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EMBEDDING DENDRIFORM ALGEBRA INTO ITS UNIVERSAL ENVELOPING ROTA-BAXTER ALGEBRA

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Abstract. In this paper, by using Gröbner-Shirshov bases for Rota-Baxter algebras, we prove that every dendriform algebra over a field of characteristic 0 can be embedded into its universal enveloping Rota-Baxter algebra.

1. Introduction

Let F be a field. A dendriform algebra (see [10]) is an F-module D with two binary operations $\prec$ and $\succ$ such that for any $x, y, z \in D$,

$$\begin{aligned} &(x \prec y) \prec z = x \prec (y \prec z + y \succ z) \\ &(x \succ y) \prec z = x \succ (y \prec z) \\ &(x \prec y + x \succ y) \succ z = x \succ (y \succ z). \end{aligned} \tag{1.1}$$

Let A be an associative algebra over F. Let an F-linear operator $P : A \to A$ satisfy the Rota-Baxter identity

$$P(x)P(y) = P(P(x)y) + P(xP(y)). \tag{1.2}$$

Then A is called a Rota-Baxter algebra.

The free Rota-Baxter algebra generated by a nonempty set X, denoted by $RB(X)$, was given by K. Ebrahimi-Fard and L. Guo [7] and the free dendriform algebra generated by X, denoted by $D(X)$, was first made explicit by J.-L. Loday in [10].

Suppose that $(D, \prec, \succ)$ is a dendriform algebra over F with a linear basis $X = \{x_i | i \in I\}$. Let $x_i \prec x_j = \{x_i \prec x_j\}, x_i \succ x_j = \{x_i \succ x_j\}$, where $\{x_i \prec x_j\}$ and $\{x_i \succ x_j\}$ are linear combinations of $x \in X$. Then D has an expression by generators and defining relations

$$D = D(X | x_i \prec x_j = \{x_i \prec x_j\}, x_i \succ x_j = \{x_i \succ x_j\}, x_i, x_j \in X).$$

Denote by

$$U(D) = RB(X | x_iP(x_j) = \{x_i \prec x_j\}, P(x_i)x_j = \{x_i \succ x_j\}, x_i, x_j \in X).$$

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Then $U(D)$ is the universal enveloping Rota-Baxter algebra of D; see [7].

The study of Rota-Baxter algebras originated from the probability study of Glenn Baxter in 1960 and was developed further by Cartier and the school of Rota in the 1960s and 1970s. This structure appeared also in the Lie algebra context as the operator form of the classical Yang-Baxter equation started in the 1980s. Since then, Rota-Baxter algebra has experienced a quite remarkable renaissance and found important theoretical developments and applications in mathematical physics, operads, number theory and combinatorics; see, for example, [1, 3, 5, 6, 8, 13].

The dendriform algebra was introduced by J.-L. Loday [10] in 1995 with motivation from algebraic K-theory, and was further studied in connection with several areas in mathematics and physics, including operads, homology, Hopf algebras, Lie and Leibniz algebras, combinatorics, arithmetic and quantum field theory; see [7, 11].

In the theory of Lie algebras, the Poincaré-Birkhoff-Witt theorem (see [12], frequently contracted to the PBW theorem) is a fundamental result giving an explicit description of the universal enveloping algebra of a Lie algebra. The term "PBW type theorem" or even "PBW theorem" may also refer to various analogues of the original theorem. I. P. Shestakov [14] proved that an Akivis algebra can be embedded into its universal enveloping non-associative algebra. M. Aymon and P.-P. Grivel [2] proved that a Leibniz algebra can be embedded into its universal enveloping diassociative algebra. P. S. Kolesnikov [9] proved that every (finite dimensional) Leibniz algebra can be embedded into current conformal algebra over the algebra of linear transformations of a (finite dimensional) linear space. As a corollary, a new proof of the theorem on injective embeddings of a Leibniz algebra into an diassociative algebra is obtained and, more explicitly, an analogue of the PBW theorem for Leibniz algebras in [10].

In this paper we study the functor from Rota-Baxter algebras to dendriform algebras given by the formulas $x \prec y := xP(y),\ x \succ y := P(x)y$. The identities defining a dendriform algebra are a consequence of the associativity of the product in a Rota-Baxter algebra and of the Rota-Baxter identity $P(x)P(y) = xP(y) + P(x)y$, which reads $P(x)P(y) = P(x \prec y + x \succ y)$. This functor is a forgetful functor; hence it admits a left adjoint, denoted by U, and given, as usual, by the quotient of the free Rota-Baxter algebra by the relations which identify the two dendriform structures. From the universal property of the enveloping functor U it comes immediately that there is a natural map $D \to U(D)$ for any dendriform algebra D. It is the unit of the adjunction. Similar to a classical problem involving associative and Lie algebras, L. Guo posts the following conjecture: each dendriform algebra can be embedded into its universal enveloping Rota-Baxter algebra; i.e., the map $D \to U(D)$ is injective.

