DERIVED EQUIVALENCE INDUCED BY INFINITELY GENERATED n-TILTING MODULES

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(Communicated by Harm Derksen)

Abstract. Let $T_R$ be a right $n$-tilting module over an arbitrary associative ring $R$. In this paper we prove that there exists an $n$-tilting module $T'_R$ equivalent to $T_R$ which induces a derived equivalence between the unbounded derived category $D(R)$ and a triangulated subcategory $E_{\perp}$ of $D(\text{End}(T'))$ equivalent to the quotient category of $D(\text{End}(T'))$ modulo the kernel of the total left derived functor $-\otimes_R'$. If $T_R$ is a classical $n$-tilting module, we have that $T = T'$ and the subcategory $E_{\perp}$ coincides with $D(\text{End}(T))$, recovering the classical case.

Introduction

Tilting theory generalizes the classical Morita theory of equivalences between module categories. Originated in the works of Gel’fand and Ponomarev, Brenner and Butler, Happel and Ringel [6, 9, 19], it has been generalized in various directions. In the recent literature, given an associative ring $R$ with $0 \neq 1$, a right $R$-module $T_R$ (possibly infinitely generated) is said to be $n$-tilting if the following conditions are satisfied:

(T1) there exists a projective resolution of right $R$-modules

$0 \rightarrow P_n \rightarrow ... \rightarrow P_1 \rightarrow P_0 \rightarrow T \rightarrow 0$;

(T2) $	ext{Ext}^i_R(T, T^{(\alpha)}) = 0$ for each $i > 0$ and each cardinal $\alpha$;

(T3) there exists a coresolution of right $R$-modules

$0 \rightarrow R \rightarrow T_0 \rightarrow T_1 \rightarrow ... \rightarrow T_m \rightarrow 0$,

where the $T_i$'s are direct summands of arbitrary direct sums of copies of $T$.

If the projectives $P_i$'s in (T1) can be assumed finitely generated, then the $n$-tilting module $T_R$ is said to be classical $n$-tilting.

Infinitely generated tilting modules arise naturally; they are objects of interest in themselves and also in the context of representation theory of Artin algebras. Relevant examples are studied in [28, 24, 2]. Moreover, they play an important role in connection with homological conjectures. In [3] it is proved that the little finitistic dimension of a Noetherian ring is finite if and only if there is an $n$-tilting...
module representing in a canonical way the category of finitely presented modules of finite projective dimension. Even in the case of finite dimensional algebras it could be possible that this tilting module is necessarily infinitely generated.

Let us denote by $S = \text{End}(T_R)$ the endomorphism ring of $T$ and by $KE_i(T)$ and $KT_i(T)$, $0 \leq i \leq n$, the following classes:

$$KE_i(T) = \{ M \in \text{Mod}-R : \text{Ext}^i_R(T, M) = 0 \text{ for each } 0 \leq j \neq i \},$$

$$KT_i(T) = \{ N \in \text{Mod}-S : \text{Tor}^i_S(N, T) = 0 \text{ for each } 0 \leq j \neq i \}.$$

In 1986 Miyashita proved that if $T_R$ is a classical $n$-tilting, then the functors $\text{Ext}^i_R(T, -)$ and $\text{Tor}^i_S(-, T)$ induce equivalences between the classes $KE_i(T)$ and $KT_i(T)$.

In the same year, works of several authors showed that the natural context for studying equivalences induced by classical tilting modules is that of derived categories. In particular Cline, Parshall and Scott, generalizing a result of Happel, proved that a classical $n$-tilting module $T_R$ provides a derived equivalence between the bounded derived categories $D^b(R)$ and $D^b(S)$ of bounded cochain complexes of right $R$- and $S$-modules.

In the context of infinite dimensional tilting theory, Facchini in 1988 proved that, over a commutative domain, the divisible module $\partial$ introduced by Fuchs is an infinitely generated 1-tilting module and it provides a pair of equivalences

$$\xymatrix{KE_0(\partial) \ar[r]^{\text{Hom}(\partial, \cdot)}_{-\otimes \partial} & KT_0(\partial) \cap \text{I-Cot}, \quad KE_1(\partial) \ar[r]^{\text{Ext}^1(\partial, \cdot)}_{\text{Tor}_1(\cdot, \partial)} & KT_1(\partial) \cap \text{I-Cot} \cap \text{I-Cot} \cap \text{I-Cot}}$$

