

A PL-MANIFOLD OF NONNEGATIVE CURVATURE HOMEOMORPHIC TO $S^2 \times S^2$ IS A DIRECT METRIC PRODUCT

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ABSTRACT. Let M^4 be a PL-manifold of nonnegative curvature that is homeomorphic to a product of two spheres, $S^2 \times S^2$. We prove that M is a direct metric product of two spheres endowed with some polyhedral metrics. In other words, M is a direct metric product of the surfaces of two convex polyhedra in \mathbb{R}^3 .

The classical H. Hopf hypothesis states: for any Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature the curvature cannot be strictly positive at all points. The result of this paper can be viewed as a PL-version of Hopf's hypothesis. It confirms the remark of M. Gromov that the condition of nonnegative curvature in the PL-case appears to be stronger than nonnegative sectional curvature of Riemannian manifolds and analogous to the condition of a nonnegative curvature operator.

1. INTRODUCTION AND DEFINITIONS

This paper presents a structure result for polyhedral 4-manifolds with curvature bounded from below. The classical H. Hopf hypothesis states: for any Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature the curvature cannot be strictly positive at all points. There is no quick answer to this question: it is known that a Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature need not be a product metric [Wil07, pp. 25–26]. There is an important distinction between a condition on the sectional curvature and a condition on the curvature operator. It is known from the work of Hamilton that a 4-manifold with a positive curvature operator is diffeomorphic to the sphere \mathbb{S}^4 or the real projective space $\mathbb{R}P^4$ [Ham86]. Compare it with a corollary of Synge's theorem: the fundamental group of a Riemannian 4-manifold with positive sectional curvature is either \mathbb{Z}_2 or trivial. For more references, [Wil07] is an extensive survey of similar results for positive and nonnegative curvature. For work on the classical Hopf conjecture, see e.g. [Bou75, Kur93].

If the Hopf conjecture included a stronger premise of a nonnegative curvature operator, it would be much easier to solve. Indeed, any smooth Riemannian 4-manifold with nonnegative curvature operator and homeomorphic to $S^2 \times S^2$ must be isometric to a product of two round 2-spheres [Cho06, Theorem 7.34]. Furthermore, according to [Che86], M. Gromov has pointed out that PL-manifolds of

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nonnegative curvature are analogous to smooth manifolds with nonnegative curvature operator. As a confirmation of this remark, this paper solves the polyhedral case of the Hopf conjecture:

Main Theorem. *A PL-manifold of nonnegative curvature homeomorphic to $S^2 \times S^2$ is a direct metric product.*

To fix the terminology: a PL-manifold is a locally finite simplicial complex whose simplices are all metrically flat (convex hulls of finite sets of points in a Euclidean space) and which is also a topological manifold. In the compact case, “locally finite” implies “finite”, so we are working with some finite simplicial decomposition.

A (metric) singularity in a PL-manifold M^n is a point $x \in M$ that has no flat neighborhood. Metric singularities comprise M_s , the singular locus. $M \setminus M_s$ is a flat Riemannian manifold. More specifically, a singularity is said to have codimension k if its tangent cone is isometric to a direct product of \mathbb{R}^{n-k} with a k -dimensional polyhedral cone, yet there is no such product for \mathbb{R}^{n-k+1} and a $(k-1)$ -dimensional polyhedral cone. We will be interested in the case when M is also an Alexandrov space of nonnegative curvature. This condition is known to be equivalent to the following formulation: the link of M_s at each singularity of codimension 2 is a circle of length $< 2\pi$. The length of this circle is referred to as the conical angle at this singularity of codimension 2.

Given M , a PL-manifold of nonnegative curvature homeomorphic to $S^2 \times S^2$, the claim is that M is a direct metric product. The proof is carried out in three stages, so the text contains three more sections in addition to “Introduction and definitions”. In section 2 we decompose $M \setminus M_s$ into a local metric product in a consistent way. That is, we establish the existence of two parallel distributions of oriented 2-planes α and β (2-distributions for brevity), foliating $M \setminus M_s$ and orthogonal to each other. In section 3 we classify singularities to decompose a neighborhood of an arbitrary singularity $x \in M_s$ or, equivalently, the tangent cone of M at x . We will prove that the tangent cone at an arbitrary singularity is a product of two 2-cones if the codimension is 4 and is a product of a 2-cone with a Euclidean plane if the codimension is 2. It will be essential that M turns out to be a polyhedral Kähler manifold, and the work of Dmitri Panov on this subject [Pan09] will be referred to many times throughout section 3. The product decomposition of the singularities extends the 2-distributions defined on $M \setminus M_s$ to a priori singular foliations, defined on all of M and transverse to each other outside of singularities. Furthermore, the local product decomposition that M has at any point is aligned along these foliations. Finally, in section 4 we argue that M is a direct metric product globally, not just locally. In particular, the leaves of the singular foliations are closed. Since the classical de Rham decomposition theorem is only stated for the smooth case, we smooth the metric on M , apply the theorem, and then argue that the decomposition also holds for M with the original polyhedral metric.