In this paper, we prove the following theorem.

Theorem 1.1. *Every dendriform algebra over a field of characteristic 0 can be embedded into its universal enveloping Rota-Baxter algebra. In other words, such a dendriform algebra is isomorphic to a dendriform subalgebra of a Rota-Baxter algebra.*

The Composition-Diamond lemma for Rota-Baxter algebras was established by L. A. Bokut, Yuqun Chen and Xueming Deng in a recent paper [4]. We will use this lemma to prove the above theorem.

2. Preliminaries

In this section, we introduce some notations which are related to Gröbner-Shirshov bases for Rota-Baxter algebras; see [4].

Let X be a nonempty set, $S(X)$ the free semigroup generated by X without identity and P a symbol of a unary operation. For any two nonempty sets Y and Z, denote by

$$\begin{aligned}\Lambda_P(Y,Z) = &(\bigcup_{r\geq 0}(YP(Z))^rY) \cup (\bigcup_{r\geq 1}(YP(Z))^r)\\ &\cup (\bigcup_{r\geq 0}(P(Z)Y)^rP(Z)) \cup (\bigcup_{r\geq 1}(P(Z)Y)^r),\end{aligned}$$

where for a set T, $T^r = \{t_1\cdots t_r | t_i \in T,\ 1 \leq i \leq r\}$ and T^0 means the empty set.

Define

$$\begin{array}{rcl}\Phi_0 & = & S(X)\\ \vdots & & \vdots\\ \Phi_n & = & \Lambda_P(\Phi_0, \Phi_{n-1})\\ \vdots & & \vdots\end{array}$$

Then

$$\Phi_0 \subset \cdots \subset \Phi_n \subset \cdots .$$

Let

$$\Phi(X) = \bigcup_{n\geq 0} \Phi_n.$$

Clearly, $P(\Phi(X)) \subset \Phi(X)$. If $u \in X \cup P(\Phi(X))$, then u is called prime. For any $u \in \Phi(X)$, u has a unique form $u = u_1u_2\cdots u_n$ where u_i is prime, $i = 1, 2, \ldots, n$, and u_i, u_{i+1} cannot both have forms as $P(u'_i)$ and $P(u'_{i+1})$.

For any $u \in \Phi(X)$ and for a set $T \subseteq X \cup \{P\}$, denote by $deg_T(u)$ the number of occurrences of $t \in T$ in u. Let

$$Deg(u) = (deg_{\{P\}\cup X}(u), deg_{\{P\}}(u)).$$

We order $Deg(u)$ lexicographically.

In the following, we always assume that F is a field of characteristic 0.

Let $F\Phi(X)$ be a free F-module with F-basis $\Phi(X)$. Extend linearly

$$P : F\Phi(X) \to F\Phi(X),\ u \mapsto P(u),$$

where $u \in \Phi(X)$.

Now we define the multiplication in $F\Phi(X)$.

Firstly, for $u, v \in X \cup P(\Phi(X))$, define

$$u \cdot v = \begin{cases} P(P(u')\cdot v') + P(u' \cdot P(v')), & \text{if } u = P(u'), v = P(v');\\ uv, & \text{otherwise.}\end{cases}$$

Secondly, for any $u = u_1u_2\cdots u_s, v = v_1v_2\cdots v_l \in \Phi(X)$, where u_i, v_j are prime, $i = 1, 2, \ldots, s, j = 1, 2, \ldots, l$, define

$$u \cdot v = u_1u_2\cdots u_{s-1}(u_s \cdot v_1)v_2\cdots v_l.$$

Equipped with the above concepts, $F\Phi(X)$ is the free Rota-Baxter algebra generated by X; see [7].

We denote by $RB(X)$ the free Rota-Baxter algebra generated by X.

Let N^+ be the set of positive integers.

Let the notation be as before. We have to order $\Phi(X)$. Let X be a well-ordered set. Let us define an ordering $>$ on $\Phi(X)$ by induction on the Deg-function.

