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AN UPPER BOUND ON THE NUMBER OF F-JUMPING COEFFICIENTS OF A PRINCIPAL IDEAL

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Abstract. We prove a result relating the Jacobian ideal and the generalized test ideal associated to a principal ideal in $R = k[x_1, \ldots, x_n]$ with $[k : k^p] < \infty$ or in $R = k[[x_1, \ldots, x_n]]$ with an arbitrary field k of characteristic $p > 0$. As a consequence of this result, we establish an upper bound on the number of F-jumping coefficients of a principal ideal with an isolated singularity.

1. Introduction

In characteristic zero, one can define invariants, called *jumping coefficients* in [ELSV04], attached to an ideal sheaf on a smooth variety via multiplier ideals. These jumping coefficients are positive rational numbers which encode interesting geometric and algebraic information (see [Laz04, Chapter 9] for details). In [ELSV04] the following connection between Jacobian and multiplier ideals is established.

Theorem 1.1 (Proposition 3.8[1] in [ELSV04]). *Given $f \in \mathbb{C}[x_1, \ldots, x_n]$, one has*

$$\mathrm{Jac}(f) \subseteq \mathcal{J}((f)^{1-\epsilon}), \text{ for all } \epsilon > 0,$$

where $\mathrm{Jac}(f) = (f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ *and* $\mathcal{J}((f)^{1-\epsilon})$ *is the multiplier ideal of the pair* $(\mathbb{C}[x_1, \ldots, x_n], (f)^{1-\epsilon})$.

As a consequence, one has

Corollary 1.2. *If $f \in R = \mathbb{C}[x_1, \ldots, x_n]$ has an isolated singularity, then f has at most* $\dim_{\mathbb{C}}(\frac{R}{\mathrm{Jac}(f)}) + 1$ *jumping coefficients in* $[0, 1]$.

The purpose of this paper is to extend these results to characteristic $p > 0$.

In characteristic $p > 0$, Hara and Yoshida introduced, in [HY03], an analogue of the multiplier ideals, namely, the generalized test ideals. Generalized test ideals are defined in any noetherian ring of characteristic $p > 0$, but the definition is less technical when the ring R is an F-finite regular ring or an excellent regular local ring. When R is an F-finite regular ring (see [BMS08]) or an excellent regular local ring (see [KLZ09]), for each ideal J of R and each positive integer e, there exists a

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[1]In the statement of [ELSV04, Proposition 3.8], f is assumed to have an isolated singularity. However, this assumption is not needed in the proof of the statement.

unique smallest ideal $I_e(J)$ such that $J \subseteq (I_e(J))^{[p^e]}$. Then, for each nonnegative real number t, the generalized test ideal, $\tau(J^t)$, can be defined as

$$\bigcup_e I_e(J^{\lceil tp^e \rceil}).$$

In this context, we say that c is an *F-jumping coefficient* of J if $\tau(J^c) \subsetneq \tau(J^{c'})$ for all $c' < c$. It is proved, in [BMS09] when R is an F-finite regular ring and in [KLZ09] when R is an excellent regular local ring, that F-jumping coefficients of each principal ideal of R consist of a discrete set of positive rational numbers.

Let $R = k[x_1, \dots, x_n]$ with $[k : k^p] < \infty$ or $R = k[[x_1, \dots, x_n]]$ with k an arbitrary field of characteristic $p > 0$. Let $\frac{\partial f}{\partial x_i}$ denote the partial derivative of $f \in R$ with respect to x_i and let $\mathrm{Jac}(f) = (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. Our main theorem of this paper is the following analogue of Theorem 1.1.

Theorem 1.3 (Main Theorem)**.** *Let R be as above. Then*

$$\mathrm{Jac}(f) \subseteq \tau((f)^{1-\epsilon}),$$

for all $\epsilon > 0$.

When k is a perfect field of characteristic $p > 0$, the singular locus of $R/(f)$ is determined by $\mathrm{Jac}(f)$. In particular, when f (or equivalently $R/(f)$) has an isolated singularity, $\dim_k(\frac{R}{\mathrm{Jac}(f)})$ is finite. Note that in this case every chain of ideals between R and $\mathrm{Jac}(f)$ has length at most $\dim_k(\frac{R}{\mathrm{Jac}(f)})$. Therefore, as a consequence of our Main Theorem, we have

Corollary 1.4. *Let k be a perfect field of characteristic $p > 0$, and let $R = k[x_1, \dots, x_n]$ or $k[[x_1, \dots, x_n]]$. Then, for each $f \in R$ with isolated singularity, there are at most $\dim_k(\frac{R}{\mathrm{Jac}(f)}) + 1$ F-jumping coefficients of f in $[0, 1]$.*

