LANGLANDS RECIPROCITY FOR THE EVEN-DIMENSIONAL NONCOMMUTATIVE TORI

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Abstract. We conjecture an explicit formula for the higher-dimensional Dirichlet character; the formula is based on the $K$-theory of the so-called noncommutative tori. It is proved that our conjecture is true for the two-dimensional and one-dimensional (degenerate) noncommutative tori. In the second case, one gets a noncommutative analog of the Artin reciprocity law.

Introduction

The aim of this paper is to bring some evidence in favor of the following analog of the Langlands reciprocity [5]:

**Conjecture 1** (Langlands conjecture for noncommutative tori). Let $K$ be a finite extension of the rational numbers $\mathbb{Q}$ with the Galois group $\text{Gal}(K/\mathbb{Q})$. For an irreducible representation $\sigma_{n+1} : \text{Gal}(K/\mathbb{Q}) \to \text{GL}_{n+1}(\mathbb{C})$, there exists a $2n$-dimensional noncommutative torus with real multiplication, $A_{RM}^{2n}$, such that $L(\sigma_{n+1}, s) \equiv L(A_{RM}^{2n}, s)$, where $L(\sigma_{n+1}, s)$ is the Artin $L$-function and $L(A_{RM}^{2n}, s)$ an $L$-function attached to the $A_{RM}^{2n}$. Moreover, $A_{RM}^{2n}$ is the image of an $n$-dimensional abelian variety $V_n(K)$ under the (generalized) Teichmüller functor $F_n$.

For the notation and terminology we refer the reader to sections 1 and 3; the noncommutative torus $A_{RM}^{2n}$ can be regarded as a substitute of the “automorphic cuspidal representation $\pi_{\sigma_{n+1}}$ of the group $\text{GL}(n+1)$” in terms of the Langlands theory. Roughly speaking, Conjecture 1 says, that the Galois extensions of the field of rational numbers come from the even-dimensional noncommutative tori with real multiplication. Note that the noncommutative tori are intrinsic to the problem, since they classify the irreducible (infinite-dimensional) representations of the Lie group $\text{GL}(n+1)$. Such representations are the heart of the Langlands program [5]. Our conjecture is supported by the following evidence.

**Theorem 1.** Conjecture 1 is true for $n = 1$ (resp., $n = 0$) and the $K$ abelian extension of an imaginary quadratic field $k$ (resp., the rational field $\mathbb{Q}$).

The structure of the paper is as follows. Minimal necessary notation is introduced in section 1, and a brief summary of the Teichmüller functor(s) is given in section 3. Theorem 1 is proved in section 2.

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1. Preliminaries

1.1. Noncommutative tori.

A. The k-dimensional noncommutative torus (4, 12). A noncommutative k-torus is the universal C*-algebra generated by k unitary operators $u_1, \ldots, u_k$. The operators do not commute with each other, but their commutators $u_i u_j u_k^{-1} u_l^{-1}$ are fixed scalar multiples $\exp(2\pi i \theta_{ij})$, $\theta_{ij} \in \mathbb{R}$, of the identity operator. The k-dimensional noncommutative torus, $A^k_\Theta$, is defined by a skew symmetric real matrix $\Theta = (\theta_{ij})$, $1 \leq i, j \leq k$. Further, we think of the $A^k_\Theta$ as a noncommutative topological space whose algebraic K-theory yields $K_0(A^k_\Theta) \cong \mathbb{Z}^{2k-1}$ and $K_1(A^k_\Theta) \cong \mathbb{Z}^{2k-1}$. The canonical trace $\tau$ on the C*-algebra $A^k_\Theta$ defines a homomorphism from $K_0(A^k_\Theta)$ to the real line $\mathbb{R}$. Under the homomorphism, the image of $K_0(A^k_\Theta)$ is a $\mathbb{Z}$-module whose generators $\tau = (\tau_i)$ are polynomials in $\theta_{ij}$. (More precisely, $\tau = \exp(\Theta)$, where the exterior algebra of $\theta_{ij}$ is nilpotent.) Recall that the C*-algebras $A$ and $A'$ are said to be stably isomorphic (Morita equivalent) if $A \otimes K \cong A' \otimes K$ for the C*-algebra $K$ of compact operators. Such an isomorphism indicates that the C*-algebras are homeomorphic as noncommutative topological spaces. By a result of Rieffel and Schwarz [13], the noncommutative tori $A^k_\Theta$ and $A^k_{\Theta'}$ are stably isomorphic if the matrices $\Theta$ and $\Theta'$ belong to the same orbit of a subgroup $SO(k, k \mid \mathbb{Z})$ of the group $GL_{2k}(\mathbb{Z})$, which acts on $\Theta$ by the formula