For any $u, v \in \Phi(X)$, if $Deg(u) > Deg(v)$, then $u > v$. If $Deg(u) = Deg(v) = (n, m)$, then we define $u > v$ by induction on (n, m).

If $(n, m) = (1, 0)$, then $u, v \in X$ and we use the ordering on X. Suppose that for (n, m) the ordering is defined where $(n, m) \geq (1, 0)$. Let $(n, m) < (n', m') = Deg(u) = Deg(v)$. If $u, v \in P(\Phi(X))$, say $u = P(u')$ and $v = P(v')$, then $u > v$ if and only if $u' > v'$ by induction. Otherwise $u = u_1u_2\cdots u_l$ and $v = v_1v_2\cdots v_s$ where $l > 1$ or $s > 1$, and $u > v$ if and only if $(u_1, u_2, \ldots, u_l) > (v_1, v_2, \ldots, v_s)$ lexicographically by induction.

It is clear that $>$ is a well ordering on $\Phi(X)$; see [4]. Throughout this paper, we will use this ordering.

Let $\star$ be a symbol and $\star \notin X$. By a $\star$-Rota-Baxter word we mean any expression in $\Phi(X \cup \{\star\})$ with only one occurrence of $\star$. The set of all $\star$-Rota-Baxter words on X is denoted by $\Phi^{\star}(X)$.

Let u be a $\star$-Rota-Baxter word and $s \in RB(X)$. Then we call

$$u|_s = u|_{\star \mapsto s}$$

an s-Rota-Baxter word. For short, we call $u|_s$ an s-word.

Note that the ordering $>$ is monomial in the sense that for any $u, v \in \Phi(X)$, $w \in \Phi^{\star}(X)$,

$$u > v \Longrightarrow \overline{w|_u} > \overline{w|_v},$$

where $w|_u = w|_{\star \mapsto u}$ and $w|_v = w|_{\star \mapsto v}$; see [4], Lemma 3.4.

If $\overline{u|_s} = u|_{\overline{s}}$, then we call $u|_s$ a normal s-word.

Now, for any $0 \neq f \in RB(X)$, f has the leading term $\bar{f}$ and $f = \alpha_1\bar{f} + \sum_{i=2}^{n} \alpha_i u_i$, where $\bar{f}, u_i \in \Phi(X), \bar{f} > u_i, 0 \neq \alpha_1, \alpha_i \in F$. Denote by $lc(f)$ the coefficient of the leading term $\bar{f}$. If $lc(f) = 1$, then we call f monic.

Let $f, g \in RB(X)$ be monic with $\overline{f} = u_1u_2\cdots u_n$ where each u_i is prime. Then, there are four kinds of compositions:

(1) If $u_n \in P(\Phi(X))$, then we define the composition of right multiplication as $f \cdot u$, where $u \in P(\Phi(X))$.

(2) If $u_1 \in P(\Phi(X))$, then we define the composition of left multiplication as $u \cdot f$, where $u \in P(\Phi(X))$.

(3) If there exists a $w = \overline{f}a = b\overline{g}$ where fa is a normal f-word and bg is a normal g-word, $a, b \in \Phi(X)$ and $deg_{\{P\}\cup X}(w) < deg_{\{P\}\cup X}(\overline{f}) + deg_{\{P\}\cup X}(\overline{g})$, then we define the intersection composition of f and g with respect to w as $(f, g)_w = f \cdot a - b \cdot g$.

(4) If there exists a $w = \overline{f} = u|_{\overline{g}}$ where $u \in \Phi^{\star}(X)$, then we define the inclusion composition of f and g with respect to w as $(f, g)_w = f - u|_g$.

We call w in $(f, g)_w$ the ambiguity with respect to f and g.

Let $S \subset RB(X)$ be a set of monic polynomials. Then the composition h is called trivial modulo (S, w), denoted by $h \equiv 0\, mod(S, w)$, if

$$h = \sum_i \alpha_i u_i|_{s_i},$$

where each $\alpha_i \in F$, $s_i \in S$, $u_i|_{s_i}$ is a normal s_i-word and $u_i|_{\overline{s_i}} < w$ ($u_i|_{\overline{s_i}} \leq \bar{h}$ if h is a composition of left (right) multiplication).

In general, for any two polynomials p and q, $p \equiv q\, mod(S, w)$ means that $p - q \equiv 0\, mod(S, w)$.

The set S is called a Gröbner-Shirshov basis in $RB(X)$ if each composition is trivial modulo S and corresponding w.

