1}{2}p\right) &= \text{Ref}\left((R_\mu(Y) - \{\mu(Y) - p\}) \cap R_\mu(Y), \frac{1}{2}p\right) \\ &= \text{Ref}\left(\text{Ref}\left(R_\mu(Y), \frac{1}{2}p\right) \cap R_\mu(Y), \frac{1}{2}p\right) \\ &= R_\mu(Y) \cap \text{Ref}\left(R_\mu(Y), \frac{1}{2}p\right) \\ &= R_\mu(Y) \cap (R_\mu(Y) - \{\mu(Y) - p\}) = Q_\mu^p(Y), \end{aligned}$$

where the first and last equalities follow from the definition of Q_μ^p , and the second and second to last equalities follow from Lemma 3.2. The third equality holds because a reflection of intersections equals the intersection of reflections and, in addition, a reflection of a reflection across the same point is the original set. \square

Here we present the major ideas of the proof of Theorem 2.3. First, as shown later, after Theorem 2.3 is proven for equivalent measures μ_1 and μ_2 , this condition can be removed. So, we make the following assumption in Lemmas 3.5, 3.7-3.13.

Assumption 3.4. The measures μ_1 and μ_2 are finite, atomless, and equivalent.

Under Assumption 3.4, let $f(x) = \frac{d\mu_2}{d\mu_1}(x)$ be a Radon-Nikodým derivative of μ_2 with respect to μ_1 . Since f is defined μ_1 -a.e., we fix any of its versions. We shall frequently use notation similar to

$$\{f(x) < l\} = \{x \in X : f(x) < l\}.$$

Second, under Assumption 3.4, for any $a \in [0, \mu_1(X)]$, we denote

$$(3.1) \quad l_a = \min \{l \geq 0 : \mu_1(\{f(x) \leq l\}) \geq a\}.$$

Observe that the minimum in (3.1) exists. Indeed, let

$$l_a = \inf \{l \geq 0 : \mu_1(\{f(x) \leq l\}) \geq a\}.$$

We need to show that $\mu_1(\{f(x) \leq l_a\}) \geq a$. If $l_a = \infty$, then $\mu_1(\{f(x) \leq \infty\}) = \mu_1(X) \geq a$. Let $l_a < \infty$. Consider a sequence $l^k \searrow l_a$, $k = 1, 2, \dots$. Then

$\bigcap_{k=1}^{\infty} \{f(x) \leq l^k\} = \{f(x) \leq l_a\}$ and $\{f(x) \leq l^k\} \supseteq \{f(x) \leq l^{k+1}\}$. Therefore $\mu_1(\{f(x) \leq l_a\}) = \lim_{k \rightarrow \infty} \mu_1(\{f(x) \leq l^k\}) \geq a$.

Third, it is possible to construct the maximal set $Z^* = X \setminus M^*$, where M^* can be defined explicitly. Let $X^l = \{f(x) = l\}$. If $\mu_1(X^l) = 0$ for all $l \in [0, \infty)$, then the definition of M^* is easier and we explain it first. In this case, there exists $a^* \in [0, \mu_1(X)]$ such that $\mu_2(M^*) = \mu_2(X) - p_2$, and M^* can be defined as

$$(3.2) \quad M^* = \{l_{a^*} \leq y < l_{a^* + (\mu_1(X) - p_1)}\}.$$

In the general situation, the number a^* can be chosen to satisfy

$$\mu_2(\{l_{a^*} < y < l_{a^* + (\mu_1(X) - p_1)}\}) \leq \mu_2(X) - p_2 \leq \mu_2(\{l_{a^*} \leq y \leq l_{a^* + (\mu_1(X) - p_1)}\})$$

and

$$(3.3) \quad M^* = \{l_{a^*} < y < l_{a^* + (\mu_1(X) - p_1)}\} \cup Z^1 \cup Z^2,$$

for Z^i , $i = 1, 2$, being some measurable subsets of X^{l^i} , where $l^1 = l_{a^*}$ and $l^2 = l_{a^* + (\mu_1(X) - p_1)}$. In particular, if $\mu_1(X^{l^1}) = 0$, let $Z^1 = X^{l^1}$, and if $\mu_1(X^{l^2}) = 0$, let $Z^2 = \emptyset$. If $\mu_1(X^{l^1}) = \mu_1(X^{l^2}) = 0$, then (3.3) reduces to (3.2). It is easy to show that the number of l such that $\mu_1(X^l) = 0$ is countable, but we do not use this fact.

The proof of Theorem 2.3 is based on several lemmas.

Lemma 3.5. *Under Assumption 3.4, the numbers l_a , $a \in [0, \mu_1(X)]$ have the following properties: (a) $\mu_1(\{f(x) < l_a\}) \leq a \leq \mu_1(\{f(x) \leq l_a\})$; (b) $l_a \leq l_{a'}$ if $a \leq a'$.*

Proof. For (a), by definition, $a \leq \mu_1(\{f(x) \leq l_a\})$. To prove that $\mu_1(\{f(x) < l_a\}) \leq a$, assume that $\mu_1(\{f(x) < l_a\}) > a$. If $l_a = 0$, then $\mu_1(\{f(x) < l_a\}) = 0 > a$, which contradicts the assumption that $a \geq 0$. If $l_a > 0$, let $\epsilon_k \searrow 0$, $k = 1, 2, \dots$, be a sequence of positive numbers. Then, for $k = 1, 2, \dots$,

$$\mu_1(\{f(x) < l_a\}) = \mu_1(\{f(x) \leq l_a - \epsilon_k\}) + \mu_1(\{l_a - \epsilon_k < f(x) < l_a\}) > a.$$

Let $D_k = \{l_a - \epsilon_k < f(x) < l_a\}$. We observe that $D_{k+1} \subseteq D_k$ and $\bigcap_{u=1}^{\infty} D_k = \emptyset$. Therefore, $\lim_{k \rightarrow \infty} \mu_1(D_k) = 0$. Thus, $\mu_1(\{f(x) \leq l_a - \epsilon\}) > a$ for some $\epsilon > 0$ and this contradicts (3.1). These contradictions imply the lemma.

For (b), assume $l_a > l_{a'}$. Then $\mu_1(\{f(x) \leq l_{a'}\}) \geq a' \geq a$, and this contradicts (3.1). \square

Note that for each $l \in [0, \infty)$, there exists a subfamily

$$\{W_b(X^l) \in \mathcal{F}_{X^l} : b \in [0, \mu_1(X^l)]\}$$

such that $W_b(X^l) \subset W_{b'}(X^l) \subseteq X^l$ whenever $b < b' \leq \mu_1(X^l)$ and $\mu_1(W_b(X^l)) = b$ for each $b \in [0, \mu_1(X^l)]$. This fact follows from Ross [10, Theorem 2(LT3)]. We set $W_0(X^l) = \emptyset$. From now on we fix a family of $W_b(X^l)$ for each $l \in [0, \infty)$.

Definition 3.6. Under Assumption 3.4, for each a , define the following set:

$$(3.4) \quad L_a = \{f(x) < l_a\} \cup W_c(X^{l_a}),$$

where $c = a - \mu_1(\{f(x) < l_a\})$.