$$
\Theta' = (A\Theta + B) / (C\Theta + D),
$$

where $(A, B, C, D) \in GL_{2k}(\mathbb{Z})$ and the matrices $A, B, C, D \in GL_k(\mathbb{Z})$ satisfy the conditions

$$
A^t D + C^t B = I, \quad A^t C + C^t A = 0 = B^t D + D^t B.
$$

(Here $I$ is the unit matrix, and $t$ at the upper right of a matrix means a transpose of the matrix.) The group $SO(k, k \mid \mathbb{Z})$ can be equivalently defined as a subgroup of the group $SO(k, k \mid \mathbb{R})$ consisting of linear transformations of the space $\mathbb{R}^{2k}$ which preserve the quadratic form $x_1 x_{k+1} + x_2 x_{k+2} + \cdots + x_k x_{2k}$. The canonical trace $\tau$ on the $\mathbb{C}^k$ unitary operators does not commute with each other, but their commutators $\exp(2\pi i \theta_{ij})$, $\theta_{ij} \in \mathbb{R}$, of the identity operator.

B. The even-dimensional normal tori. Further, we restrict ourselves to the case $k = 2n$ (the even-dimensional noncommutative tori). It is known that by the orthogonal linear transformations every (generic) real even-dimensional skew symmetric matrix can be brought to the normal form

$$
\Theta_0 = \begin{pmatrix}
0 & \theta_1 & & \\
-\theta_1 & 0 & & \\
& & \ddots & \\
& & & 0 & \theta_n \\
& & & -\theta_n & 0
\end{pmatrix},
$$

where $\theta_i > 0$ are linearly independent over $\mathbb{Q}$. We shall consider the noncommutative torus $A^{2n}_{\Theta_0}$, given by the matrix (2); we refer to the family as a normal family. Recall that any noncommutative torus has a canonical trace $\tau$ which defines a homomorphism from $K_0(A^{2n}_{\Theta_0}) \cong \mathbb{Z}^{2n-1}$ to $\mathbb{R}$. It follows from [4] that the image of $K_0(A^{2n}_{\Theta_0})$ under the homomorphism has a basis, given by the formula

$$
\tau(K_0(A^{2n}_{\Theta_0})) = \mathbb{Z} + \theta_1 \mathbb{Z} + \cdots + \theta_n \mathbb{Z} + \sum_{i=n+1}^{2n-1} p_i(\theta) \mathbb{Z},
$$

where $p_i(\theta) \in \mathbb{Z}[1, \theta_1, \ldots, \theta_n]$.

C. The real multiplication (3). The noncommutative torus $A^{2n}_{\Theta_0}$ is said to have a real multiplication if the endomorphism ring $\text{End} (\tau(K_0(A^{2n}_{\Theta_0})))$ exceeds the trivial ring $\mathbb{Z}$. Since any endomorphism of the $\mathbb{Z}$-module $\tau(K_0(A^{2n}_{\Theta_0}))$ is the...
multiplication by a real number, it is easy to deduce that all the entries of $\Theta = (\theta_{ij})$ are algebraic integers. (Indeed, the endomorphism is described by an integer matrix which defines a polynomial equation involving $\theta_{ij}$.) Thus, the noncommutative tori with real multiplication are a countable subset of all $k$-dimensional tori; any element of the set we shall denote by $A_{RM}^k$. Notice that for the even-dimensional noncommutative tori, the polynomials $p_i(\theta)$ produce the algebraic integers in the extension of $\mathbb{Q}$ by $\theta_i$. Any such integer is a linear combination (over $\mathbb{Z}$) of the $\theta_i$. Thus, the trace formula reduces to $\tau(K_0(A_{RM}^{2n})) = \mathbb{Z} + \theta_1\mathbb{Z} + \cdots + \theta_n\mathbb{Z}$.