Theorem 2.1 (Composition-Diamond lemma for Rota-Baxter algebras [4])**.** *Let $RB(X)$ be a free Rota-Baxter algebra over a field of characteristic 0 and let S be a set of monic polynomials in $RB(X)$, $>$ the monomial ordering on $\Phi(X)$ defined as before and $Id(S)$ the Rota-Baxter ideal of $RB(X)$ generated by S. Then the following statements are equivalent:*

(1) *S is a Gröbner-Shirshov basis in $RB(X)$.*
(2) *$f \in Id(S) \Rightarrow \bar{f} = u|_{\bar{s}}$ for some $u \in \Phi^{\star}(X)$, $s \in S$.*
(3) *$Irr(S) = \{u \in \Phi(X) | u \neq v|_{\bar{s}}, s \in S, v|_s$ is a normal s-word$\}$ is an F-basis of $RB(X|S) = RB(X)/Id(S)$.*

If a subset S of $RB(X)$ is not a Gröbner-Shirshov basis, then one can add all nontrivial compositions of polynomials of S to S. Continuing this process repeatedly, we finally obtain a Gröbner-Shirshov basis S^{comp} that contains S. Such a process is called the Shirshov algorithm.

3. The proof of Theorem 1.1

In this section, we assume that $RB(X)$ is the free Rota-Baxter algebra generated by $X = \{x_i \mid i \in I\}$.

Lemma 3.1. *For any $u, v \in \Phi(X)$, we have $\overline{P(u)P(v)} = max\{\overline{P(P(u)v)}, \overline{P(uP(v))}\}$.*

Proof. By Rota-Baxter formula (1.2), we may assume that $P(P(u)v) = \sum n_i u_i$, $P(uP(v)) = \sum m_j v_j$, where $n_i, m_j \in N^+$, $u_i, v_j \in \Phi(X)$. Since the characteristic of F is 0, the result follows. □

Denote by

$$F_1 = \{x_i P(x_j) - \{x_i \prec x_j\} \mid i, j \in I\},$$
$$F_2 = \{P(x_i) x_j - \{x_i \succ x_j\} \mid i, j \in I\},$$
$$Irr(F_1 \cup F_2) = \{u \in \Phi(X) \mid u \neq v|_{\bar{s}}, s \in F_1 \cup F_2, v|_s \text{ is a normal } s\text{-word}\},$$
$$\Phi_1(X) = \Phi(X) \cap Irr(F_1 \cup F_2).$$

For a polynomial $f = \sum_{i=1}^{n} \alpha_i u_i \in RB(X)$, where each $0 \neq \alpha_i \in F$, $u_i \in \Phi(X)$, denote the set $\{u_i,\ 1 \leq i \leq n\}$ by $supp(f)$.

Lemma 3.2. 1) *Let $f = P(x_i)u$, $g = vP(x_j)$, where $i, j \in I$, $u, v \in \Phi_1(X) \setminus X$. Then $f \equiv \sum \alpha_i u_i\ mod(F_1 \cup F_2, \overline{f})$ and $g \equiv \sum \beta_i v_i\ mod(F_1 \cup F_2, \overline{g})$, where for any i, $\alpha_i, \beta_i \in F$, $u_i, v_i \in \Phi_1(X) \setminus X$.*

2) *Let $f = P(u)P(v)$, $g = P(v')P(u')$, where $u, u' \in \Phi_1(X) \setminus X$, $v, v' \in \Phi_1(X)$. Then $f \equiv \sum \alpha_i P(u_i)\ mod(F_1 \cup F_2, \overline{f})$ and $g \equiv \sum \beta_i P(v_i)\ mod(F_1 \cup F_2, \overline{g})$, where for any i, $\alpha_i, \beta_i \in F$, $u_i, v_i \in \Phi_1(X) \setminus X$.*

Proof. 1) We prove only the case when $f = P(x_i)u \equiv \sum \alpha_i u_i\ mod(F_1 \cup F_2, \overline{f})$. The other case is similar.

We use induction on $n = deg_{\{P\} \cup X}(u)$. Since $u \in \Phi_1(X) \setminus X$, we have $n \geq 2$.

Assume that $n = 2$. Then either $u = x_jx_k$ or $u = P(x)$, $x_j, x_k, x \in X$. If $u = x_jx_k$, then we have $f = P(x_i)x_jx_k \equiv \{x_i \succ x_j\}x_k\ mod(F_1 \cup F_2, \overline{f})$, and $supp(\{x_i \succ x_j\}x_k) \subset \Phi_1(X) \setminus X$. If $u = P(x)$, then $f = P(x_i)P(x) = P(P(x_i)x) + P(x_iP(x)) \equiv P(\{x_i \succ x\}) + P(\{x_i \prec x\})\ mod(F_1 \cup F_2, \overline{f})$, and $supp(P(\{x_i \succ x\})), supp(P(\{x_i \prec x\})) \subset \Phi_1(X) \setminus X$.