Note that property (a) in Lemma 3.5 guarantees that $c \in [0, \mu_1(X^l)]$.

Lemma 3.7. *Under Assumption 3.4, the sets $L_a \in \mathcal{F}$, $a \in [0, \mu_1(X)]$, have the following properties: (a) $\mu_1(L_a) = a$; (b) $\{f(x) < l_a\} \subseteq L_a \subseteq \{f(x) \leq l_a\}$; (c) $L_a \subset L_{a'} \subseteq X$ if $a < a' \leq \mu_1(X)$.*

Proof. For (a), $\mu_1(L_a) = \mu_1(\{f(x) < l_a\}) + \mu_1(W_c(X^{l_a})) = \mu_1(\{f(x) < l_a\}) + a - \mu_1(\{f(x) < l_a\}) = a$. Property (b) follows from $W_c(X^{l_a}) \subseteq X^{l_a}$ and (3.4). For (c), if $l_a = l_{a'}$, then $c < c'$, where $c' = a' - \mu_1(\{f(x) < l_a\})$, and thus

$$L_a = \{f(x) < l_a\} \cup W_c(X^{l_a}) \subset \{f(x) < l_a\} \cup W_{c'}(X^{l_a}) = L_{a'}.$$

If $l_a < l_{a'}$, then $L_a \subseteq \{f(x) \leq l_a\} \subset \{f(x) < l_{a'}\} \subseteq L_{a'}$. □

Let $M_{a,d} = L_{a+d} \setminus L_a$. For each $d \in [0, \mu_1(X)]$ and each $a \in [0, \mu_1(X) - d]$, denote $g_d(a) = \mu_2(M_{a,d}) = \int_{M_{a,d}} f(x) \mu_1(dx)$. The function $g_d(a)$ is nondecreasing and continuous in $a \in [0, \mu_1(X) - d]$ for each $d \in [0, \mu_1(X)]$. However, we will not use the fact that it is nondecreasing. So we only prove the continuity in the following lemma.

Lemma 3.8. *Under Assumption 3.4, $g_d(a)$ is continuous in $a \in [0, \mu_1(X) - d]$ for each $d \in [0, \mu_1(X)]$.*

Proof. We show that $\mu_2(L_{a+d})$ is continuous in $a \in ([0, \mu_1(X) - d]$ for any $d \in [0, \mu_1(X)]$. Since $g_d(a) = L_{a+d} - L_a$, this implies the lemma. Consider a sequence $\{a^k : k = 1, 2, \dots\}$, where $a_k \in [0, \mu_1(X) - d]$. Let $a^k \nearrow a$. Then $L_{a^k+d} \subset L_{a^{k+1}+d} \subset B \subseteq L_{a+d}$, where $B = \bigcup_{k=1}^\infty L_{a^k+d}$. Therefore, $\mu_i(L_{a^k+d}) \nearrow \mu_i(B)$ and $\mu_i(L_{a+d}) = \mu_i(B) + \mu_i(L_{a+d} \setminus B)$, $i = 1, 2$. Since $\mu_1(L_{a^k+d}) = a^k + d \nearrow a + d = \mu_1(L_{a+d})$, we have $\mu_1(B) = a + d$ and $\mu_1(L_{a+d} \setminus B) = 0$. Since μ_1 and μ_2 are equivalent measures, $\mu_2(L_{a+d} \setminus B) = 0$ and $\mu_2(L_{a^k+d}) \nearrow \mu_2(L_{a+d})$.

Now let $a^k \searrow a$. Then $L_{a^k+d} \supset L_{a^{k+1}+d} \supset D \supseteq L_{a+d}$, where $D = \bigcap_{k=1}^\infty L_{a^k+d}$, and $\mu_i(L_{a^k+d}) \searrow \mu_i(D)$, $\mu_i(L_{a+d}) = \mu_i(D) - \mu_i(D \setminus L_{a+d})$ for $i = 1, 2$. Similar to the previous case, $\mu_1(L_{a^k+d}) = a^k + d \searrow a + d = \mu_1(L_{a+d})$, so $\mu_1(D) = a + d$, $\mu_2(D \setminus L_{a+d}) = \mu_1(D \setminus L_{a+d}) = 0$, and $\mu_2(D) = \mu_2(L_{a+d})$. Thus, $\mu_2(L_{a^k+d}) \searrow \mu_2(L_{a+d})$. □

Observe that a point $q \in \mathbb{R}^2$ is on the upper (lower) boundary of $R_\mu(X)$ if and only if $q \in R_\mu(X)$ and $q'_2 \leq q_2$ ($q'_2 \geq q_2$) for any $q' \in R_\mu(X)$ with $q'_1 = q_1$.

Lemma 3.9. *Under Assumption 3.4, a point $q \in \mathbb{R}^2$ is on the lower boundary of $R_\mu(X)$ if and only if $0 \leq q_1 \leq \mu_1(X)$ and $q_2 = \mu_2(L_{q_1})$, and it is on the upper boundary of $R_\mu(X)$ if and only if $0 \leq q_1 \leq \mu_1(X)$ and $q_2 = \mu_2(X \setminus L_{\mu_1(X)-q_1})$.*

Proof. For the lower boundary, let $q_2 = \mu_2(L_{q_1})$. Since $q_1 = \mu_1(L_{q_1})$, we have $q = \mu(L_q) \in R_\mu(X)$. For any set $Z \in \mathcal{F}$ with $\mu_1(Z) = q_1$, define disjoint sets $Z_1 = Z \setminus L_{q_1}$, $Z_2 = L_{q_1} \setminus Z$, and $M = Z \cap L_{q_1}$. Then $Z = Z_1 \cup M$, $L_{q_1} = Z_2 \cup M$, and $\mu_1(Z_1) = \mu_1(Z_2)$, since $\mu_1(Z) = q_1 = \mu_1(L_{q_1})$. Furthermore, $Z_1 \subseteq \{f(x) \geq l_{q_1}\}$ and $Z_2 \subseteq \{f(x) \leq l_{q_1}\}$. Therefore,

$$\begin{aligned} \mu_2(Z_1) &= \int_{Z_1} f(x) \mu_1(dx) \geq l_{q_1} \int_{Z_1} \mu_1(dx) \\ &= l_{q_1} \int_{Z_2} \mu_1(dx) \geq \int_{Z_2} f(x) \mu_1(dx) = \mu_2(Z_2). \end{aligned}$$

So $\mu_2(Z) = \mu_2(Z_1) + \mu_2(M) \geq \mu_2(Z_2) + \mu_2(M) = \mu_2(L_{q_1})$, and thus q is on the lower boundary of $R_\mu(X)$.

If q is on the lower boundary of $R_\mu(X)$, then $q_2 \leq \mu_2(L_{q_1})$. Since $q \in R_\mu(X)$, there exists $Z \in \mathcal{F}$ with $\mu(Z) = q$. But, as proved above, $\mu_2(Z) \geq \mu_2(L_{q_1})$ for any $Z \in \mathcal{F}$ with $\mu_1(Z) = q_1$. Thus $q_2 \geq \mu_2(L_{q_1})$. Therefore, $q_2 = \mu_2(L_{q_1})$.