It will be useful to recall the definition of a polyhedral Kähler manifold [Pan09]. Let M be an oriented PL-manifold of even dimension $d = 2n$. The definition will be more concise if we suppose that all conical angles at codimension 2 faces are either 2π (no singularity) or not a multiple of 2π . This is a valid assumption since we are interested in manifolds that have nonnegative curvature. Consider the image G of $\pi_1(M^{2n} \setminus M_s^{2n})$ in $SO(2n)$ generated by holonomies of the metric on the complement of the singular locus. A manifold is called polyhedral Kähler if the group G is contained in a subgroup of $SO(2n)$ conjugate to $U(n)$.

2. FINDING PARALLEL 2-DISTRIBUTIONS

In this section we remove all singularities from our consideration and focus on $M \setminus M_s$, a flat Riemannian manifold. The goal of this section is to find two parallel 2-distributions on $M \setminus M_s$ that are orthogonal to each other. Since $M \setminus M_s$ is a flat Riemannian manifold, one can study differential forms on it, even though smooth differential forms on M itself are not well-defined. Every parallel form (i.e. $\nabla\omega = 0$) on $M \setminus M_s$ is harmonic, L^2 , closed and co-closed, as is verified by taking the differential and the codifferential in local flat coordinates and integrating in local coordinates. This can be done using a finite flat atlas coming from the PL-structure.

Theorem 2.1. *Let M be a PL-manifold of nonnegative curvature, homeomorphic to $S^2 \times S^2$ and let M_s be the singular locus of M . Then there are two parallel, mutually orthogonal 2-distributions on $M \setminus M_s$.*

The idea is to find a parallel degenerate 2-form on $M \setminus M_s$ with a 2-dimensional kernel. Then one of the 2-distributions will be given by this kernel, the other being the orthogonal complement. If M is indeed a direct metric product, then we should certainly have such 2-forms, namely the areas of the signed projections onto the first and the second factor respectively. We are also interested in obtaining a symplectic 2-form on $M \setminus M_s$ to show that M is polyhedral Kähler. This will prove that M has no singularities of codimension 3 [Pan09, Proposition 3.5].

The main tool here is a theorem of J. Cheeger that applies in particular to PL-manifolds of nonnegative curvature [Che86]. We are using his result in the following form:

Theorem 2.2 (J. Cheeger). *Let M^n be a PL-manifold of nonnegative curvature. Let H^i be the space of L^2 -harmonic i -forms on $M \setminus M_s$ that are closed and co-closed. Then $\dim(H^i) = b^i(M)$, the i -th Betti number. Moreover, all forms in H^i are parallel.*

What it means for our present discussion, given that $b^2(S^2 \times S^2) = 2$, is that the vector space of parallel harmonic 2-forms on $M \setminus M_s$ is 2-dimensional. We are nearly done with the proof of Theorem 2.1. It remains to find a parallel harmonic 2-form $\omega \in H^2$ such that $\omega \neq 0$, yet $\omega \wedge \omega = 0$. Such a form would yield the required 2-distributions as $\ker(\omega)$ and $\ker(\omega)^\perp$, respectively. However, to prove the existence of such a form we would like to be able to say that H^2 has the standard cohomological product structure given by the wedge product, not just the right dimension as we have concluded from J. Cheeger’s theorem. In order to establish this we will use the notion of piecewise smooth differential forms on polytopes [Zvo08, section 5.2].

Let us recall the main definitions. Then we will proceed to prove Lemma 2.8, thus completing the proof of Theorem 2.1.