2. A result on differential operators

In this section we consider differential operators over $R = \mathbb{Z}[x_1, \dots, x_n]$ (or $R = \mathbb{Z}[[x_1, \dots, x_n]]$, respectively): let $D_{m,i} : R \to R$ be the $\mathbb{Z}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$-linear (or $\mathbb{Z}[[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]]$-linear, respectively) map that sends x_i^ℓ to $\binom{\ell}{m} x_i^{\ell-m}$ and let $D_{0,i}$ be the identity map. We may write $D_{m,i}$ as

$$D_{m,i} = \frac{1}{m!} \frac{\partial^m}{\partial x_i^m} : R \to R.$$

Even though the following proposition is stated over a field k in [Lyu10], the same proof works over $\mathbb{Z}$.

Proposition 2.1 (Proposition 2.1 in [Lyu10])**.** *For each $f \in R$, we have*

$$D_{m,i} \cdot f = \sum_{\ell=0}^{m} D_{\ell,i}(f) \cdot D_{m-\ell,i} \text{ in } \mathrm{End}_{\mathbb{Z}}(R);$$

i.e., given $f, g \in R$, we have

$$D_{m,i}(fg) = \sum_{\ell=0}^{m} D_{\ell,i}(f) D_{m-\ell,i}(g) \text{ in } R.$$

Our main result of this section is the following identity.

Theorem 2.2. *Given any $f \in R$ and a positive integer m, we have*

$$\sum_{\ell=0}^{m}(\ell-1)D_{\ell,i}(f)D_{m-\ell,i}(f^{m-1})=0. \tag{1}$$

Proof. Writing $D_{m,i}$ as $\frac{1}{m!}\frac{\partial^m}{\partial x_i^m}$ and multiplying (1) by $m!$, we can rewrite (1) as

$$\sum_{\ell=0}^{m}(\ell-1)\binom{m}{\ell}\frac{\partial^\ell f}{\partial x_i^\ell}\frac{\partial^{m-\ell}f^{m-1}}{\partial x_i^{m-\ell}}=0.$$

To ease our notation, we will write $\frac{\partial^\ell f}{\partial x_i^\ell}$ as $\partial^\ell(f)$ and $\frac{\partial^{m-\ell}f^{m-1}}{\partial x_i^{m-\ell}}$ as $\partial^{m-\ell}(f^{m-1})$. Hence the above equation becomes

$$\sum_{\ell=0}^{m}(\ell-1)\binom{m}{\ell}\partial^\ell(f)\partial^{m-\ell}(f^{m-1})=0. \tag{2}$$

We first expand

$$\sum_{\ell=0}^{m}(\ell-1)\binom{m}{\ell}\partial^\ell(f)\partial^{m-\ell}(f^{m-1})$$

as

$$-f\partial^m(f^{m-1})-\sum_{\ell=2}^{m}\binom{m}{\ell}\partial^\ell(f)\partial^{m-\ell}(f^{m-1})+\sum_{\ell=2}^{m}\ell\binom{m}{\ell}\partial^\ell(f)\partial^{m-\ell}(f^{m-1}). \tag{3}$$

Use the fact that $\ell\binom{m}{\ell}=m\binom{m-1}{\ell-1}$ to rewrite (3) as
(4)

$$-f\partial^m(f^{m-1})-\sum_{\ell=2}^{m}\binom{m}{\ell}\partial^\ell(f)\partial^{m-\ell}(f^{m-1})+\sum_{\ell=2}^{m}m\binom{m-1}{\ell-1}\partial^\ell(f)\partial^{m-\ell}(f^{m-1}).$$

Note that Leibniz's rule implies that

$$\partial^m(f^m)=\partial^m(ff^{m-1})=\sum_{l=0}^{m}\binom{m}{l}\partial^l(f)\partial^{m-l}(f^{m-1})$$

and rewrite (4) as

$$\begin{aligned}&-f\partial^m(f^{m-1})-\big(\partial^m(f^m)-f\partial^m(f^{m-1})-m\partial(f)\partial^{m-1}(f^{m-1})\big)\\ &\qquad+\sum_{\ell=1}^{m-1}m\binom{m-1}{\ell}\partial^\ell(\partial(f))\partial^{m-1-\ell}(f^{m-1}).\end{aligned} \tag{5}$$