The statement on the upper boundary follows from the symmetry of $R_\mu(X)$. \square

Lemma 3.10. *Under Assumption 3.4, given $u = (u_1, u_2) \in R_\mu(X)$, there exists $a^* \in [0, \mu_1(X) - u_1]$ such that $\mu(M_{a^*, u_1}) = u$.*

Proof. Since $\mu_1(L_0) = 0$ and μ_1 and μ_2 are equivalent, $\mu_2(L_0) = 0$. Therefore, $g_{u_1}(0) = \mu_2(L_{u_1} \setminus L_0) = \mu_2(L_{u_1}) - \mu_2(L_0) = \mu_2(L_{u_1})$. Similarly, $\mu_2(X \setminus L_{\mu_1(X)}) = 0$, because $\mu_1(X \setminus L_{\mu_1(X)}) = \mu_1(X) - \mu_1(L_{\mu_1(X)}) = 0$. Thus

$$\begin{aligned} g_{u_1}(\mu_1(X) - u_1) &= \mu_2(L_{\mu_1(X)} \setminus L_{\mu_1(X) - u_1}) = \mu_2(L_{\mu_1(X)}) - \mu_2(L_{\mu_1(X) - u_1}) \\ &= \mu_2(X) - \mu_2(X \setminus L_{\mu_1(X)}) - \mu_2(L_{\mu_1(X) - u_1}) \\ &= \mu_2(X \setminus L_{\mu_1(X) - u_1}). \end{aligned}$$

According to Lemma 3.9, the point $(u_1, g_{u_1}(0))$ is on the lower boundary of the range $R_\mu(X)$ and the point $(u_1, g_{u_1}(\mu_1(X) - u_1))$ is on the upper boundary of the range $R_\mu(X)$. So $u_2 \in [g_{u_1}(0), g_{u_1}(\mu_1(X) - u_1)]$. Since $g_{u_1}(a)$ is continuous in $a \in [0, \mu_1(X) - u_1]$, there exists a^* , such that $g_{u_1}(a^*) = u_2$. That is, $\mu_2(M_{a^*, u_1}) = u_2$. By definition, $\mu_1(M_{a^*, u_1}) = u_1$. Therefore, $\mu(M_{a^*, u_1}) = u$. \square

Note that Lemmas 3.5 and 3.7-3.10 hold if one replaces everywhere the set X with any measurable subset $Z \in \mathcal{F}$. In particular, expressions such as $\{f(x) < l\}$ should be replaced with $\{x \in Z : f(x) < l\}$. We define explicitly

$$(3.5) \quad l_a(Z) = \min \{l \geq 0 : \mu_1(\{x \in Z : f(x) \leq l\}) \geq a\}.$$

Let $Z^l = \{x \in Z : f(x) = l\}$. As follows from Ross [10, Theorem 2(LT3)], for each $l \in [0, \infty)$, there exists a family

$$\{W_b(Z^l) \in \mathcal{F}_{Z^l} : b \in [0, \mu_1(Z^l)]\}$$

such that $W_b(Z^l) \subset W_{b'}(Z^l) \subseteq Z^l$ whenever $b < b' \leq \mu_1(Z^l)$ and $\mu_1(W_b(Z^l)) = b$ for each $b \in [0, \mu_1(Z^l)]$. Again, we fix a family of $W_b(Z^l)$ for each $l \in [0, \infty)$ and each Z , and define

$$L_a(Z) = \{x \in Z : f(x) < l_a\} \cup W_c(Z^{l_a}),$$

where $c = a - \mu_1(\{x \in Z : f(x) < a\})$. Note that $l_a(X) = l_a$ and $L_a(X) = L_a$, for each $a \in [0, \mu_1(X)]$. In the following two lemmas and their proofs, for a given $u \in R_\mu(X)$, we consider a point $a^* \in [0, \mu_1(X) - u_1]$ with $\mu(M_{a^*, u_1}) = u$ and the set $Z = X \setminus M_{a^*, u_1}$. The existence of a^* is stated in Lemma 3.10. Later it will become clear that Z is the maximal subset with the vector measure $p = \mu(X) - u$ and M_{a^*, u_1} is the the minimal subset with the vector measure $p = u$.

Lemma 3.11. *Let Assumption 3.4 hold. For a given $u = (u_1, u_2) \in R_\mu(X)$, consider $a^* \in [0, \mu_1(X) - u_1]$ with $\mu(M_{a^*, u_1}) = u$. Then*

$$(3.6) \quad \mu_2(L_a(Z)) = \begin{cases} \mu_2(L_a) & \text{if } a \in [0, a^*]; \\ \mu_2(L_{a+u_1} \setminus M_{a^*, u_1}) & \text{if } a \in (a^*, \mu_1(X) - u_1]. \end{cases}$$

Proof. First, consider the case $a \in [0, a^*]$. We have $Z = X \setminus M_{a^*, u_1} \supseteq L_{a^*} \supseteq L_a = \{f(x) < l_a\} \cup W_c(X^{l_a})$, where $c = a - \mu_1(\{f(x) < l_a\})$. In addition, $\{f(x) < l_a\} \cup W_c(X^{l_a}) \subseteq \{f(x) \leq l_a\}$. Therefore

$$\begin{aligned} \mu_1(\{x \in Z : f(x) \leq l_a\}) &= \mu_1(Z \cap \{f(x) \leq l_a\}) \\ &\geq \mu_1(Z \cap (\{f(x) < l_a\} \cup W_c(X^{l_a}))) = \mu_1(\{f(x) < l_a\} \cup W_c(X^{l_a})) = a. \end{aligned}$$

Thus, (3.5) implies that $l_a(Z) \leq l_a$. On the other hand, take an arbitrary $l < l_a$. Since $Z \subseteq X$,

$$\mu_1(\{x \in Z : f(x) \leq l\}) \leq \mu_1(\{f(x) \leq l\}) < a.$$

Therefore, $l_a(Z) > l$ for all $l < l_a$. Thus, $l_a(Z) \geq l_a$. We conclude that $l_a(Z) = l_a$.

Denote $A = \{f(x) < l_a\}$. Since $Z \supseteq L_a \supseteq A$ and $l_a(Z) = l_a$, then $\{x \in Z : f(x) < l_a(Z)\} = A$. By definition, each of the sets L_a and $L_a(Z)$ is the union of two disjoint subsets: $L_a = A \cup W_c(X^{l_a})$ and $L_a(Z) = A \cup W_b(Z^{l_a})$ with $c = a - \mu_1(A) = b$. Thus, since $X^{l_a} \supseteq Z^{l_a}$ and $f(x) = l_a$ when $x \in X^{l_a}$, we have $\mu_2(W_c(X^{l_a})) = \mu_2(W_b(Z^{l_a})) = l_a c$. So, $\mu_2(L_a(Z)) = \mu_2(A) + \mu_2(W_b(Z^{l_a})) = \mu_2(A) + \mu_2(W_c(X^{l_a})) = \mu_2(L_a)$.