Definition 2.3 (see [Zvo08, Definition 5.1]). A polytope in a real vector space is an intersection of a finite number of open or closed half-spaces such that its interior is nonempty. Replacing, in the above intersection, some of the closed half-spaces by their boundary hyperplanes, we obtain a face of the polytope.

Definition 2.4 (see [Zvo08, Definition 5.2]). A polytopal complex is a finite set X of polytopes in real vector spaces, together with gluing functions satisfying the following conditions. (i) Each gluing function is an affine map that identifies a

polytope $P_1 \in X$ with a face of another polytope $P_2 \in X$. (For brevity, we will say that P_1 is a face of P_2 .) (ii) If P_1 is a face of P_2 , which is a face of P_3 , then P_1 is a face of P_3 , and the corresponding gluing functions form a commutative diagram. (iii) If $P_1 \in X$ is identified with a face of $P_2 \in X$, no other polytope $P_1 \in X$ can be identified with the same face of P_2 .

Clearly, a finite simplicial complex is also a polytopal complex. In fact, there is more than one way to make a polytopal complex out of a given simplicial complex. It is convenient to agree that the simplices of codimension zero and one do become polytopes composing the polytopal complex, whereas all the other simplices of higher codimension do not.

We proceed to define differential forms on polytopal complexes. A differential form on a polytope is simply a differential form with smooth coefficients defined in some neighborhood of the polytope in the ambient vector space [Zvo08].

Definition 2.5 (see [Zvo08, Definition 5.3]). A differential k -form on a polytopal complex is a set of differential k -forms defined on all the polytopes such that restricting the k -form to a face of a polytope coincides with the k -form on the face.

We have the following results:

Proposition 2.6 (see [Zvo08, Proposition 5.6]). *The de Rham cohomology groups of a polytopal complex X are canonically identified, as real vector spaces, with its usual cohomology groups over \mathbb{R} .*

Proposition 2.7 (see [Zvo08, Proposition 5.9]). *The algebra structure of the de Rham cohomology of a polytopal complex X (given by the multiplication of forms) coincides with the usual algebra structure of the cohomology of X .*

This gives enough background to study piecewise smooth differential forms on $M = S^2 \times S^2$. Once we take the classes of closed forms modulo exact ones, the wedge product structure is the same as the standard cohomological product for $S^2 \times S^2$, as desired.

Lemma 2.8. *One can find a parallel, harmonic, L^2 , closed and co-closed 2-form ω on $M \setminus M_s$ such that $\omega \neq 0$, yet $\omega \wedge \omega = 0$ locally at every point. In particular, $\ker(\omega)$ is 2-dimensional.*

Proof. View M as a polytopal complex (Definition 2.4). The polytopes forming it are the 4- and 3-dimensional simplices of M . The 3-dimensional ones become faces of the 4-dimensional ones. Simplices of lower dimension do not become polytopes.

The singular cohomology of $S^2 \times S^2$ with coefficients in \mathbb{R} contains an element $h \neq 0$ that squares to zero: $h^2 = 0$. There are two such elements up to rescaling, just as in the de Rham cohomology of $S^2 \times S^2$ there are up to rescaling two nonzero elements that integrate to zero on one or the other of the S^2 factors in the product. Proposition 2.7 therefore implies that there is a piecewise smooth closed 2-form η on M such that $\eta \wedge \eta$ is cohomologous to zero, yet η itself is not cohomologous to zero.

Every parallel harmonic form $\omega \in H^2$ can also be viewed as a piecewise smooth form on M . This follows from a rather literal reading of Definition 2.5 as well as from the comment preceding this definition. That is, we agreed that without loss of generality M as a polytopal complex contains only polytopes of dimensions 3 and 4. From Cheeger's theorem, $\dim(H^2) = 2$. Furthermore, every nonzero $\omega \in H^2$ is

not cohomologous to zero as a piecewise smooth form on M . Indeed, we can take the wedge product of ω with its Hodge star. The computation is carried out in local flat coordinates. We obtain a volume form. This is a volume form in either sense: it is clear that every piecewise smooth volume form on M is also a (parallel) volume form on $M \setminus M_s$, and vice versa. Therefore, we have 2 dimensions of closed piecewise smooth 2-forms on M modulo exact forms and 2 dimensions of parallel harmonic forms in H^2 . These two sets have to coincide. Taking ω that is representing the element h from the cohomology of $S^2 \times S^2$ given by $h \neq 0, h^2 = 0$, we obtain a nonzero ω such that $\omega \wedge \omega$ is cohomologous to zero.