For $n > 2$, there are three cases to consider.

(1) $u = x_ju_1,\ x_j \in X$. Then there are two subcases to consider.

  (a) $u_1 = x_ku_2,\ x_k \in X$. Then $f = P(x_i)u \equiv \{x_i \succ x_j\}x_ku_2\ \ mod(F_1 \cup F_2, \overline{f})$, where $supp(\{x_i \succ x_j\}x_ku_2) \subset \Phi_1(X) \setminus X$.

  (b) $u_1 = P(v)u_2$. Since $u \in \Phi_1(X) \setminus X$, we get that $v \notin X$. Thus, $f = P(x_i)u \equiv \{x_i \succ x_j\}P(v)u_2\ mod(F_1 \cup F_2, \overline{f})$, where $supp(\{x_i \succ x_j\}P(v)u_2) \subset \Phi_1(X) \setminus X$.

(2) $u = P(u_1)$. Then

$$f = P(x_i)u = P(x_i)P(u_1) = P(P(x_i)u_1) + P(x_iP(u_1)).$$

Let

$$P(x_i)u_1 \equiv \sum \gamma_i w_i\ \ mod(F_1 \cup F_2, \overline{P(x_i)u_1})$$

and

$$x_iP(u_1) \equiv \sum \gamma_i' w_i'\ \ mod(F_1 \cup F_2, \overline{x_iP(u_1)}),$$

where all $w_i, w_i' \in \Phi_1(X)$. By using Lemma 3.1, $\overline{f} = \overline{P(x_i)P(u_1)} \geq \overline{P(P(x_i)u_1)} = P(\overline{P(x_i)u_1}) \geq P(w_j)$ and similarly, $\overline{f} \geq P(w_j')$ for any j, j'. Then $f \equiv \sum \gamma_iP(w_i) + \sum \gamma_i'P(w_i')\ mod(F_1 \cup F_2, \overline{f})$, where $P(w_i), P(w_i') \in \Phi_1(X) \setminus X$.

(3) $u = P(u_1)u_2$, where u_2 is not empty. Then $u_1 \notin X$, and $u_2 = x_ju_3$ for some $x_j \in X$ since $u \in \Phi_1(X) \setminus X$. Therefore, $f = P(x_i)u = P(x_i)P(u_1)u_2 = P(P(x_i)u_1)u_2 + P(x_iP(u_1))u_2$. For $P(x_iP(u_1))u_2$, we have $P(x_iP(u_1))u_2 \in \Phi_1(X) \setminus X$. For $P(P(x_i)u_1)u_2$, since $u_1 \notin X$, by induction on n, we get that $P(x_i)u_1 \equiv \sum \gamma_iv_i\ mod(F_1 \cup F_2, \overline{P(x_i)u_1})$, where $v_i \in \Phi_1(X) \setminus X$. By using Lemma 3.1, $\overline{f} = \overline{P(x_i)P(u_1)u_2} \geq \overline{P(P(x_i)u_1)u_2} = P(\overline{P(x_i)u_1})u_2 \geq P(v_i)u_2$. As a result, $P(P(x_i)u_1)u_2 \equiv \sum \gamma_iP(v_i)u_2\ mod(F_1 \cup F_2, \overline{f})$ and $P(v_i)u_2 \in \Phi_1(X) \setminus X$.

2) We only prove the case $f = P(u)P(v)$. The other case is proved similarly.

We use induction on $n = deg_{\{P\}\cup X}(P(u)P(v))$. Since $u \in \Phi_1(X) \setminus X$, we have $n \geq 5$.

Assume that $n = 5$. Then either $u = x_ix_j$ and $v = x$ or $u = P(x_i)$ and $v = x$, where $x_i, x_j, x \in X$.

If $u = x_ix_j$ and $v = x$, then we have $f = P(u)P(v) = P(x_ix_j)P(x) = P(P(x_ix_j)x) + P(x_ix_jP(x)) \equiv P(P(x_ix_j)x) + P(x_i\{x_j \prec x\})\ mod(F_1 \cup F_2, \overline{f})$, and $(\{P(x_ix_j)x\} \cup supp(x_i\{x_j \prec x\})) \subset \Phi_1(X) \setminus X$.