Second, consider the case $a \in (a^*, \mu_1(X) - u_1]$. Observe that $M_{a^*, u_1} \subseteq L_{a^*+u_1} \subseteq L_{a+u_1} = \{f(x) < l_{a+u_1}\} \cup W_c(X^{l_{a+u_1}})$, where $c = a + u_1 - \mu_1(\{f(x) < l_{a+u_1}\})$. In addition, $\{f(x) < l_{a+u_1}\} \cup W_c(X^{l_{a+u_1}}) \subseteq \{f(x) \leq l_{a+u_1}\}$. Therefore,

$$\begin{aligned} \mu_1(\{x \in Z : f(x) \leq l_{a+u_1}\}) &= \mu_1(\{f(x) \leq l_{a+u_1}\} \cap Z) = \mu_1(\{f(x) \leq l_{a+u_1}\} \setminus M_{a^*, u_1}) \\ &\geq \mu_1(\{f(x) < l_{a+u_1}\} \cup W_c(X^{l_{a+u_1}}) \setminus M_{a^*, u_1}) = a + u_1 - u_1 = a. \end{aligned}$$

Thus, (3.5) implies that $l_a(Z) \leq l_{a+u_1}$. On the other hand, note that $M_{a^*, u_1} \subseteq \{f(x) \leq l_a(Z)\}$. Indeed, since $a > a^*$, we have $l_a(Z) \geq l_{a^*}(Z) = l_{a^*}$. Assume $l_{a^*} \leq l_a(Z) < l_{a^*+u_1}$. Then $\{x \in Z : f(x) \leq l_a(Z)\} = \{f(x) \leq l_a(Z)\} \setminus M_{a^*, u_1} = L_{a^*}$, and $a = \mu_1(\{x \in Z : f(x) \leq l_a(Z)\}) = \mu_1(L_{a^*}) = a^*$, which is a contradiction. Therefore, $l_a(Z) \geq l_{a^*+u_1}$ and $M_{a^*, u_1} \subseteq \{f(x) \leq l_{a^*+u_1}\} \subseteq \{f(x) \leq l_a(Z)\}$. Thus, $\{x \notin Z : f(x) \leq l_a(Z)\} = \{x \in M_{a^*, u_1} : f(x) \leq l_a(Z)\} = M_{a^*, u_1}$ and

$$\mu_1(\{f(x) \leq l_a(Z)\}) = \mu_1(\{x \in Z : f(x) \leq l_a(Z)\}) + \mu_1(M_{a^*, u_1}) \geq a + u_1,$$

where the last step follows from property (b) in Lemma 3.7. Formula (3.1) implies that $l_a(Z) \geq l_{a+u_1}$. Therefore, $l_a(Z) = l_{a+u_1}$.

Consider again the identity $L_{a+u_1} = \{f(x) < l_{a+u_1}\} \cup W_c(X^{l_{a+u_1}})$, where the sets in the union are disjoint and $c = (a + u_1) - \mu_1(\{f(x) < l_{a+u_1}\})$. Similarly, $L_a(Z) = \{x \in Z : f(x) < l_{a+u_1}\} \cup W_b(Z^{l_{a+u_1}})$, where $b = a - \mu_1(\{x \in Z : f(x) < l_{a+u_1}\})$. Since $l_a(Z) = l_{a+u_1}$ and $\{f(x) < l_{a+u_1}\} \supset M_{a^*, u_1}$, we have $b = a - \mu_1(\{x \in Z : f(x) < l_{a+u_1}\}) = a - \mu_1(\{f(x) < l_{a+u_1}\} \setminus M_{a^*, u_1}) = (a + u_1) - \mu_1(\{f(x) < l_{a+u_1}\}) = c$. Thus,

$$\begin{aligned} \mu_2(L_a(Z)) &= \mu_2(\{x \in Z : f(x) < l_a(Z)\}) + \mu_2(W_b(Z^{l_a(Z)})) \\ &= \mu_2(\{x \in Z : f(x) < l_{a+u_1}\}) + l_{a+u_1} \mu_1(W_b(Z^{l_{a+u_1}})) \\ &= \mu_2(\{f(x) < l_{a+u_1}\}) - \mu_2(M_{a^*, u_1}) + l_{a+u_1} \mu_1(W_c(X^{l_{a+u_1}})) \\ &= \mu_2(L_{a+u_1}) - \mu_2(M_{a^*, u_1}) = \mu_2(L_{a+u_1} \setminus M_{a^*, u_1}), \end{aligned}$$

where the second equality holds because $l_a(Z) = l_{a+u_1}$, $f(x) = l_{a+u_1}$ for $x \in X^{l_{a+u_1}}$, and $Z^{l_{a+u_1}} \subseteq X^{l_{a+u_1}}$ (in fact $Z^{l_{a+u_1}} = X^{l_{a+u_1}}$, but we do not use this). The

third equality holds because of $\{x \in Z : f(x) < l_{a+u_1}\} = \{f(x) < l_{a+u_1}\} \setminus M_{a^*,u_1}$, $\{f(x) < l_{a+u_1}\} \supset M_{a^*,u_1}$, and $b = c$. The fourth equality follows from $l_{a+u_1}\mu_1(W_c(X^{l_{a+u_1}})) = \mu_2(W_c(X^{l_{a+u_1}}))$. \square

Lemma 3.12. *Let Assumption 3.4 hold. For a given $u = (u_1, u_2) \in R_\mu(X)$, consider $a^* \in [0, \mu_1(X) - u_1]$ with $\mu(M_{a^*,u_1}) = u$. Let $q = (q_1, q_2)$ be on the lower (upper) boundary of $R_\mu(Z)$. If $q_1 \in [0, a^*]$ ($q_1 \in [0, \mu_1(X) - u_1 - a^*]$), then q is on the lower (upper) boundary of $R_\mu(X)$, and if $q_1 \in (a^*, \mu_1(X) - u_1]$ ($q_1 \in [\mu_1(X) - u_1 - a^*, \mu_1(X) - u_1]$), then $r = \mu(X) - u - q$ is on the upper (lower) boundary of $R_\mu(X)$.*

Proof. When q is on the lower boundary of $R_\mu(Z)$, according to Lemma 3.9, $\mu_2(L_{q_1}(Z)) = q_2$. If $q_1 \in [0, a^*]$, then by Lemma 3.11, $\mu_2(L_{q_1}) = \mu_2(L_{q_1}(Z)) = q_2$, and Lemma 3.9 implies that q is on the lower boundary of $R_\mu(X)$.

If $q_1 \in (a^*, \mu(X) - u_1]$, then for $r = (r_1, r_2)$,

$$\begin{aligned} r_2 &= \mu_2(X) - u_2 - q_2 = \mu_2(X) - \mu_2(M_{a^*,u_1}) - \mu_2(L_{q_1}(Z)) \\ &= \mu_2(X) - (\mu_2(M_{a^*,u_1}) + \mu_2(L_{q_1+u_1} \setminus M_{a^*,u_1})) = \mu_2(X) - \mu_2(L_{q_1+u_1}) \\ &= \mu_2(X \setminus L_{q_1+u_1}) = \mu_2(X \setminus L_{\mu_1(X)-r_1}), \end{aligned}$$

where the first and last equalities follow from the definition of r , the second equality follows from Lemma 3.10, the third equality follows from Lemma 3.11, and the fourth equality follows from $q_1 > a^*$. According to Lemma 3.9, r is on the upper boundary of $R_\mu(X)$.