1.2. $L$-function of noncommutative tori. We consider even-dimensional noncommutative tori with real multiplication. Denote by $A$ a positive integer matrix whose normalized Perron-Frobenius eigenvector coincides with the vector $\theta = (1, \theta_1, \ldots, \theta_n)$ such that $A$ is not a power of a positive integer matrix. In other words, $A\theta = \lambda A\theta$, where $A \in GL_{n+1}(\mathbb{Z})$ and $\lambda A$ is the corresponding eigenvalue. (Explicitly, $A$ can be obtained from vector $\theta$ as the matrix of minimal period of the Jacobi-Perron continued fraction of $\theta$ [2]). Let $p$ be a prime number; take the matrix $A^p$ and consider its characteristic polynomial $char(A^p) = x^{n+1} + a_1x^n + \cdots + a_nx + 1$. We introduce the following notation:

$$L_{p}^{n+1} := \begin{pmatrix} a_1 & a_2 & \cdots & a_n & p \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}. \tag{3}$$

A local zeta function of the $A_{RM}^{2n}$ is defined as the reciprocal of $det(I_{n+1} - L_{p}^{n+1}z)$; in other words,

$$\zeta_p(A_{RM}^{2n}, z) := \frac{1}{1 - a_1z + a_2z^2 - \cdots - a_nz^n + p2^{n+1}}, \quad z \in \mathbb{C}. \tag{4}$$

An $L$-function of $A_{RM}^{2n}$ is a product of the local zeros over all but a finite number, of primes $L(A_{RM}^{2n}, s) = \prod_p \{ trz(A) - (n+1)z \} \zeta_p(A_{RM}^{2n}, p^{-s})$, $s \in \mathbb{C}$.

Remark 1. It will be shown that for $n = 0$ and $n = 1$ formula (4) fits Conjecture [1]. For $n \geq 2$ it is an open problem based on an observation that the crossed product $A_{RM}^{2n} \rtimes L_p^{n+1}$ is a proper noncommutative analog of the (higher-dimensional) Tate module, where the matrix $L_{p}^{n+1}$ corresponds to the Frobenius automorphism of the module [13], p. 172.

2. Proof of Theorem [1]

2.1. Case $n = 1$. Each one-dimensional abelian variety is a nonsingular elliptic curve. Choose this curve to have complex multiplication by (an order in) the imaginary quadratic field $k$ and denote such a curve by $E_{CM}$. Then, by theory of complex multiplication, the (maximal) abelian extension of $k$ coincides with the minimal field of definition of the curve $E_{CM}$, i.e. $E_{CM} \cong E(K)$ [14]. The Teichmüller functor $F := F_1$ maps $E(K)$ into a two-dimensional noncommutative torus with real multiplication (section 3); we shall denote the torus by $A_{RM}^{2}$. To calculate the corresponding $L$-function $L(A_{RM}^{2}, s)$, let $A$ be a $2 \times 2$ positive integer matrix whose normalized Perron-Frobenius eigenvector is $(1, \theta_1)$. For a prime...
Let $E_{CM}$ be an elliptic curve with complex multiplication by an order $R$ in the ring of integers of the imaginary quadratic field $k$. Then $A_{RM} = F(E_{CM})$ is a noncommutative torus with real multiplication by the order $\mathfrak{R}$ in the ring of integers of a real quadratic field $\mathfrak{r}$ (section 3). Let $tr (\alpha) = \alpha + \bar{\alpha}$ be the trace function of a (quadratic) algebraic number field.
Recall that each \( b \) is a real quadratic integer with index \( \xi \). The fundamental units of \( \alpha \) are \( (a + d, 1, -1, 0) \) when \( n = c = -ad \). To calculate \( \alpha \) in \( \mathfrak{R} \) corresponding to \( a \), we apply formula \( (5) \), which gives us

\[
F : \left( \begin{array}{cc} a + d & c - ad \\ 1 & 0 \end{array} \right) \mapsto \left( \begin{array}{cc} a + d & c - ad \\ -1 & 0 \end{array} \right).
\]

In a given base \( \{1, n\} \) of \( \mathbb{Z} + n \mathbb{Z} \), one can write \( \omega \lambda_1 = (a + d) \lambda_1 + (c - ad) \lambda_2 \) and \( \omega \lambda_2 = -\lambda_1 \). It is an easy exercise to verify that \( \omega \) is a real quadratic integer with \( \text{tr} (\omega) = a + d \); the latter coincides with \( \text{tr} (\alpha) \).