If $u = P(x_i)$ and $v = x$, then $f = P(u)P(v) = P(P(x_i))P(x) = P(P(P(x_i))x) + P(P(x_i)P(x)) = P(P(P(x_i))x) + P(P(P(x_i)x)) + P(P(x_iP(x))) \equiv P(P(P(x_i))x) + P(P(\{x_i \succ x\})) + P(P(\{x_i \prec x\}))\ mod(F_1 \cup F_2, \overline{f})$, and $(\{P(P(x_i))x\} \cup supp(P(\{x_i \succ x\})) \cup supp(P(\{x_i \prec x\}))) \subset \Phi_1(X) \setminus X$.

For $n > 5$, since $f = P(u)P(v) = P(uP(v)) + P(P(u)v)$ and by Lemma 3.1, it is sufficient to prove that $P(uP(v)) \equiv \sum \alpha_iP(u_i)\ mod(F_1 \cup F_2, \overline{P(uP(v))})$, $P(P(u)v) \equiv \sum \alpha_iP(v_i)\ mod(F_1 \cup F_2, \overline{P(P(u)v)})$, where $u_i, v_i \in \Phi_1(X) \setminus X$.

For $P(uP(v))$, there are two cases to consider.

(1) $u = u_1 x_i,\ x_i \in X$. Then there are two subcases to consider.
   (a) $v \notin X$. Then $P(uP(v)) = P(u_1 x_i P(v))$ and $u_1 x_i P(v) \in \Phi_1(X) \setminus X$.
   (b) $v = x_j \in X$. Then $P(uP(v)) = P(u_1 x_i P(x_j)) \equiv P(u_1\{x_i \prec x_j\})$. If $u_1 = u_2 x$ for some $x \in X$, then $P(uP(v)) \equiv P(u_2 x\{x_i \prec x_j\})$ where $supp(u_2 x\{x_i \prec x_j\}) \subset \Phi_1(X) \setminus X$. If $u_1 = u_2 P(u_3)$, then $u = u_1 x_i = u_2 P(u_3) x_i$ and $u_3 \notin X$. Then $P(uP(v)) \equiv P(u_1\{x_i \prec x_j\}) \equiv P(u_2 P(u_3)\{x_i \prec x_j\})$ where $supp(u_2 P(u_3)\{x_i \prec x_j\}) \subset \Phi_1(X) \setminus X$.

(2) $u = u_1 P(u_2)$. Then there are two subcases to consider.
   (a) $u_2 = x_i \in X$. Since $u \in \Phi_1(X) \setminus X$, we have $u = P(x_i)$. As a result, $P(uP(v)) = P(P(x_i)P(v))$. Since $P(v) \notin X$, the result follows from 1).
   (b) $u_2 \notin X$. Then $u_2 \in \Phi_1(X) \setminus X$ and $P(uP(v)) = P(u_1 P(u_2) P(v))$. By induction on n, $P(u_2)P(v) \equiv \sum \alpha_i P(v_i) \quad mod(F_1 \cup F_2, \overline{P(u_2)P(v)})$, where $v_i \in \Phi_1(X) \setminus X$. Then $P(uP(v)) = P(u_1 P(u_2) P(v)) \equiv \sum \alpha_i P(u_1 P(v_i))$ and $u_1 P(v_i) \in \Phi_1(X) \setminus X$.

For $P(P(u)v)$, there are also two cases to consider.

(1) $v = x_i v_1,\ x_i \in X$. Then $P(P(u)v) = P(P(u) x_i v_1)$ and $P(u) x_i v_1 \in \Phi_1(X) \setminus X$.
(2) $v = P(v_1) v_2$. Then $P(P(u)v) = P(P(u)P(v_1)v_2)$ with $v_1 \in \Phi_1(X)$. By induction on n, we get that $P(u)P(v_1) \equiv \sum \alpha_i P(u_i)\, mod(F_1 \cup F_2, \overline{P(u)P(v_1)})$, where $u_i \in \Phi_1(X) \setminus X$. Then $P(P(u)v) \equiv \sum \alpha_i P(P(u_i) v_2)$ and $P(u_i) v_2 \in \Phi_1(X) \setminus X$.