between the category $KE_0(\partial)$ of all divisible modules and the category $KT_0(\partial) \cap \text{I-Cot}$ of all $1$-reduced I-cotorsion modules, and the category $KE_1(\partial)$ of all reduced modules and the category $KT_1(\partial) \cap \text{I-Cot}$ of all $I$-divisible I-cotorsion modules, respectively. This equivalence generalizes both the Harrison and Matlis equivalences. In 1995 Colpi and Trlifaj started the study in general of 1-tilting modules. They realized that it can be useful to “change slightly” the tilting module to realize a good equivalence theory. They proved that if $T_R$ is a 1-tilting module, there exists another 1-tilting module $T'_R$ equivalent to $T_R$ (i.e. $KE_0(T) = KE_0(T')$), with endomorphism ring $S' = \text{End}(T')$, such that the functors $\text{Hom}_R(T', -)$ and $-\otimes_{S'}T'$ induce an equivalence between $KE_0(T) = KE_0(T')$ and its image class in $\text{Mod}-S'$. Moreover $T'$ results in a finitely presented $S'$-module. In 2001 Gregorio and Tonolo extended this result proving the existence of a pair of equivalences

$$\xymatrix{KE_i(T') \ar[r]^{\text{Ext}^i(T', -)}_{\text{Tor}_i(-, T')} & KT_i(T') \cap \text{Cost}(T'), \quad i = 0, 1}$$

where $\text{Cost}(T')$ is the class of costatic right $S'$-modules (see [17]).

In 2009 Bazzoni gives a better understanding of the whole situation in the setting of derived categories proving that for a 1-tilting module $T_R$ it is possible to find an equivalent 1-tilting module $T'$ which induces a derived equivalence between the unbounded derived category $D(R)$ and the quotient category of $D(S')$ modulo the full triangulated subcategory $\text{Ker}(-\otimes_{S'}^L T')$, namely the kernel of the total left derived functor of the functor $-\otimes_{S'}^L T'$. 

In this paper we generalize Bazzoni’s result to a general \( n \)-tilting module \( T_R \). We prove the existence of a good \( n \)-tilting module \( T_R' \) equivalent to \( T_R \) (see Definition 1.1), which, also in such a case, provides a derived equivalence between the unbounded derived category \( \mathcal{D}(R) \) and a triangulated subcategory \( \mathcal{E}_\perp \) of \( \mathcal{D}(\text{End}(T')) \). The category \( \mathcal{E}_\perp \) turns out to be equivalent to the quotient category of \( \mathcal{D}(\text{End}(T')) \) modulo the kernel of the total left derived functor \(-\otimes_{S'} T'\). Moreover, as done in [25] in the contravariant case, we interpret the derived equivalence at the level of stalk complexes obtaining on the underlying module categories a generalization of the Miyashita equivalences.

1. \( n \)-tilting classes

In 2004 Bazzoni (see [4]) proved that \( T_R \) is an \( n \)-tilting module if and only if the classes
\[
T^-\infty := \{ M_R : \text{Ext}^i_R(T, M) = 0 \text{ for each } i > 0 \}
\]
and
\[
\text{Gen}_n(T) := \{ M_R : \exists T^{(\alpha_i)} \to \ldots \to T^{(\alpha_1)} \to M \to 0, \text{ for some cardinals } \alpha_i \}
\]
coincide.

**Definition 1.1.** Two \( n \)-tilting right \( R \)-modules \( T_R \) and \( T'_R \) are said to be equivalent if \( \text{Gen}_n(T_R) = \text{Gen}_n(T'_R) \).

An arbitrary direct sum of copies of an \( n \)-tilting module is an \( n \)-tilting module equivalent to the original one. Therefore equivalent tilting modules can have completely different endomorphism rings.

**Definition 1.2.** We say that \( T_R \) is a good \( n \)-tilting module if it is \( n \)-tilting and it satisfies the condition (T3') there is an exact sequence
\[
0 \to R \to T_0 \to T_1 \to \ldots \to T_n \to 0
\]
where the \( T_i \)'s are direct summands of finite direct sums of copies of \( T \).

It is easy to verify that a classical \( n \)-tilting module is good (see e.g. [16, p. 189]).

**Proposition 1.3.** For any \( n \)-tilting module \( T_R \) there exists an equivalent good \( n \)-tilting module \( T'_R \) such that
\[
KE_i(T) = KE_i(T') \text{ for each } i \geq 0.
\]

*Proof.* Let \( T_R \) be an \( n \)-tilting module. If it is classical, then \( T \) already satisfies (T3'). Otherwise, from condition (T3) let us consider \( T' = T_0 \oplus \ldots \oplus T_n \). Since \( T' \) is a direct summand of a direct sum of copies of \( T \), we have
\[
\text{Gen}_n(T') \subseteq \text{Gen}_n(T) = T^-\infty \subseteq T'^{-\infty},
\]
and \( T' \) satisfies properties (T1) and (T2) of tilting modules. Since by construction it satisfies also property (T3'), we have \( \text{Gen}_n(T') = T'^{-\infty} \) and \( T' \) is the wanted good \( n \)-tilting module equivalent to \( T \).