Let \( \omega \in \mathfrak{R} \) be an endomorphism of the pseudo-lattice \( \mathbb{Z} + n \mathbb{Z} \) of degree \( \deg (\omega) := \omega \bar{\omega} = n \). The endomorphism maps the pseudo-lattice to a sub-lattice of index \( n \). Any such endomorphism has a form \( \mathbb{Z} + (n \theta) \mathbb{Z} \) \([3]\), p. 131. Let us calculate \( \omega \) in a base \( \{1, n \theta \} \) when it is given by the matrix \( (a + d, c - ad, -1, 0) \). In this case \( n = c = -ad \) and \( \omega \) induces an automorphism \( \omega^* = (a + d, 1, -1, 0) \) of the sublattice \( \mathbb{Z} + (n \theta) \mathbb{Z} \) according to the matrix equation

\[
\begin{pmatrix} a + d & n \\ -1 & 0 \end{pmatrix} \left( \begin{array}{c} 1 \\ n \theta \end{array} \right) = \left( \begin{array}{c} a + d \\ -1 \end{array} \right). 
\]

Thus, one gets a map \( \rho : \mathfrak{R} \to \mathfrak{R}^* \) given by the formula \( \omega = (a + d, n, -1, 0) \mapsto \omega^* = (a + d, 1, -1, 0) \), where \( \mathfrak{R}^* \) is the group of units of \( \mathfrak{R} \). Since \( \text{tr} (\omega^*) = a + d = \text{tr} (\omega) \) and \( \omega^* = \rho (\omega) \), one gets the following:

**Corollary 2.** For all \( \omega \in \mathfrak{R} \), it holds that \( \text{tr} (\omega) = \text{tr} (\rho (\omega)) \).

Note that \( \mathfrak{R}^* = \{ \pm \varepsilon^k \mid k \in \mathbb{Z} \} \), where \( \varepsilon > 1 \) is a fundamental unit of the order \( \mathfrak{R} \subseteq \mathcal{O}_k \). Here \( \mathcal{O}_k \) means the ring of integers of a real quadratic field \( \mathbb{K} = \mathbb{Q} (\theta) \).

Choosing a sign in front of \( \varepsilon^k \), the following index map is defined as \( \iota : \mathfrak{R} \to \mathfrak{R}^* \to \mathbb{Z} \). Let \( \alpha \in \mathfrak{R} \) and \( \deg (\alpha) = -n \). To calculate the \( \iota (\alpha) \), recall some notation from Hasse \([6]\), §16.5.C. Let \( \mathbb{Z} / n \mathbb{Z} \) be a cyclic group of order \( n \). For brevity, let \( I = \mathbb{Z} + n \mathbb{Z} \) be a pseudo-lattice and \( I_n = \mathbb{Z} + (n \theta) \mathbb{Z} \) be its sublattice of index \( n \). The fundamental units of \( I \) and \( I_n \) are \( \varepsilon \) and \( \varepsilon_n \), respectively. By \( \mathfrak{G}_n \) one understands a subgroup of \( \mathbb{Z} / n \mathbb{Z} \) of prime residue classes \( \text{mod} \ n \). The \( \mathfrak{g}_n \subset \mathfrak{G}_n \) is a subgroup of the nonzero divisors of the \( \mathfrak{G}_n \). Finally, let \( g_n \) be the smallest number such that it divides \( |\mathfrak{G}_n / \mathfrak{g}_n| \) and \( \varepsilon^n \in I_n \). (The notation drastically simplifies in the case when \( n = p \) is a prime number.)

**Lemma 4.** \( \iota (\alpha) = g_n \).

**Proof.** Notice that \( \deg (\omega) = -\deg (\alpha) = n \), where \( \omega = F (\alpha) \). Then the map \( \rho \) defines \( I \) and \( I_n \); one can now apply the calculation of \([6]\), pp. 296-300. Namely, Theorem XIII’ on p. 298 yields the required result. (We kept the notation of the original.)