If q is on the upper boundary of $R_\mu(Z)$, then, because of symmetry, $r = \mu(X) - u - q$ is on the lower boundary of $R_\mu(Z)$. If $q_1 \in [\mu_1(X) - u_1 - a^*, \mu_1(X) - u_1]$, then $\mu_1(X) - u_1 - q_1 \in [0, a^*]$. From the first part of the proof, $r = \mu(X) - u - q$ is on the lower boundary of $R_\mu(X)$. If $q_1 \in [0, \mu_1(X) - u_1 - a^*]$, then $\mu_1(X) - u_1 - q_1 \in (a^*, \mu_1(X) - u_1]$. Again, from the first part of the proof, $\mu(X) - u - (\mu(X) - u - q_1) = q_1$ is on the upper boundary of $R_\mu(X)$. \square

Lemma 3.13. *Under Assumption 3.4, for any vector $p \in R_\mu(X)$, there exists a maximal set $Z^* \in \mathcal{S}_p(X)$ and, in addition, $R_\mu^p(X) = Q_\mu^p(X)$.*

Proof. For $u = \mu(X) - p$, consider a^* defined in Lemma 3.10. For $Z^* = X \setminus M_{a^*,\mu_1(X)-p_1}$, the following three statements are true: (1) $R_\mu(Z^*) \subseteq R_\mu^p(X)$; (2) $R_\mu^p(X) \subseteq Q_\mu^p(X)$; (3) $Q_\mu^p(X) \subseteq R_\mu(Z^*)$.

For (1), $\mu(Z^*) = \mu(X \setminus M_{a^*,\mu_1(X)-p_1}) = \mu(X) - \mu(M_{a^*,\mu_1(X)-p_1}) = \mu(X) - (\mu(X) - p) = p$, where the second to the last equality follows from Lemma 3.10. Thus, $R_\mu(Z^*) = R_\mu^p(X)$.

For (2), assume that there exists a vector $q \in R_\mu^p(X)$ such that $q \notin Q_\mu^p(X)$. Then Definition 2.1 implies that either $q \notin R_\mu(X)$ or $q \notin (R_\mu(X) - \{\mu(X) - p\})$. However $q \in R_\mu^p(X) \subseteq R_\mu(X)$. Therefore, $q \notin R_\mu(X) - \{\mu(X) - p\}$, which is equivalent to $p - q \notin \{\mu(X)\} - R_\mu(X) = R_\mu(X)$, where the equality follows from Lemma 3.1. Since $R_\mu^p(X) \subseteq R_\mu(X)$, we have $p - q \notin R_\mu^p(X)$. By Lemma 3.3, $R_\mu^p(X)$ is centrally symmetric with the center $\frac{p}{2}$. Therefore $q \notin R_\mu^p(X)$. The above contradiction implies (2).

For (3), assume $q \in Q_\mu^p(X)$, but $q \notin R_\mu(Z^*)$. By Lyapunov's theorem, $R_\mu(Z^*)$ is a convex compactum. Let $q^u = (q_1, q_2^u)$ and $q^l = (q_1, q_2^l)$ be the intersection points of the vertical line $\mu_1 = q_1$ and the upper and lower boundaries of $R_\mu(Z^*)$ respectively. Then one of the following must be true: $q_2 > q_2^u$ or $q_2 < q_2^l$. Without

loss of generosity, we consider the former case. Since q_u is on the upper boundary of $R_\mu(Z^*)$, according to Lemma 3.12, one of the following is true: (a) q^u is on the upper boundary of $R_\mu(X)$ or (b) $r = p - q^u$ is on the lower boundary of $R_\mu(X)$. For (a), $q_2 > q_2^u$ implies $q \notin R_\mu(X)$. Thus $q \notin Q_\mu^p(X)$. This contradicts our assumption. For (b), we let $r' = p - q$. Obviously, $r'_1 = r_1$ and $r'_2 < r_2$. This implies that r' is below the lower boundary point r . Thus, $r' \notin R_\mu(X)$ and $r' \notin Q_\mu^p(X)$. But according to Lemma 3.3, this means $q \notin Q_\mu^p(X)$, which contradicts our assumption. Statements (1)-(3) imply the lemma. \square

Let D be a two-by-two invertible matrix with positive entries, and $A \subseteq \mathbb{R}^2$. We denote by AD the set $\{pD : p \in A\}$. For a vector measure $\mu = (\mu_1, \mu_2)$, let $\nu = \mu D$ be the vector measure $(\nu_1, \nu_2) = (D_{11}\mu_1 + D_{21}\mu_2, D_{12}\mu_1 + D_{22}\mu_2)$. Then the measures ν_1 and ν_2 are equivalent.

Lemma 3.14. (a) $R_\mu(Y)D = R_\nu(Y)$ for all $Y \in \mathcal{F}$; (b) $R_\mu^p(X)D = R_\nu^{pD}(X)$ for all $p \in R_\mu(X)$; (c) $Q_\mu^p(X)D = Q_\nu^{pD}(X)$ for all $p \in R_\mu(X)$.

Proof. (a) For any point $q \in R_\nu(Y)$, there exists a set $Z \in \mathcal{F}_Y$ such that $\nu(Z) = q$. Since $\mu(Z) = qD^{-1}$ and $qD^{-1} \in R_\mu(Y)$, we have $q \in R_\mu(Y)D$. For any point $q \in R_\mu(Y)D$, we have $qD^{-1} \in R_\mu(Y)$. Thus there exists a set $Z \in \mathcal{F}_Y$ such that $\mu(Z) = qD^{-1}$ and $\nu(Z) = q$. Therefore, $\nu(Z) \in R_\nu(Y)$.

(b) For any point $q \in R_\nu^{pD}(X)$, there exist sets $Y \in \mathcal{F}$ and $Z \in \mathcal{F}_Y$ such that $\nu(Y) = pD$ and $\nu(Z) = q$. So $\mu(Y) = p$ and $\mu(Z) = qD^{-1}$. Thus, $qD^{-1} \in R_\mu^p(X)$ and therefore, $q \in R_\mu^p(X)D$. For any point $q \in R_\mu^p(X)D$, we have $qD^{-1} \in R_\mu^p(X)$. So there exist sets $Y \in \mathcal{F}$ and $Z \in \mathcal{F}_Y$ such that $\mu(Y) = p$ and $\mu(Z) = qD^{-1}$, and consequently $\nu(Y) = pD$ and $\nu(Z) = q$. Thus $q \in R_\nu^{pD}(X)$.

(c) According to Definition 2.1, $Q_\mu^p(X)D = (R_\mu(X)D - \{\mu(X)D - pD\}) \cap R_\mu(X)D = (R_\nu(X) - \{\nu(X) - pD\}) \cap R_\nu(X) = Q_\nu^{pD}(X)$. \square

Proof of Theorem 2.3. According to Lemma 3.13, Theorem 2.3 holds under Assumption 3.4, which states that μ_1 and μ_2 are equivalent. If μ_1 and μ_2 are not equivalent, consider $\nu = \mu D$. Since ν_1 and ν_2 are equivalent, $Q_\mu^p(X) = Q_\nu^{pD}(X)D^{-1} = R_\nu^{pD}(X)D^{-1} = R_\mu^p(X)$, where the first equality and the last equality are by Lemma 3.14, and the second equality is due to Lemma 3.13. Furthermore, according to Lemma 3.13, there exists a maximal set Z^* such that $R_\nu(Z^*) = R_\nu^{pD}(X)$. Therefore, $R_\mu(Z^*) = R_\nu(Z^*)D^{-1} = R_\nu^{pD}(X)D^{-1} = R_\mu^p(X)$. \square

Now consider Theorem 2.4. For $A \subseteq \mathbb{R}^m$ and $b \in \mathbb{R}^m$, let $A + b = \{a + b : a \in A\}$ and $A - b = A + (-b)$. Observe that $A - b = A - \{b\}$. Recall that $A \oplus B = \bigcup_{b \in B} (A + b)$ is called the Minkowski addition and $A \ominus B = \bigcap_{b \in B} (A - b)$ is called the Minkowski subtraction, where $A, B \subseteq \mathbb{R}^m$.