By definition, $\omega \wedge \omega$ is cohomologous to zero if and only if $\int_M \omega \wedge \omega = 0$. Since ω is parallel, so is $\omega \wedge \omega$. M is orientable and its volume form is nonzero, thus $\int_M \omega \wedge \omega = 0$ implies $\omega \wedge \omega = 0$ locally at every point. □

Cheeger’s theorem (Theorem 2.2) does not give any results regarding the multiplicative structure (wedge product). The above proof generalizes to the following corollary:

Corollary 2.9. *Let M be an orientable PL-manifold of nonnegative curvature and let M_s be its singular locus. Let H^i be the space of L^2 -harmonic (parallel) i -forms on $M \setminus M_s$ that are closed and co-closed. Then the multiplicative structure on these forms given by the wedge product in local flat coordinates is the same as the standard cohomology of the same manifold M .*

This proves Theorem 2.1. Specifically, $\alpha = \ker(\omega)$ is one of the 2-distributions, and it can be oriented using ω itself. $\beta = \ker(\omega)^\perp$ is the other 2-distribution. These two 2-distributions allow us to give a more precise description of all parallel harmonic 2-forms on $M \setminus M_s$. The signed areas of the projections onto α and β are represented by two parallel harmonic forms that span $H^2(M)$. In appropriate local coordinates these two forms are just $dx_1 \wedge dx_2$ and $dx_3 \wedge dx_4$. The two 2-distributions are the kernels of the forms. The sum of these two forms, $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$, is a symplectic form with repeating eigenvalues $\pm i$. This form yields a pseudocomplex structure on $M \setminus M_s$.

Corollary 2.10. *M is a polyhedral Kähler manifold. The definition from [Pan09] was recalled in the introduction.*

Proof. Let J be the matrix of the 2-form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ expressed in an appropriate positively oriented orthogonal basis. The matrices representing the holonomies preserving the form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ commute with J . Commuting with J is equivalent to being in $GL(2, \mathbb{C})$. However, $O(4) \cap GL(2, \mathbb{C}) = U(2)$. If the basis turns out to be negatively oriented, use $dx_1 \wedge dx_2 - dx_3 \wedge dx_4$ instead. Either way, the image of the holonomies of $M \setminus M_s$ is in a subgroup of $SO(4)$ conjugate to $U(2)$ — precisely what the definition of a polyhedral Kähler manifold says. □

3. CLASSIFYING SINGULARITIES

The two 2-distributions we have found behave nicely but are defined only on the nonsingular part $M \setminus M_s$. The next step would be to focus on one singularity $p \in M_s$ at a time and to factor a neighborhood of p . We would like to show that such a neighborhood can be factored as a direct metric product, primarily because of the two mutually orthogonal 2-distributions on $M \setminus M_s$ that we have found in the previous section. Note that it is often easier to think about factoring the tangent

cone of M at p , even though we really need to factor a neighborhood. After factoring we would also like to show that the 2-distributions can be extended to singular foliations passing through the given singularity p and still totally geodesic.

Crucially, we have just established that M is a polyhedral Kähler manifold (Corollary 2.10), thus a complex surface. We can therefore use the tools developed by Dmitri Panov in his work on polyhedral Kähler manifolds [Pan09].

Lemma 3.1 (D. Panov). *Let M^4 be a $4_{\mathbb{R}}$ -dimensional polyhedral Kähler manifold. Then there are no singularities of codimension 3. That is, all singularities have to have codimension 2 or 4.*

Proof. See [Pan09, Proposition 3.5]. □

There can only be finitely many singularities of codimension 4, since they all have to be vertices of the simplicial decomposition of M . The locus of singularities of codimension 2 with the induced intrinsic metric is a flat 2-dimensional manifold, by definition of singularities of codimension 2. This manifold is a subset of the two-skeleton Σ^2 . Furthermore, those finitely many codimension 4 singularities cannot be isolated from the rest of the singular locus.