**Corollary 3.** \( \iota (\psi_{E (K)} (\mathfrak{P})) = p \).
This paper has been retracted by the author
2.2. Case \( n = 0 \). When \( n = 0 \), one gets a one-dimensional (degenerate) noncommutative torus. Such an object, \( \mathcal{A}_Q \), can be obtained from the 2-dimensional torus \( \mathcal{A}_Q^2 \) by forcing \( \theta = p/q \in \mathbb{Q} \) to be a rational number (hence our notation). One can always assume \( \theta = 0 \) and, thus, \( \tau(K_0(\mathcal{A}_Q)) = \mathbb{Z} \). To calculate matrix \( L_1^p \), notice that the group of automorphisms of the \( \mathbb{Z} \)-module \( \tau(K_0(\mathcal{A}_Q)) = \mathbb{Z} \) is trivial, i.e., it is a multiplication by \( \pm 1 \); hence our \( 1 \times 1 \) (real) matrix \( A \) is either 1 or \(-1\). Since \( A \) must be positive, we conclude that \( A = 1 \). However, \( A = 1 \) is not a prime matrix if one allows the complex entries. Indeed, for any \( N > 1 \) matrix \( A' = \zeta_N \) gives us \( A = (A')^N \), where \( \zeta_N = e^{2\pi i/N} \) is the \( N \)-th root of unity. Therefore, \( A = \zeta_N \) and \( L_1^p = \text{tr} \ (A^p) = A^p = \zeta_N^p \). A degenerate noncommutative torus, corresponding to the matrix \( A = \zeta_N \), shall be written as \( \mathcal{A}_Q^N \). In turn, such a torus is the image (under the Teichmüller functor) of a zero-dimensional abelian variety, which we denote by \( V_0^N \). Suppose that \( Gal(\mathbb{K}/\mathbb{Q}) \) is abelian and let \( \sigma : Gal(\mathbb{K}/\mathbb{Q}) \to \mathbb{C}^\times \) be a homomorphism. Then, by the Artin reciprocity \([1]\), there exists an integer \( N \) and a Dirichlet character \( \chi_\sigma : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \) such that \( \sigma(F_p) = \chi_\sigma(p) \); choose our zero-dimensional variety to be \( V_0^N \). In view of the notation, \( L_1^p = \zeta_N^p ; \) on the other hand, it is verified directly that \( \zeta_N^p = e^{2\pi i p/N} = \chi_\sigma(p) \). Thus, \( L_1^p = \chi_\sigma(p) \). To obtain a local zeta function, we substitute \( a_1 = L_1^p \) into the formula (4) and obtain

\[
\zeta_p(\mathcal{A}_Q^N,s) = \frac{1}{1 - \chi_\sigma(p)^s},
\]

where \( \chi_\sigma(p) \) is the Dirichlet character. Therefore, \( L(\mathcal{A}_Q^N,s) \equiv L(s,\chi_\sigma) \) is the Dirichlet \( L \)-series. Such a series, by construction, coincides with the Artin \( L \)-series of the representation \( \sigma : Gal(\mathbb{K}/\mathbb{Q}) \to \mathbb{C}^\times \). Case \( n = 0 \) of Theorem 2 follows.

3. Teichmüller functors

Denote by \( \Lambda \) a lattice of rank \( 2n \). Recall that an \( n \)-dimensional (principally polarized) abelian variety, \( V_n \), is the complex torus \( \mathbb{C}^n/\Lambda \), which admits an embedding into a projective space \([9]\).

3.1. Abelian varieties of dimension \( n = 1 \).

A. Basic example. Let \( n = 1 \) and consider the complex torus \( V_1 \cong \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \); it always embeds (via the Weierstrass \( \wp \) function) into a projective space as a non-singular elliptic curve. Let \( \mathbb{H} = \{ \tau = x + iy \in \mathbb{C} \mid y > 0 \} \) be the upper half-plane and \( \partial \mathbb{H} = \{ \theta \in \mathbb{R} \mid y = 0 \} \) its (topological) boundary. We identify \( V_1(\tau) \) with the points of \( \mathbb{H} \) and \( \mathcal{A}_Q^2 \) with the points of \( \partial \mathbb{H} \). Let us show that the boundary is natural. The latter means that the action of the modular group \( SL_2(\mathbb{Z}) \) extends to the boundary, where it coincides with the stable isomorphisms of tori. Indeed, conditions (4) are equivalent to