Lemma 3.15. Let $A_1, A_2, B_1, B_2 \subseteq \mathbb{R}^2$ be convex and compact sets such that $A_1 \oplus B_1 = A_2 \oplus B_2$ and $B_1 \subseteq B_2$. Then $A_2 \subseteq A_1$.

Proof. According to [11, Lemma 3.1.8], if $A, B \subseteq \mathbb{R}^2$ are convex and compact sets, then $(A \oplus B) \ominus B = A$. Thus if $a \in A_2$, then $a \in (A_2 \oplus B_2) \ominus B_2$, and consequently $a \in (A_1 \oplus B_1) \ominus B_2$. So $a \in (A_1 \oplus B_1) - b$, for any $b \in B_2$. Since $B_1 \subseteq B_2$, we have $a \in (A_1 \oplus B_1) - b$, for any $b \in B_1$, and thus $a \in (A_1 \oplus B_1) \ominus B_1 = A_1$. \square

Proof of Theorem 2.4. Now let Z^* be the maximal set with the measure p . Then $\mu(X \setminus Z^*) = \mu(X) - p$. Consider any set M such that $\mu(M) = \mu(X) - p$. Obviously,

$R_\mu(M) \oplus R_\mu(X \setminus M) = R_\mu(X \setminus Z^*) \oplus R_\mu(Z^*) = R_\mu(X)$. In addition, $R_\mu(X \setminus M) \subseteq R_\mu(Z^*)$ by definition. Thus according to Lemma 3.15, $R_\mu(X \setminus Z^*) \subseteq R_\mu(M)$.

Similarly, let $M^* = X \setminus Z^*$ be the minimal set with the measure $\mu(X) - p$. Then $\mu(Z^*) = p$. Consider any set Z such that $\mu(Z) = p$. Obviously, $R_\mu(Z) \oplus R_\mu(X \setminus Z) = R_\mu(Z^*) \oplus R_\mu(X \setminus Z^*) = R_\mu(X)$. In addition, $R_\mu(X \setminus Z^*) = R_\mu(M^*) \subseteq R_\mu(X \setminus Z)$ by definition. Thus according to Lemma 3.15, $R_\mu(Z) \subseteq R_\mu(Z^*)$. \square

With Theorem 2.4, the existence of the minimal subset $M^* \in \mathcal{S}_\mu^q(X)$ immediately follows from the existence of the maximal subset $Z^* \in \mathcal{S}_\mu^{\mu(X)-q}(X)$. Furthermore, $R_\mu(M^*) = (R_\mu(M^*) \oplus R_\mu(Z^*)) \ominus R_\mu(Z^*) = R_\mu(X) \ominus Q_\mu^{\mu(X)-q}(X)$.

Corollary 3.16. *For a two-dimensional finite atomless vector measure $\mu = (\mu_1, \mu_2)$ and for a vector $q \in R_\mu(X)$, there exists a minimal set $M^* \in \mathcal{S}_\mu^q(X)$. In addition, $R_\mu(M^*) = R_\mu(X) \ominus Q_\mu^{\mu(X)-q}(X)$.*

4. COUNTEREXAMPLE FOR 3D MEASURES

In this section, we present an example of a measurable space (X, \mathcal{F}) endowed with a three-dimensional atomless finite measure $\nu = (\nu_1, \nu_2, \nu_3)$ and a vector $p \in R_\nu(X)$ such that a maximal subset of X with the measure p does not exist. Theorem 2.4 implies that the minimum set does not exist either in this example.

Recall that, with respect to a measure μ , sets A and B are said to be equal up to null sets (denoted by $A \simeq B$) if $\mu(A \setminus B) = \mu(B \setminus A) = 0$. Also recall that $X^l = \{f(x) = l\}$.

Proposition 4.1. *Let $\mu = (\mu_1, \mu_2)$ satisfy Assumption 3.4 and let $Y \in \mathcal{F}$. If $\mu_1\{X^{l_{\mu_1}(Y)}\} = 0$ and $\mu_2(Y) = \mu_2(L_{\mu_1}(Y))$, then $Y \simeq L_{\mu_1}(Y)$.*

Proof. Assume that $Y \simeq L_{\mu_1}(Y)$ does not hold. We define three disjoint sets $Z_1 = Y \setminus L_{\mu_1}(Y)$, $Z_2 = L_{\mu_1}(Y) \setminus Y$, and $M = Y \cap L_{\mu_1}(Y)$. Observe that $Y = Z_1 \cup M$ and $L_{\mu_1}(Y) = Z_2 \cup M$. These equalities and $\mu_1(Y) = \mu_1(L_{\mu_1}(Y))$ imply $\mu_1(Z_1) = \mu_1(Z_2)$. Furthermore, $Z_1 \subseteq \{f(x) \geq l_{\mu_1}(Y)\}$ and $Z_2 \subseteq \{f(x) < l_{\mu_1}(Y)\}$, because according to (3.4), $L_{\mu_1}(Y) = \{f(x) < l_{\mu_1}(Y)\}$ when $\mu_1\{X^{l_{\mu_1}(Y)}\} = 0$. Therefore,

$$\begin{aligned} \mu_2(Z_1) &= \int_{Z_1} f(x) \mu_1(dx) \geq l_{\mu_1}(Y) \int_{Z_1} \mu_1(dx) \\ &= l_{\mu_1}(Y) \int_{Z_2} \mu_1(dx) > \int_{Z_2} f(x) \mu_1(dx) = \mu_2(Z_2). \end{aligned}$$

So $\mu_2(Y) = \mu_2(Z_1) + \mu_2(M) > \mu_2(Z_2) + \mu_2(M) = \mu_2(L_{\mu_1}(Y))$. This contradiction implies the proposition. \square

Example 4.2. Let $X = [0, 1]$ and \mathcal{F} be the Borel σ -field. Consider the three-dimensional vector measure $\nu(dx) = (\nu_1, \nu_2, \nu_3)(dx) = (1, 2x, \rho(x)) dx$, where

$$\rho(x) = \begin{cases} 4x & \text{if } x \in [0, \frac{1}{2}); \\ 4x - 2 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Consider the points $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $q^1 = (\frac{1}{4}, \frac{1}{16}, \frac{1}{8})$, and $q^2 = (\frac{1}{4}, \frac{5}{32}, \frac{1}{16})$. It is easy to show that $q^1, q^2 \in R_\nu^p(X)$. Indeed let $Z^1 = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, $Z^2 = [0, \frac{1}{8}] \cup [\frac{3}{8}, \frac{5}{8}] \cup [\frac{7}{8}, 1]$, $W^1 = [0, \frac{1}{4}] \subseteq Z^1$, and $W^2 = [0, \frac{1}{8}] \cup [\frac{1}{2}, \frac{5}{8}] \subseteq Z^2$, and we have $\nu(Z^1) = \nu(Z^2) = p$, $\nu(W^1) = q^1$, and $\nu(W^2) = q^2$. Since Z^1 and Z^2 are not

equal up to a null set, Proposition 4.3 implies that there does not exist a set Z such that $\nu(Z) = p$ and $q^1, q^2 \in R_\mu(Z)$.