The proof is complete. □

Lemma 3.3. *Let $S = F_1 \cup F_2 \cup F_3$, where*

$$\begin{aligned} F_3 \quad = \quad & \{u_0 P(v_1) u_1 \cdots P(v_n) u_n \mid u_0, u_n \in X^*,\ u_i \in X^* \backslash \{1\}, 1 \le i < n, \\ & v_j \in \Phi_1(X) \setminus X,\ 1 \le j \le n;\ |u_0| \ge 2 \quad if \quad n = 0\}, \end{aligned}$$

for any $u \in X^*$, $|u|$ is the length of u, and X^* is the free monoid generated by X.

Then S is a Gröbner-Shirshov basis in $RB(X)$.

Proof. The ambiguities of all possible compositions of the polynomials in S are listed below:

$f_1 \wedge f_2$**:** $f_1 \in F_1,\ f_2 \in F_2$, and $w = x_i P(x_j) x_k,\ i, j, k \in I$.
$f_2 \wedge f_1$**:** $f_1 \in F_1,\ f_2 \in F_2$, and $w = P(x_i) x_j P(x_k),\ i, j, k \in I$.
$f_2 \wedge f_3$**:** $f_2 \in F_2,\ f_3 \in F_3$, and $w = P(x_i) x_j u_0 P(v_1) u_1 \cdots P(v_n) u_n,\ u_0, u_n \in X^*,\ u_k \in X^* \backslash \{1\},\ v_l \in \Phi_1(X) \setminus X,\ i, j \in I,\ n \ge 0,\ 1 \le k < n,\ 1 \le l \le n$. When $n = 0$, $|u_0^{(1)}| \ge 1$.
$f_3 \wedge f_1$**:** $f_1 \in F_1,\ f_3 \in F_3$, and $w = u_0 P(v_1) u_1 \cdots P(v_n) u_n x_i P(x_j),\ u_0, u_n \in X^*,\ u_k \in X^* \backslash \{1\},\ v_l \in \Phi_1(X) \setminus X,\ i, j \in I,\ n \ge 0,\ 1 \le k < n,\ 1 \le l \le n$. When $n = 0$, $|u_0^{(1)}| \ge 1$.
$f_3 \wedge f_3'$**:** $f_3, f_3' \in F_3$. There are three ambiguities: one is for the intersection composition, and two are for the inclusion composition.

All possible compositions of left and right multiplication are: $f_1 P(u)$, $P(u) f_2$, $f_3 P(u)$ and $P(u) f_3$, where $f_i \in F_i,\ u \in \Phi(X),\ i = 1, 2, 3$.

Now we prove that all the compositions are trivial.

For $f_1 \wedge f_2$, let $f = x_iP(x_j) - \{x_i \prec x_j\}$, $g = P(x_j)x_k - \{x_j \succ x_k\},\ i, j, k \in I$. Then $w = x_iP(x_j)x_k$ and

$$\begin{aligned}(f, g)_w &= x_iP(x_j)x_k - \{x_i \prec x_j\}x_k - (x_iP(x_j)x_k - x_i\{x_j \succ x_k\})\\ &= x_i\{x_j \succ x_k\} - \{x_i \prec x_j\}x_k\\ &\equiv 0 \ \ mod(F_3, w).\end{aligned}$$

For $f_2 \wedge f_1$, let $f = P(x_i)x_j - \{x_i \succ x_j\}$, $g = x_jP(x_k) - \{x_j \prec x_k\},\ i, j, k \in I$. Then $w = P(x_i)x_jP(x_k)$ and by equation (1.1),

$$\begin{aligned}(f, g)_w &= P(x_i)x_jP(x_k) - \{x_i \succ x_j\}P(x_k) - P(x_i)(x_jP(x_k) - \{x_j \prec x_k\})\\ &= P(x_i)\{x_j \prec x_k\} - \{x_i \succ x_j\}P(x_k)\\ &\equiv \{x_i \succ \{x_j \prec x_k\}\} - \{\{x_i \succ x_j\} \prec x_k\}\\ &\equiv 0 \ \ mod(S, w).\end{aligned}$$

For $f_3 \wedge f_1$, let $f = u_0P(v_1)u_1 \cdots P(v_n)u_nx_i$, $g = x_iP(x_j) - \{x_i \prec x_j\},\ u_0, u_n \in X^*, u_k \in X^*\backslash\{1\}, v_l \in \Phi_1(X) \setminus X, i, j \in I, n \geq 0, 1 \leq k < n, 1 \leq l \leq n$, and $|u_0| \geq 1$ if $n = 0$. Then $w = u_0P(v_1)u_1 \cdots P(v_n)u_nx_iP(x_j)$ and

$$\begin{aligned}(f, g)_w &= u_0P(v_1)u_1 \cdots P(v_n)u_n\{x_i \prec x_j\}\\ &\equiv 0 \ \ mod(S, w).\end{aligned}$$

For $f_2 \wedge f_3$, the proof is similar to $f_3 \wedge f_1$.