Finally, since \( \text{KerExt}^j(T, -) = \text{KerExt}^j(T_0 \oplus \ldots \oplus T_n, -) = \text{KerExt}^j(T', -) \), we conclude that \( KE_i(T) = KE_i(T') \) for each \( i \geq 0 \). \( \square \)
A good n-tilting module has an endomorphism ring $S$ sufficiently large to permit building a good equivalence theory between the unbounded derived categories $\mathcal{D}(R)$ and $\mathcal{D}(S)$. In the sequel we will work directly with good n-tilting modules.

**Proposition 1.4.** Let $T_R$ be a good n-tilting module and $S = \text{End}(T_R)$. Then $sT$ has a projective resolution

$$0 \to Q_n \to ... \to Q_0 \to sT \to 0,$$

where the $Q_i$'s are direct summands of a finite direct sum of copies of $S$, $\text{Ext}_S^1(T, T) = 0$ for each $i \geq 0$, and $R \cong \text{End}(sT)$.

**Proof.** By Definition there is an exact sequence

$$0 \to R \to T_0 \to T_1 \to ... \to T_n \to 0$$

with the $T_i$'s direct summands of $T^n_m$ for a suitable $m \in \mathbb{N}$. Denote by $K_i$ the kernel of the map $T_i \to T_{i+1}$, $1 \leq i \leq n - 1$. Applying the contravariant functor $\text{Hom}_R(-, T)$ we get easily by dimension shifting that

$$0 = \text{Ext}_R^1(K_j, T) \text{ for each } 1 \leq j \leq n - 1, \text{ and } i \geq 1.$$

Therefore we have the exact sequence

$$(†) \quad 0 \to \text{Hom}_R(T_n, T) \to \text{Hom}_R(T_{n-1}, T) \to ... \to \text{Hom}_R(T_1, T) \to \text{Hom}_R(T_0, T) \to sT \to 0,$$

where each $\text{Hom}_R(T_i, T)$ is a direct summand of $\text{Hom}_R(T^n_m, T) = S^n$ and hence a finitely generated projective $S$-module. Given a right $R$-module $M$, let us denote for simplicity by $M^*$ the left $S$-module $\text{Hom}_R(M, T)$, by $M^{**}$ the right $R$-module $\text{Hom}_S(M^*, T)$, and by $\delta_M$ the evaluation map $M \to M^{**}$. The modules $K_i^*$ are the cokernels of the morphisms $\text{Hom}_R(T_{i+1}, T) \to \text{Hom}_R(T_i, T)$, $1 \leq i \leq n - 1$. Applying to $(†)$ the contravariant functor $\text{Hom}_S(-, T)$ we get the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}_S(T, T) = R^{**} & \longrightarrow & T_0^{**} & \longrightarrow & K_1^{**} & \longrightarrow & \text{Ext}_S^1(T, T) & \longrightarrow & 0 \\
\downarrow \delta_R & & \downarrow \delta_0 & & \downarrow \delta_{K_1} & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & T_0 & \longrightarrow & K_1 & \longrightarrow & 0 \\
\vdots & & & & & & & & & \\
0 & \longrightarrow & K_{n-1}^{**} & \longrightarrow & T_{n-1}^{**} & \longrightarrow & T_n^{**} & \longrightarrow & \text{Ext}_S^1(K_{n-1}^*, T) & \longrightarrow & 0 \\
\downarrow \delta_{K_{n-1}} & & \downarrow \delta_{T_{n-1}} & & \downarrow \delta_{T_n} & & \downarrow & & \\
0 & \longrightarrow & K_{n-1} & \longrightarrow & T_{n-1} & \longrightarrow & T_n & \longrightarrow & 0
\end{array}$$

Since the $\delta_{T_i}$'s are isomorphisms we get

$$\text{Ext}_S^1(T, T) = 0 \text{ and } 0 = \text{Ext}_S^1(K_i^*, T) \cong \text{Ext}_S^{i+1}(T, T) \text{ for each } 1 \leq i \leq n - 1,$$

and $R \cong \text{Hom}_S(T, T)$.

\[\square\]

**Lemma 1.5** (Lemmas 1.8, 1.9 [27]). Let $T_R$ be a good n-tilting and $S = \text{End} T$. For any right $R$-module $M$ in $T^{\perp}$ and any right projective $S$-module $P_S$, we have

1. $\text{Tor}_i^S(\text{Hom}_R(T, M), T) = 0$ for each $i > 0$;
2. $\text{Hom}_R(T, M) \otimes_S T \cong M$, $f \otimes t \mapsto f(t)$;
(3) $\text{Ext}^i_R(T, P \otimes_S T) = 0$ for each $i > 0$.