\[
A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix},
\]

where \( ad - bc = 1 \), \( a, b, c, d \in \mathbb{Z} \) and \( \Theta' = (A\Theta + B)/(C\Theta + D) = (0, a\Theta + b)/(c\Theta + d) \). Therefore, \( \Theta' = (a\Theta + b)/(c\Theta + d) \) for a matrix \( (a, b, c, d) \in SL_2(\mathbb{Z}) \). Thus, the action of \( SL_2(\mathbb{Z}) \) extends to the boundary \( \partial \mathbb{H} \), where it induces stable isomorphisms of the noncommutative tori.

B. The Teichmüller functor \([10]\). There exists a continuous map \( F_1 : \mathbb{H} \to \partial \mathbb{H} \), which sends isomorphic complex tori to the stably isomorphic noncommutative
tori. An exact result is this: Let \( \phi \) be a closed form on the torus whose trajectories define a measured foliation. According to the Hubbard-Masur theorem (applied to the complex tori), this foliation corresponds to a point \( \tau \in \mathbb{H} \). The map \( F_1 : \mathbb{H} \to \partial \mathbb{H} \) is defined by the formula \( \tau \mapsto \theta = \int_{\gamma_2} \phi / \int_{\gamma_1} \phi \), where \( \gamma_1 \) and \( \gamma_2 \) are generators of the first homology of the torus. The following is true: (i) \( \mathbb{H} = \partial \mathbb{H} \times (0, \infty) \) is a trivial fiber bundle whose projection map coincides with \( F_1 \); (ii) \( F_1 \) is a functor which sends isomorphic complex tori to the stably isomorphic noncommutative tori. We shall refer to \( F_1 \) as the Teichmüller functor. Recall that the complex torus \( \mathbb{C}/(\mathbb{Z}+\tau \mathbb{Z}) \) is said to have a complex multiplication if the endomorphism ring of the lattice \( \Lambda = \mathbb{Z}+\tau \mathbb{Z} \) exceeds the trivial ring \( \mathbb{Z} \). The complex multiplication happens if and only if \( \tau \) is an algebraic number in an imaginary quadratic field. The following is true: \( F_1(V_1^{CM}) = A_{RM}^2 \), where \( V_1^{CM} \) is a torus with complex multiplication.

3.2. Abelian varieties of dimension \( n \geq 1 \).

A. The Siegel upper half-space \((\mathbb{H}_n)\). The space

\[
\mathbb{H}_n := \{ \tau = (\tau_i) \in \mathbb{C}^{n(n+1)} \mid \text{Im} (\tau_i) > 0 \}
\]

of symmetric \( n \times n \) matrices with complex entries is called a Siegel upper half-space. The points of \( \mathbb{H}_n \) are one-to-one with the \( n \)-dimensional principally polarized abelian varieties. Let \( Sp(2n, \mathbb{R}) \) be the symplectic group. It acts on \( \mathbb{H}_n \) by the linear fractional transformations \( \tau \mapsto \tau' = (a\tau + b)/(c\tau + d) \), where \( (a, b, c, d) \in Sp(2n, \mathbb{R}) \) and \( a, b, c, d \) are the \( n \times n \) matrices with real entries. The abelian varieties \( V_n \) and \( V'_n \) are isomorphic if and only if \( \tau \) and \( \tau' \) belong to the same orbit of the group \( Sp(2n, \mathbb{Z}) \); the action is discontinuous on \( \mathbb{H}_n \). Denote by \( \Sigma_{2n} \) a space of the \( 2n \)-dimensional normal noncommutative tori. The following lemma is critical.