Now

$$\begin{aligned}&m\partial(f)\partial^{m-1}(f^{m-1})+\sum_{\ell=1}^{m-1}m\binom{m-1}{\ell}\partial^\ell(\partial(f))\partial^{m-1-\ell}(f^{m-1})\\ &=m\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}\partial^\ell(\partial(f))\partial^{m-1-\ell}(f^{m-1})\\ &=\partial^{m-1}\left(m(\partial f)f^{m-1}\right)\\ &=\partial^{m-1}\partial f^m=\partial^m f^m,\end{aligned}$$

and (5) simplifies to

$$-f\partial^m(f^{m-1})-\partial^m(f^m)+f\partial^m(f^{m-1})+\partial^m f^m=0.$$

□

3. Proof of the Main Theorem

Throughout this section, R is either $k[x_1, \ldots, x_n]$ with $[k : k^p] < \infty$ or $k[[x_1, \ldots, x_n]]$ with k an arbitrary field of characteristic $p > 0$. In either case, by reducing the operators $D_{m,i}$ mod p, we get differential operators over k, which will still be denoted $D_{m,i}$. Note that the identity (1) also holds over k and that each differential operator $D_{m,i}$ is R^{p^e}-linear when $m < p^e$.

We begin with an easy observation.

Lemma 3.1. *Given any $f \in R$, we have*

$$D_{p^e,i}(f^{p^e}) = (\frac{\partial f}{\partial x_i})^{p^e}.$$

Proof. Since $D_{p^e,i}$ is k-linear and f is a k-linear combination of monomials, it suffices to consider the case when f is a monomial. Also, since $D_{p^e,i}$ is $k[x_1, \ldots, \hat{x}_i, \ldots, x_n]$-linear (or $k[[x_1, \ldots, \hat{x}_i, \ldots, x_n]]$-linear), it suffices to consider the case when $f = x_i^t$ for some $t \in \mathbb{N}$. We have

$$D_{p^e,i}(f^{p^e}) = \binom{tp^e}{p^e} x_i^{(t-1)p^e} \text{ and } (\frac{\partial f}{\partial x_i})^{p^e} = t^{p^e} x_i^{(t-1)p^e}.$$

But it is well-known that $\binom{tp^e}{p^e} \equiv t \pmod{p}$ (cf. [Eis95, Lemma 15.22 (Lucas's Theorem)]) and Fermat's Little Theorem implies that $t^{p^e} \equiv t \pmod{p}$. □

We can now prove our Main Theorem.

Proof of Main Theorem. Clearly it is enough to prove that

$$\mathrm{Jac}(f) \subseteq \tau((f)^{1-\frac{1}{p^e}})$$

for all integers $e > 0$.

Using the definition of $\tau((f)^{1-\frac{1}{p^e}})$ and the fact that R is noetherian, we can write $\tau((f)^{1-\frac{1}{p^e}}) = I_{e'}\left((f)^{\lceil p^{e'}(1-\frac{1}{p^e})\rceil}\right) = I_{(e'-e)+e}\left((f)^{p^{e'-e}(p^e-1)}\right)$ for some $e' \geq e$ and [BMS08][Lemma 2.4(4)] implies that this equals $I_e\left((f)^{p^e-1}\right)$.

Thus $\tau((f)^{1-\frac{1}{p^e}}) = I_e(f^{p^e-1})$ is the smallest ideal J such that $f^{p^e-1} \subseteq J^{[p^e]}$, so it suffices to prove that for any ideal J with $f^{p^e-1} \subseteq J^{[p^e]}$, we have

$$\frac{\partial f}{\partial x_i} \in J.$$

Since $D_{t,i}$ with $t < p^e$ is R^{p^e}-linear, we have

$$D_{t,i}(f^{p^e-1}) \in J^{[p^e]}, \text{ for all } t < p^e.$$

Setting $m = p^e$ in the identity (1), we have that

$$fD_{p^e,i}(f^{p^e-1}) = \sum_{\ell=2}^{p^e} (\ell-1) D_{\ell,i}(f) D_{p^e-\ell,i}(f^{p^e-1}) \in J^{[p^e]}.$$

According to Proposition 2.1,

$$D_{p^e,i}(f^{p^e}) = D_{p^e,i}(ff^{p^e-1}) = \sum_{\ell=0}^{p^e} D_{\ell,i}(f) D_{p^e-\ell,i}(f^{p^e-1}),$$

and hence $D_{p^e,i}(f^{p^e}) \in J^{[p^e]}$. Combining this with Lemma 3.1, we see that

$$(\frac{\partial f}{\partial x_i})^{p^e} \in J^{[p^e]}$$

and consequently,

$$\frac{\partial f}{\partial x_i} \in J$$

since R is regular. This finishes the proof of our Main Theorem. □

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