If $T_R$ is a classical $n$-tilting module, then

(4) $P \cong \text{Hom}_R(T, P \otimes_S T)$, $p \mapsto (f : t \mapsto p \otimes t)$.

Proof. Everything except condition (3) follows by the quoted lemmas in [27]. If $P \leq S^{(\alpha)}$ we have

$$\text{Ext}^i_R(T, P \otimes_S T) \leq \text{Ext}^i_R(T, S^{(\alpha)} \otimes_S T) = \text{Ext}^i_R(T, T^{(\alpha)}) = 0. \quad \square$$

2. Tilting equivalences in derived categories

In the sequel, for any ring $R$, we denote by $\mathcal{K}(R)$ the homotopy category of unbounded complexes of right $R$-modules and by $\mathcal{D}(R)$ the associated derived category. Given an object $M \in \text{Mod-}R$, we continue to denote by $M$ also the stalk complex in $\mathcal{D}(R)$ associated to $M$, i.e. the complex with $M$ concentrated in degree zero. Any complex $C^* \in \mathcal{D}(R)$ admits a $K$-injective resolution, i.e. a complex $I^\bullet C^*$ quasi-isomorphic to $C^*$ whose terms are injective modules such that $\text{Hom}_{\mathcal{K}(R)}(N^\bullet, I^\bullet C^*) = 0$ for each exact complex $N^\bullet$. Similarly, any complex $C^* \in \mathcal{D}(R)$ admits a $K$-projective resolution, i.e. a complex $pC^*$ quasi-isomorphic to $C^*$ whose terms are projective modules such that $\text{Hom}_{\mathcal{K}(R)}(pC^*, N^\bullet) = 0$ for each exact complex $N^\bullet$ (see for instance [7], [22]). This result guarantees the existence of the total derived functor of any additive functor defined on module categories.

Given any covariant left exact functor $H : \text{Mod-}R \rightarrow \text{Mod-}S$, we denote by $\mathcal{K}H$ its total right derived functor defined on $\mathcal{D}(R)$. For any $C^* \in \mathcal{D}(R)$, $\mathcal{K}H(C^*)$ coincides with the complex $H(I^\bullet C^*)$, where we still denote by $H$ its extension to $\mathcal{K}(R)$. Similarly, for any right exact covariant functor $G : \text{Mod-}S \rightarrow \text{Mod-}R$, we denote by $LG$ its total left derived functor defined on $\mathcal{D}(S)$. For any $N^\bullet \in \mathcal{D}(S)$, $LG(N^\bullet)$ coincides with the complex $G(pN^\bullet)$.

A module $M$ in $\text{Mod-}R$ is called $H$-acyclic if $R^iHM := H^i(\mathcal{K}H M) = 0$ for any $i \neq 0$. The abelian group $R^iHM$ coincides with the usual $i$-th derived functor $H^{(i)}(-)$ of $H$ evaluated in $M$. Analogously $G$-acyclic objects are defined and $L^iG(-) := H^i((LG)(-)) = G^{(-i)}(-)$. Following the proof of [21] Corollary I.5.3.7], in case the functor $H$ has finite homological dimension, the class $\mathfrak{ m}$ of the complexes with $H$-acyclic components satisfies the conditions 1 and 2 of [21] Theorem I.5.1]; therefore for any complex $M^*$ in $\mathcal{D}(R)$, we have

$$\mathcal{K}H M^* = H(J^*),$$

where $J^*$ is a complex in $\mathfrak{ m}$ quasi-isomorphic to $M^*$. The analogous result holds for the left derived functor of $G$, in case $G$ has finite homological dimension.

In view of these considerations, by Lemma [15] we have the following result:

Corollary 2.1. Let $T_R$ be a good $n$-tilting module with endomorphism ring $S$. Then for each injective module $I_R$ and each projective module $P_S$ we have

(1) $\text{Hom}_R(T, I) = - \otimes_S T$-acyclic;

(2) $P \otimes_S T$ is $\text{Hom}_R(T, -)$-acyclic.