Proposition 3.2. *A singular point $x \in M_s$ of any codimension cannot be isolated from the rest of the singular locus.*

Proof. Assume for a contradiction that $x \in M_s$ is isolated from the rest of the singular locus. Technically, x has a flat pinched neighborhood. However, then the link at x is homeomorphic to S^3 and has constant curvature 1. Therefore, it is the standard 3-sphere and the cone over the link is \mathbb{R}^4 . Hence, M is not a singularity by definition, a contradiction. □

Note that this argument fails in the 2-dimensional case: S^1 is not simply connected. Indeed, singularities of a 2-dimensional PL-manifold are always of the highest possible codimension and yet always isolated from one another.

At this point we can conclude that the singular locus of M consists of several 2-simplices that are also faces in the simplicial decomposition of M . The vertices of these triangles may be singularities of codimension 4, but at all other points the singular locus is a flat 2-dimensional manifold. We consider a singularity $x \in M_s$ of codimension 2. By definition, its tangent cone can be factored as a product of \mathbb{R}^2 with a 2-cone. This factoring is unique, as one of the factors can be recovered by considering all geodesics passing through x . Moreover, the fibers of the factoring are aligned along the 2-distributions α and β :

Proposition 3.3. *Let $x \in M_s$ be a singularity of codimension 2. Consider the unique factoring of a neighborhood of x as $C \times \mathbb{R}^2$. Then the fibers of this factoring are parallel (respectively, perpendicular) to the two 2-distributions α and β that have been found in section 2.*

Proof. This proof is by a direct geometric argument and does not rely on Panov's paper [Pan09]. Consider a real vector field near x stretching the metric away from the singular locus. Let p be an arbitrary nonsingular point near x , and let the field at p be given by $\frac{1}{2} \nabla (\text{dist}^2(p, M_s))$, where ∇ is the gradient and $\text{dist}(p, M_s) > 0$ is the distance from p to the singular locus. For every 2-cone that is a fiber in the local product structure near p , the trajectories of the vector field are emanating

from the vertex of the 2-cone and are moving away in all directions tangent to the cone.

Take a neighborhood of p splitting as $C \times \mathbb{R}^2$ and intersect it with $M \setminus M_s$. The intersection splits as the product of a pinched 2-cone with \mathbb{R}^2 . Let γ be a generator of the fundamental group of this intersection. Let ω be a parallel 2-form on $M \setminus M_s$ with a 2-dimensional kernel. It follows from the previous section that this kernel is either α or β .

Choose two nonzero tangent vectors u and v at p . Let u be parallel to $(*, \mathbb{R}^2)$ and perpendicular to the gradient vector field defined above, and let v be parallel to $(C, *)$. Travel along γ and parallel transport u and v along the way. After a single loop around x observe that u is unchanged but that v is turned by some angle and becomes \hat{v} . The holonomy of γ preserves ω just as any other parallel form, so $\omega(u, v) = \omega(u, \hat{v})$, i.e. $\omega(u, v - \hat{v}) = 0$. The conical angle $< 2\pi$, so $v - \hat{v} \neq 0$. By choosing v appropriately we can make $v - \hat{v}$ have any direction in the plane of the conical fibers $(C, *)$. Thus, $\omega(u, v) = 0$ for any u parallel to $(*, \mathbb{R}^2)$ and any v parallel to $(C, *)$. Given that $\ker(\omega)$ is 2-dimensional, it is easy to see that the kernel of ω is indeed parallel to one of the fibers of $C \times \mathbb{R}^2$. □

Lemma 3.4. *Let $x \in M_s$ be a singularity of codimension 4. Then a sufficiently small neighborhood of this singularity can be factored as a product of two 2-cones, aligned along the two 2-distributions α and β that have been found in section 2.*

The proof was inspired by the proof of [Pan09, Proposition 3.9]. We will need the definitions of the enveloping map and the branching set.

Definition 3.5 (see [Pan09, Definition 3.4]). Let M be a 4-dimensional polyhedral Kähler manifold and let U be the universal cover of $M \setminus M_s$. The enveloping map E of M is defined as a locally isometric map $E : U \rightarrow \mathbb{C}^2$, where \mathbb{C} is the set of complex numbers. Equivalently, this map can be seen as a multi-valued map from M to \mathbb{C}^2 that is locally isometric outside of M_s and has infinite ramification at M_s . The image of M_s under the map is called the branching set $B(E)$ of E ; it is composed of linear holomorphic faces. Note that $B(E)$ is usually everywhere dense in \mathbb{C}^2 , but in the case when $B(E)$ is closed the restriction map $E : E^{-1}(\mathbb{C}^2 \setminus B(E)) \rightarrow \mathbb{C}^2 \setminus B(E)$ is a covering map.