Proposition 4.3. *Consider the sets X, Z^1, Z^2 , the measure ν and vectors p, q^1, q^2 from Example 4.2. Let $Z \in \mathcal{S}_\nu^p(X)$. For each $i = 1, 2$, if $q^i \in R_\nu(Z)$, then $Z \simeq Z^i$.*

Proof. Let $i = 1$. Since $q^1 \in R_\nu(Z)$, there exists a set $W^1 \in \mathcal{F}_Z$ such that $\nu(W^1) = q^1$. Define a two-dimensional vector measure $\mu = (\mu_1, \mu_2) = (\nu_1, \nu_2)$. Then $\mu(W^1) = (\frac{1}{4}, \frac{1}{16})$. Observe that, according to (3.1) and (3.4), $l_{\mu_1(W^1)} = l_{\frac{1}{4}} = \frac{1}{2}$ and $L_{\mu_1(W^1)} = L_{\frac{1}{4}} = [0, \frac{1}{4}]$. In addition, $\mu_1(X^{l_{\mu_1(W^1)}}) = 0$ and $\mu_2(W^1) = \frac{1}{16} = \mu_2(L_{\mu_1(W^1)})$. Therefore, according to Proposition 4.1, $W^1 \simeq L_{\mu_1(W^1)} = [0, \frac{1}{4}]$. On the other hand, let $Y = W^1 \cup (X \setminus Z)$. Since $W^1 \subseteq Z$, $\nu(Y) = \nu(W^1) + (\nu(X) - \nu(Z)) = q^1 + (\nu(X) - p) = (\frac{3}{4}, \frac{9}{16}, \frac{5}{8})$, and thus, $\mu(Y) = (\frac{3}{4}, \frac{9}{16})$. Observe that, according to (3.1) and (3.4), $l_{\mu_1(Y)} = l_{\frac{3}{4}} = \frac{3}{2}$ and $L_{\mu_1(Y)} = L_{\frac{3}{4}} = [0, \frac{3}{4}]$. In addition, $\mu_1(X^{l_{\mu_1(Y)}}) = 0$ and $\mu_2(Y) = \frac{9}{16} = \mu_2(L_{\mu_1(Y)})$. Therefore, according to Proposition 4.1, $Y \simeq L_{\mu_1(Y)} = L_{\frac{3}{4}}$. The above observations imply that $Z = W^1 \cup (X \setminus Y) \simeq L_{\frac{1}{4}} \cup (X \setminus L_{\frac{3}{4}}) = [0, \frac{1}{4}] \cup ([0, 1] \setminus [0, \frac{3}{4}]) = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1] = Z^1$.

Let $i = 2$. Since $q^2 \in R_\nu(Z)$, there exists a set $W^2 \in \mathcal{F}_Z$ such that $\nu(W^2) = q^2$. Define the two-dimensional vector measure $\mu = (\mu_1, \mu_2) = (\nu_1, \nu_3)$. Then $\mu(W^2) = (\frac{1}{4}, \frac{1}{16})$. Observe that, according to (3.1) and (3.4), $l_{\mu_1(W^2)} = l_{\frac{1}{4}} = \frac{1}{2}$ and $L_{\mu_1(W^2)} = L_{\frac{1}{4}} = [0, \frac{1}{8}] \cup [\frac{1}{2}, \frac{5}{8}]$. In addition, $\mu_1(X^{l_{\mu_1(W^2)}}) = 0$ and $\mu_2(W^2) = \frac{1}{16} = \mu_2(L_{\mu_1(W^2)})$. Therefore, according to Proposition 4.1, $W^2 \simeq L_{\mu_1(W^2)} = [0, \frac{1}{8}] \cup [\frac{1}{2}, \frac{5}{8}]$. On the other hand, let $Y = W^2 \cup (X \setminus Z)$. Since $W^2 \subseteq Z$, $\nu(Y) = \nu(W^2) + (\nu(X) - \nu(Z)) = q^2 + (\nu(X) - p) = (\frac{3}{4}, \frac{21}{32}, \frac{9}{16})$, and thus, $\mu(Y) = (\frac{3}{4}, \frac{9}{16})$. Observe that, according to (3.1) and (3.4), $l_{\mu_1(Y)} = l_{\frac{3}{4}} = \frac{3}{2}$ and $L_{\mu_1(Y)} = L_{\frac{3}{4}} = [0, \frac{3}{8}] \cup [\frac{1}{2}, \frac{7}{8}]$. In addition, $\mu_1(X^{l_{\mu_1(Y)}}) = 0$ and $\mu_2(Y) = \frac{9}{16} = \mu_2(L_{\mu_1(Y)})$. Therefore, according to Proposition 4.1, $Y \simeq L_{\mu_1(Y)} = L_{\frac{3}{4}}$. The above observations imply that $Z = W^2 \cup (X \setminus Y) \simeq L_{\frac{1}{4}} \cup (X \setminus L_{\frac{3}{4}}) = ([0, \frac{1}{8}] \cup [\frac{1}{2}, \frac{5}{8}]) \cup ([0, 1] \setminus ([0, \frac{3}{8}] \cup [\frac{1}{2}, \frac{7}{8}])) = [0, \frac{1}{8}] \cup [\frac{3}{8}, \frac{5}{8}] \cup [\frac{7}{8}, 1] = Z^2$. \square

5. GEOMETRIC CONSTRUCTION OF MAXIMAL RANGES

In [8], Lyapunov commented that a subset of the two-dimensional Euclidean space \mathbb{R}^2 is the range of some two-dimensional finite atomless vector measure on some measurable space if and only if it satisfies the following conditions: (1) it is convex; (2) it is closed; (3) it is centrally symmetric; (4) it contains the origin. Since the geometrically constructed set $Q_\mu^p(X)$ satisfies conditions (1)-(4), it must be the range of some two-dimensional finite atomless vector measure on some measurable space. Theorem 2.3 immediately tells us that it is the range of the vector measure μ on the measurable space (Z^*, \mathcal{F}_{Z^*}) . The second equality in Theorem 2.3 allows us to construct geometrically the set $R_\mu^p(x)$ by shifting the set $R_\mu(X)$ by $(p - \mu(X))$ and intersecting the shifted set with $R_\mu(X)$.

We consider three examples with the same set $X = [0, 1]$, but with different probability vector measures. Let $p = (0.7, 0.8)$ in all these examples.