**Lemma 7.** \( Sp(2n, \mathbb{R}) \subseteq O(n, n|\mathbb{R}) \).

**Proof.** (i) The group \( O(n, n|\mathbb{R}) \) can be defined as a subgroup of \( GL_2(\mathbb{R}) \) which preserves the quadratic form \( f(x_1, \ldots, x_{2n}) = x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n} \). We shall denote \( u_i = x_1 \) for \( 1 \leq i \leq n \) and \( v_i = x_i \) for \( n+1 \leq i \leq 2n \). Consider the following skew symmetric bilinear form \( q(u, v) = u_1v_{n+1} + \cdots + u_nv_{2n} - u_{n+1}v_1 - \cdots - u_{2n}v_n \), where \( u, v \in \mathbb{R}^{2n} \). It is known that each linear substitution \( g \in Sp(2n, \mathbb{R}) \) preserves the form \( q(u, v) \). Since \( q(u, v) = f(x_1, \ldots, x_{2n}) - u_{n+1}v_1 - \cdots - u_{2n}v_n \), one concludes that \( g \) also preserves the form \( f(x_1, \ldots, x_{2n}) \), i.e. \( g \in O(n, n|\mathbb{R}) \). It is easy to see that the inclusion is proper except in the case \( n = 1 \), i.e. when \( Sp(2, \mathbb{R}) \cong O(1, 1|\mathbb{R}) \cong SL_2(\mathbb{R}) \). The lemma follows.

(ii) We wish to give a second proof of this important fact, which is based on the explicit formulas for the block matrices \( A, B, C \) and \( D \). The fact that a symplectic linear transformation preserves the skew symmetric bilinear form \( q(u, v) \) can be written in a matrix form:

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^t \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right),
\]

where \( t \) is the transpose of a matrix. Performing the matrix multiplication, one gets the matrix identities \( ab - cd = I \), \( ac - db = 0 = b^t + c^t \). Let us show that these identities imply the Rieffel-Schwarz identities imposed on the matrices.
A, B, C and D. Indeed, in view of the formulas (11), the Rieffel-Schwarz identities can be written as

\[
\begin{bmatrix}
 a^t & 0 \\
 0 & a^t
\end{bmatrix}
\begin{bmatrix}
 d & 0 \\
 0 & d
\end{bmatrix}
+ \begin{bmatrix}
 0 & c^t \\
 -c^t & 0
\end{bmatrix}
\begin{bmatrix}
 0 & b \\
 -b & 0
\end{bmatrix}
= \begin{bmatrix}
 I & 0 \\
 0 & I
\end{bmatrix}.
\]

(13)

A step-by-step matrix multiplication in (13) shows that the identities \(a^t d - c^t b = I,\ a^t c^t - c^t a = 0 = b^t d - d^t b\) imply the identities (13). (Beware: the operation is not commutative.) Thus, any symplectic transformation satisfies the Rieffel-Schwarz identities, i.e. belongs to the group \(O(n,n;\mathbb{R})\). Lemma 7 follows. \(\square\)

B. The generalized Teichmüller functors. By Lemma 7, the action of \(Sp(2n,\mathbb{Z})\) on the \(\mathbb{H}_n\) extends to the \(\Sigma_{2n}\), where it acts by stable isomorphisms of the noncommutative tori. Thus, \(\Sigma_{2n}\) is a natural boundary of the Siegel upper half-space \(\mathbb{H}_n\). However, unless \(n = 1\), the \(\Sigma_{2n}\) is not a topological boundary of \(\mathbb{H}_n\). Indeed, \(dim_{\mathbb{Z}}(\mathbb{H}_n) = n(n+1)\) and \(dim_{\mathbb{R}}(\mathbb{H}_n) = n^2 + n - 1\), while \(dim_{\mathbb{Z}}(\Sigma_{2n}) = n\). Thus, \(\Sigma_{2n}\) is an \(n\)-dimensional subspace of the topological boundary of \(\mathbb{H}_n\). This subspace is everywhere dense in \(\partial \mathbb{H}_n\), since the \(Sp(2n,\mathbb{Z})\)-orbit of an element of \(\Sigma_{2n}\) is everywhere dense in \(\partial \mathbb{H}_n\) (13). A (conjectural) continuous map \(F_n : \mathbb{H}_n \to \Sigma_{2n}\) shall be called a **generalized Teichmüller functor**. The \(F_n\) has the following properties: (i) it sends each pair of isomorphic abelian varieties to a pair of the stably isomorphic even-dimensional normal tori; (ii) the range of \(F_n\) on the abelian varieties with complex multiplication consists of the noncommutative tori with real multiplication. As explained, such a functor has been constructed only in the case \(n = 1\). The difficulties in higher dimensions are due to the lack (so far) of a proper Teichmüller theory for the abelian varieties of dimension \(n \geq 2\).

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REFERENCES

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