It will sometimes be convenient to write $E(M_s)$ instead of $B(E)$. The remark about linear holomorphic faces significantly relies on Panov’s work [Pan09]: polyhedral Kähler manifolds are complex surfaces, and the singularities have holomorphic direction. The remark about the covering map is easy to prove. The only property to be proved is that E maps U onto $\mathbb{C}^2 \setminus B(E)$. In other words, $E(U) \cup B(E) = \mathbb{C}^2$. Indeed, removing a set of codimension 2 leaves $\mathbb{C}^2 \setminus B(E)$ connected, and E is a local isometry. Technically, if $x \in E(U)$ and $\text{dist}(x, B(E)) > \epsilon$, then the whole ϵ -neighborhood of x is a subset of $E(U)$.

Proof of Lemma 3.4. Focus on a single singularity in $x \in M_s$ that has codimension 4. Since we are only interested in the structure of M locally near x , replace M by its tangent cone at x . This 4-dimensional tangent cone is what Panov calls a polyhedral Kähler cone. It is still a PL-manifold of nonnegative curvature and by definition still a polyhedral Kähler manifold. The 2-distributions α and β can be naturally extended over the whole nonsingular part of the cone by rescaling a neighborhood of x . We can therefore concentrate on factoring this cone, which is

denoted by M until the end of the proof. The notation x is also abandoned, and O is used instead to denote the vertex of the cone. Propositions 3.2 and 3.3 and Lemma 3.1 still apply. M_s is the new singular locus that is technically also a cone, and the definitions of the enveloping map and the branching set still make perfect sense.

We set up the enveloping map in the most convenient way. There is some freedom in the choice of E . Use this freedom to make sure that E makes the images of the 2-distributions α and β parallel to the complex lines in \mathbb{C}^2 : $z_1 = 0$ and $z_2 = 0$ respectively. We denote the “vertical” line $z_1 = 0$ by L_1 and the “horizontal” line $z_2 = 0$ by L_2 . Note also that even though E can be seen as a multi-valued map on M_s , there is clearly only one value $E(O) \in B(E)$, where $O \in M_s$ is the vertex of the cone. Without loss of generality let $E(O)$ be the origin in \mathbb{C}^2 .

It is easy to argue that since M_s is a connected set, the branching set $B(E) = E(M_s)$ has to be connected as well. One can actually prove that it is path connected by a direct argument using Proposition 3.3. Furthermore, $B(E) \setminus \{(0, 0)\} = E(M_s \setminus \{O\})$ may have many connected components, yet each of these components has to be a connected subset of L_1 or L_2 . Since we know that $E(U) \cup B(E) = \mathbb{C}^2$, we can conclude that $L_1 \setminus \{(0, 0)\}$ is entirely contained either in $E(M_s) = B(E)$ or in $E(U)$, because this connected 2-dimensional set cannot be represented as a union of two disjoint open 2-dimensional sets. The same is true for $L_2 \setminus \{(0, 0)\}$.

We conclude that there are four possibilities: the branching set $E(M_s)$ can be $\{(0, 0)\}$, L_1 , L_2 , or $L_1 \cup L_2$. The first one cannot be true by Proposition 3.2. If the second or the third one were true, then all the codimension 2 singularities of M would be “aligned” with the 2-distribution α but none with β , or vice versa. The structure of the link of M at O as a 3-sphere with one singular circle would identify O as a codimension 2 singularity and M as the product of a 2-cone with a Euclidean plane, a contradiction.

Recall the remark from Definition 3.5. The branching set is not everywhere dense in \mathbb{C}^2 , so it is topologically closed. In fact, we already know that the branching set is $L_1 \cup L_2$ and also that E maps U onto $\mathbb{C}^2 \setminus (L_1 \cup L_2)$. This map is a covering map as already explained, and it is a universal covering map since U is simply connected. Therefore, U is isometric to the universal cover of $\mathbb{C}^2 \setminus (L_1 \cup L_2)$, which is a direct metric product of two copies of the universal cover of the pinched plane, $\mathbb{C} \setminus \{0\}$.