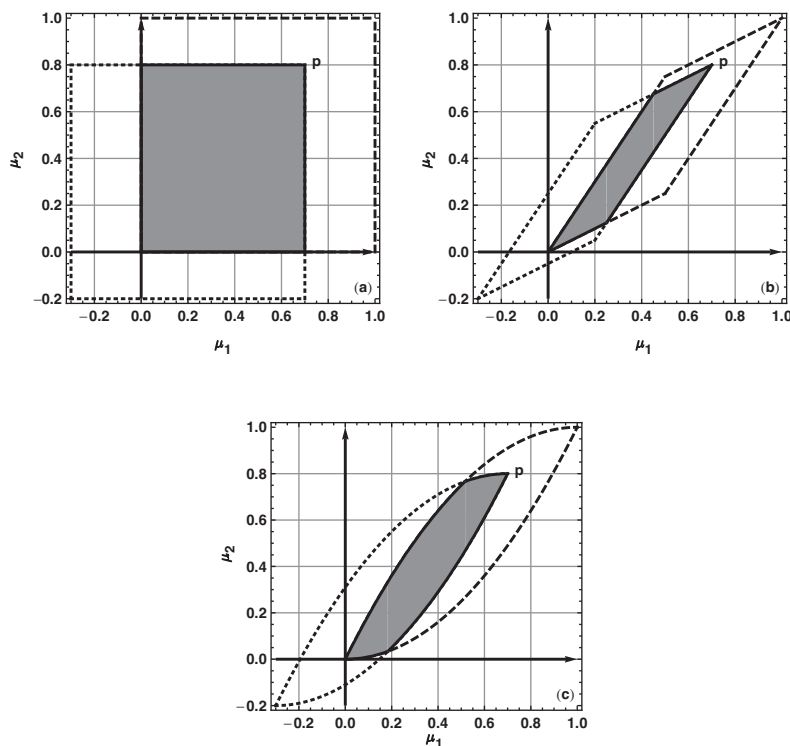


FIGURE 1. Plots (a)-(c) present the maximal subsets for the vector measures described in Examples 5.1-5.3, respectively, with $p = (0.7, 0.8)$. The area enclosed by the dashed lines is the range $R_\mu(X)$. The area enclosed by the dotted lines is obtained by parallelly shifting the dashed area by $(-0.3, -0.2)$. The shaded area is formed by the intersection of the above two areas and represents the identical sets $R_\mu(Z^*)$, $R_\mu^p(X)$ and $Q_\mu^p(X)$.

Example 5.1. Let μ_1 and μ_2 be singular. Then the range $R_\mu(X)$ is the unit square enclosed by the dashed lines in Fig. 1(a). The shaded area denotes the identical sets $R_\mu(Z^*)$, $R_\mu^p(X)$ and $Q_\mu^p(X)$ with $p = (0.7, 0.8)$.

Example 5.2. Consider the vector measure $\mu(dx) = (\mu_1, \mu_2)(dx) = (1, f(x))dx$, where

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}); \\ \frac{3}{2} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then the range of $R_\mu(X)$ is the area enclosed by the dashed lines in Fig. 1(b). The shaded area denotes the three identical sets $R_\mu(Z^*)$, $R_\mu^p(X)$ and $Q_\mu^p(X)$ with $p = (0.7, 0.8)$.

Example 5.3. Let $\mu(dx) = (\mu_1, \mu_2)(dx) = (1, 2x)dx$. Then the range $R_\mu(X)$ is the area enclosed by the dashed lines in Fig. 1(c). The shaded area denotes the three identical sets $R_\mu(Z^*)$, $R_\mu^p(X)$, and $Q_\mu^p(X)$ when $p = (0.7, 0.8)$.

6. PROOF OF THEOREM 2.5

For any $B \subseteq A$, denote $p(B) = \sum_{a \in B} p^a$, where either $A = \{1, 2, \dots\}$ or $A = \{1, \dots, n\}$ for some $n = 1, 2, \dots$.

Lemma 6.1. *Let $\mu = (\mu_1, \mu_2)$ be a two-dimensional finite atomless measure. If $p(B) \in R_\mu(X)$ for all $B \subseteq A$ and $\sum_{a \in A} p^a = \mu(X)$, then there exists a partition $\{Z^a \in \mathcal{F} : a \in A\}$ of X such that $p^a = \mu(Z^a)$ for each $a \in A$.*

Proof. Consider $p = \mu(X) - p^1$. According to Theorem 2.3, there exists a maximal subset $Z^* \in \mathcal{S}_\mu^p(X)$ and $R_\mu(Z^*) = Q_\mu^p(X)$. Let $Z^1 = X \setminus Z^*$, $X^1 = Z^*$, and $A^1 = A \setminus \{1\}$. Note that $p^1 = \mu(Z^1)$ and $p(B) \in R_\mu(X^1)$ for all $B \subseteq A^1$. Indeed, $p(B) + p^1 = p(B \cup \{1\}) \in R_\mu(X)$. Thus, $p(B) \in R_\mu(X) - \{(\mu(X) - p)\}$, and in addition $p(B) \in R_\mu(X)$. Therefore, $p(B) \in Q_\mu^p(X) = R_\mu(X^1)$.

Now for $p^2 \in \{p^a : a \in A^1\}$ there exists a maximal set $Z^* \in \mathcal{S}_\mu^p(X^1)$, where $p = \mu(X^1) - p^2$. Let $Z^2 = X^1 \setminus Z^*$, $X^2 = Z^*$, and $A^2 = A^1 \setminus \{2\}$. Then $p^2 = \mu(Z^2)$ and $p(B) \in R_\mu(X^2)$ for all $B \subseteq A^2$. The repetition of this procedure generates the desired partition $\{Z^a \in \mathcal{F} : a \in A\}$. \square

Proof of Theorem 2.5. The necessity is obvious. For the sufficiency, in view of Lemma 6.1, it is enough to prove that condition (ii) implies $p(B) \in R_\mu(X)$ for all $B \subseteq A$. If B is finite, condition (ii) implies $p(B) \in R_\mu(X)$. If B is infinite, let $B = \{a^1, a^2, \dots\}$ and $B_n = \{a^1, a^2, \dots, a^n\}$, $n = 1, 2, \dots$. Then $p(B) = \lim_{n \rightarrow \infty} p(B_n)$ and $p(B_n) \in R_\mu(X)$ for $n = 1, 2, \dots$, according to condition (ii). Since $R_\mu(X)$ is closed, $p(B) \in R_\mu(X)$. \square

Finally, we show that when $m = 2$, the Dvoretzky-Wald-Wolfowitz purification theorem for a countable image set A [4, 6] is a particular case of Theorem 2.5. Let $p^a = \int_X \pi(a|x) \mu(dx)$, $a \in A$. If these vectors p^a satisfy conditions (i) and (ii) of Theorem 2.5, then Theorem 2.5 implies that the transition probability can be purified in the case of a countable A and $m = 2$. Indeed, for (i), obviously $\sum_{a \in A} p^a = \mu(X)$. For (ii), if $B \subseteq A$, then

$$\sum_{a \in B} p^a = \sum_{a \in B} \int_X \pi(a|x) \mu(dx) = \int_X \pi(B|x) \mu(dx) \in R_\mu(X),$$

where the inclusion follows from a version of Lyapunov's theorem [1, p. 218].

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