For $f_3 \wedge f_3'$, we have $(f, g)_w = 0$.

Now, we check the compositions of left and right multiplication. We prove only the cases of $f_1P(u)$ and $P(u)f_3$, where $f_1 \in F_1,\ f_3 \in F_3$, $u \in \Phi(X)$. Others can be similarly proved.

We may assume that $u \in \Phi_1(X)$.

For $f_1P(u)$, let $f = x_iP(x_j) - \{x_i \prec x_j\}, i, j \in I$ and $w = \overline{fP(u)}$. There are two cases to consider.

(1) $u = x_k \in X$. Then by using the equation (1.1),

$$\begin{aligned}fP(u) &= x_iP(x_j)P(x_k) - \{x_i \prec x_j\}P(x_k)\\ &= x_iP(P(x_j)x_k) + x_iP(x_jP(x_k)) - \{x_i \prec x_j\}P(x_k)\\ &\equiv x_iP(\{x_j \succ x_k\}) + x_iP(\{x_j \prec x_k\}) - \{\{x_i \prec x_j\} \prec x_k\}\\ &\equiv \{x_i \prec \{x_j \succ x_k\}\} + \{x_i \prec \{x_j \prec x_k\}\} - \{\{x_i \prec x_j\} \prec x_k\}\\ &\equiv 0 \ \ mod(S, w).\end{aligned}$$

(2) $u \in \Phi_1(X) \setminus X$. Then

$$\begin{aligned}fP(u) &= x_iP(x_j)P(u) - \{x_i \prec x_j\}P(u)\\ &= x_iP(P(x_j)u) + x_iP(x_jP(u)) - \{x_i \prec x_j\}P(u).\end{aligned}$$

By Lemma 3.2, we have $P(x_j)u \equiv \sum \alpha_l u_l \, mod(F_1 \cup F_2, \overline{P(x_j)u})$, where $u_l \in \Phi_1(X) \setminus X$. Then

$$\begin{aligned}fP(u) &\equiv x_iP(\sum \alpha_l u_l) + x_iP(x_jP(u)) - \{x_i \prec x_j\}P(u)\\ &\equiv \sum \alpha_l x_iP(u_l) + x_iP(x_jP(u)) - \{x_i \prec x_j\}P(u)\\ &\equiv 0 \ \ mod(S, w).\end{aligned}$$

For $P(u)f_3$, let $f = P(v_1)u_1 \cdots P(v_n)u_n$, where $u_n \in X^*,\ u_t \in X^*\backslash\{1\},\ v_l \in \Phi_1(X) \setminus X,\ n \geq 1,\ 1 \leq t < n,\ 1 \leq l \leq n$, and let $w = \overline{P(u)f}$. Then

$$P(u)f = P(u)P(v_1)u_1 \cdots P(v_n)u_n.$$

By Lemma 3.2, we have $P(u)P(v_1) \equiv \sum \alpha_i P(w_i)\ mod(F_1 \cup F_2, \overline{P(u)P(v_1)})$, where each $w_i \in \Phi_1(X) \setminus X$. Then

$$\begin{aligned} P(u)f &\equiv \sum \alpha_i P(w_i)u_1 \cdots P(v_n)u_n \\ &\equiv 0\ \ mod(S, w). \end{aligned}$$

So, all compositions in S are trivial. The proof is complete. □

We now prove Theorem 1.1.

The proof of Theorem 1.1. Let $R = F_1 \cup F_2$. Then for any $u \notin Irr(R^{comp})$, we have $u = v|_{\bar{r}}$, where $r \in R^{comp}, v|_r$ is a normal R^{comp}-word. Then $f = v|_r \in Id(R^{comp}) = Id(R) \subseteq Id(S)$. Since S is a Gröbner-Shirshov basis in $RB(X)$, by Theorem 2.1, we have $\bar{f} = w|_{\bar{s}}$ for some $w \in \Phi^\star(X)$, $s \in S$. That is, $u = v|_{\bar{r}} = \bar{f} \notin Irr(S)$. So, we have that $Irr(R^{comp}) \supset Irr(S) \supset X$. Since $Irr(R^{comp})$ is an F-basis of $U(D)$, D can be embedded into $U(D)$. □

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