This essentially completes the proof. Forget about the enveloping map, and let $\pi : U \rightarrow M \setminus M_s$ be the universal covering map. The deck transformation group is generated by finitely many elements that can be represented by loops around codimension 2 singularities in M_s . Here “going around” means that every such loop is a generator of the fundamental group of a nonsingular neighborhood of the singularity in question, just as in the proof of Proposition 3.3. Indeed, M is a finite simplicial complex, and every element of π can be geometrically represented by a curve in $M \setminus M_s$ that only comes close to codimension 2 singularities. Choose a curve of a special kind so that it consists of small loops around codimension 2 singularities connected by, say, straight segments. Every such loop can be chosen to be tangent to either α or β . Therefore, the deck transformation group respects the product structure of U , and after factoring U by this group we obtain $M \setminus M_s$. Hence, $M \setminus M_s$ is also a direct metric product of two 2-manifolds. Take the metric completion to see that M is also a direct metric product, and the factors are easily shown to be 2-cones. \square

Recall that M is a PL manifold of nonnegative curvature homeomorphic to $S^2 \times S^2$. Now that we have learned to factor all kinds of singularities that M can contain, the 2-distributions α and β can be continued through the singularities to become a priori singular foliations going along the fibers and thus geodesic to the totally geodesic with respect to the local product structure.

Corollary 3.6. *Let M be a PL-manifold of nonnegative curvature, homeomorphic to $S^2 \times S^2$. Then at every point $p \in M$, M can be locally represented in a unique way as a product $C_1 \times C_2$ of two 2-cones with conical angles $2\pi\alpha_1 \leq 2\pi$ and $2\pi\alpha_2 \leq 2\pi$ such that this decomposition is aligned along the 2-distributions α and β that have been found in section 2.*

The 2-distributions α and β can be continued to become totally geodesic singular foliations that are aligned along the local factoring of M at any point, singular or nonsingular.

Proof. The existence follows from Proposition 3.3 and Lemmas 3.1 and 3.4. To prove uniqueness, identify the factors as codimension 2 singularities. \square

4. DECOMPOSING M INTO A DIRECT METRIC PRODUCT

We know from Corollary 3.6 that M is locally a direct metric product. We need to show that M is a direct metric product globally. The main difficulty is to verify that the leaves of the foliations α and β are closed rather than dense in M . Basically, what we need is an appropriate version of the de Rham decomposition theorem. Unfortunately, we are not aware of any sufficiently general formulation that would apply in the PL-case. For this reason an earlier version of this paper did in effect prove a generalized version of the de Rham decomposition theorem. However, that proof required too much work, yet provided only a modest generalization and also relied heavily on the condition of nonnegative curvature. Therefore, we are taking a different approach. We smooth the metric on M to make it a smooth Riemannian manifold, but we keep the foliations α and β unchanged and still totally geodesic with respect to the new metric. After that we can use the classical de Rham decomposition theorem to see that the leaves of the foliations are now and have been originally closed. M is simply connected, and we conclude that the local factorization must be global.

Lemma 4.1. *There exists a smooth Riemannian metric g on M such that the original two foliations α and β are still preserved by the holonomy of g and, furthermore, (M, g) is still locally a direct metric product with the fibers of the product being α and β that are now smooth foliations totally geodesic with respect to the smoothed metric.*

Observation. We first learn to smooth the metric on a 2-cone. Consider a two-dimensional cone C with conical angle $2\pi\alpha$. Let $\epsilon > 0$ be a parameter specifying how much we are willing to change the metric. The metric is $g = \alpha^2 r^2 d\theta^2 + dr^2 = \frac{\alpha^2 x^2 + y^2}{r^2} dx^2 + \frac{x^2 + \alpha^2 y^2}{r^2} dy^2$, which is not smooth at the origin unless $\alpha = 1$. Let $f(r)$ be a smooth function such that $f(r) = r^{\frac{2-2\alpha}{\alpha}}$ for small r . Rescale the metric g to $f(r)g$. After a change of coordinates it is easy to see that $f(r)g$ is flat.

We therefore define $f(r)$ as $r^{\frac{2-2\alpha}{\alpha}}$ for $r < \epsilon/2$, $f(r) = 1$ for $r \geq \epsilon$, and anything in between as long as $f(r)$ is smooth and positive for $r > 0$. $f(r)g$ is a smooth Riemannian metric on the same 2-cone. It coincides with the original metric outside of an ϵ -neighborhood of the origin.

Proof of Lemma 4.1. In order to “smooth” the metric on M we will construct an atlas of charts and specify the new metric using these charts. There will be three kinds of charts: charts containing singularities of codimension 4 and 2 (the first kind), charts containing only singularities of codimension 2 (the second kind), and nonsingular charts (the third kind). M is compact, so we can generalize Corollary 3.6 to obtain uniform size local decompositions at all points of M . That is, we can choose $\epsilon > 0$ sufficiently small so that the 10ϵ -neighborhood of any point in M is isometric to a neighborhood of the origin in a product of two cones, or a product of a cone and a plane, or \mathbb{R}^4 . In particular, the distance between any two singularities of codimension 4 must be at least 10ϵ .

Let $p \in M_s$ be a codimension 4 singularity. Factor a neighborhood of p as $C_1 \times C_2$. Take a 3ϵ -neighborhood of the origin $U_1 \subset C_1$ and a 3ϵ -neighborhood of the origin $U_2 \subset C_2$. Let $U_1 \times U_2$ be a chart centered at p . Smooth the metric in each factor using $f(r)$ as in the observation preceding the proof. Call this kind of chart “a chart of the first kind”. It is important that by the choice of ϵ ($3\sqrt{2} < \frac{10}{2}$) all charts of the first kind are disjoint from one another.

Now let $p \in M_s$ be a codimension 2 singularity that is not covered by a chart of the first kind. In particular, the distance from p to any codimension 4 singularity is at least 3ϵ . Factor a 3ϵ -neighborhood of p as $C \times \mathbb{R}^2$ and take a 2ϵ -neighborhood in each of the factors. Let the product of these 2ϵ -neighborhoods be a chart centered at p . Smooth the metric in the first factor using $f(r)$ as above. Call this kind of chart centered at a codimension 2 singularity “a chart of the second kind”. Such a chart of the second kind should be created for every $p \in M_s$ that is not already covered by charts of the first kind, even if p is already covered by another chart of the second kind.

Lastly, let $p \in M \setminus M_s$ be a nonsingular point that is not covered by a chart of the first kind nor by a chart of the second kind. Take an ϵ -neighborhood of p . It should have a flat metric. Make this neighborhood a chart as it is and keep the flat metric. There is nothing to smooth, as the flat metric is already smooth. This kind of chart is called “a chart of the third kind”.

Now one has to analyze the six possible types of intersections of charts such as the first kind with the second kind, the third kind with the third kind, etc., to see that the smooth metric has been defined consistently. In each of the six cases take the center p of one of the charts and recall that the 10ϵ -neighborhood of p can be factored into a product of two cones. Resolve the intersection of the charts by looking at it within a product of two cones and conclude that the new metric has been defined consistently on all of M .

It is clear that the smoothed metric has the same local product structure as before. Moreover, the fibers are exactly the same when viewed simply as sets of points. The same kind of argument using a 10ϵ -neighborhood product decomposition shows that the original foliations α and β are now smooth foliations still going along the fibers of the product structure and therefore totally geodesic with respect to the smoothed metric. \square

Theorem 4.2. *M with the original polyhedral metric is isometric to two copies of S^2 with some polyhedral metrics.*

Proof. Apply the classical de Rham decomposition theorem to M with the smoothed metric. We obtain a direct metric product of two spheres that are metrically the surfaces of two convex polyhedra with smoothed corners. The foliations go along the fibers of this factoring. Therefore the leaves of the foliations are closed, and were closed under the original metric. After all, these leaves are the same sets of points, and the smoothing of the metric did not affect the topology.

We have represented M as a topological product of two copies of S^2 . M is locally a direct metric product, and the fibers of the metric product are also the fibers of the topological product. Note that if two PL-manifolds are homeomorphic and the homeomorphism f is a local isometry, then f is an isometry. Lastly, $S^2 \times S^2$ is compact. Putting it all together, it is straightforward to complete the proof by showing that all leaves of either foliation are isometric to each other. One first proves this claim for two leaves of α that are close to each other at some point. After that, two arbitrary leaves of α can be shown to be isometric, and the same goes for β . This can be done by a direct geometric argument that basically uses compactness and simply connectedness of M several times. Finally, we do not even need the property of simply connectedness to conclude that M is a direct metric product of one leaf of α with one leaf of β . \square

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