In particular for cochain complexes $I^*$ and $P^*$ whose terms are injective right $R$-modules and projective right $S$-modules respectively, we have

$\mathcal{K} \text{Hom}(T, I^*) \otimes_S T = \text{Hom}(T, I^*) \otimes S T$ and $\mathcal{K} \text{Hom}(T, P^* \otimes_S T) = \text{Hom}(T, P^* \otimes S T)$. 


Finally, we recall that any adjoint pair of functors \((G, H)\) between categories of modules induces an adjoint pair \((L G, R H)\) between the associated unbounded derived categories. For other notation and results in derived categories we refer to \[21, 29\].

In the sequel we denote by \(H\) the functor \(\text{Hom}_R(T, -)\) and by \(G\) the functor \(- \otimes_S T\).

**Theorem 2.2.** Let \(T_R\) be a good \(n\)-tilting module and \(S = \text{End}_{\mathcal{R}} T_R\). The following hold:

1. The counit adjunction morphism
   \[
   L G \circ R H \rightarrow \text{Id}_{D(\mathcal{R})}
   \]
   is invertible.
2. The functor \(R H : D(\mathcal{R}) \rightarrow D(S)\) is fully faithful.
3. If \(\Sigma\) is the system of morphisms \(u \in D(S)\) such that \(L G u\) is invertible in \(D(\mathcal{R})\), then \(\Sigma\) admits a calculus of left fractions and the category \(D(S)[\Sigma^{-1}]\) coincides with the quotient category \(D(S)\) modulo the full triangulated subcategory \(\text{Ker}(L G)\) of the objects annihilated by the functor \(L G\).
4. There is a triangle equivalence
   \[
   D(S)[\Sigma^{-1}] \xrightarrow{\Theta} D(\mathcal{R}),
   \]
   where \(\Theta\) is the functor such that \(L G = \Theta \circ q\) with \(q\) the canonical quotient functor \(q : D(S) \rightarrow D(S)[\Sigma^{-1}]\).

**Proof.** (1) Let \(M_{\bullet}\) be a complex in \(D(\mathcal{R})\) and consider a \(K\)-injective resolution \(i M_{\bullet}\) of \(M_{\bullet}\). By Corollary 2.1 we have
\[
L G (R H (\mathcal{I}M_{\bullet})) = L G (H (i M_{\bullet})) = G (H (\mathcal{I}M_{\bullet})).
\]
Since \((\text{Hom}_R(T, I^n) \otimes S T)_{n \in \mathbb{Z}}\) and \(\mathcal{I}M_{\bullet}\) are isomorphic by Lemma 1.5 (2), we have
\[
L G (R H (\mathcal{I}M_{\bullet})) = G (H (\mathcal{I}M_{\bullet})) \cong \mathcal{I}M_{\bullet} = M_{\bullet}.
\]
Conditions (2), (3) and (4) follow by applying [15, Proposition I.1.3]. \(\square\)

Let \(\mathcal{C}\) be a triangulated category closed under arbitrary coproducts; recall that a triangle functor \(L : \mathcal{C} \rightarrow \mathcal{C}\) is a **Bousfield localization** if there exists a natural transformation \(\phi : 1_{\mathcal{C}} \rightarrow L\) such that for each \(X\) in \(\mathcal{C}\),

1. \(L(\phi_X) : L(X) \rightarrow L^2(X)\) is an isomorphism;
2. \(L(\phi_X) = \phi_{L(X)}\).

In such a case the kernel \(\mathcal{L}\) of \(L\) is a full triangulated subcategory of \(\mathcal{C}\) closed under coproducts; i.e. it is a **localizing** subcategory. The category
\[
\mathcal{L}_\perp := \{X \in \mathcal{C} : \text{Hom}_{\mathcal{C}}(\mathcal{L}, X) = 0\}
\]
is called the subcategory of \(\mathcal{L}\)-local objects. If \(\mathcal{L}_\perp\) is also closed under coproducts, then \(\mathcal{L}\) is called **smashing** [8, 7].

A localization functor \(L\) factorizes as
\[
\mathcal{C} \xrightarrow{\Phi} \mathcal{C} / \text{Ker} L \xrightarrow{\rho} \mathcal{L}_\perp \xrightarrow{i} \mathcal{C},
\]
where $q$ is the canonical quotient functor and $\rho$ is an equivalence: $(\rho \circ q, j)$ is an adjoint pair. Moreover the composition

$$L_{\perp} \xrightarrow{j} C \xrightarrow{\rho} C/\text{Ker } L$$

is an equivalence and $(q, j \circ \rho)$ is an adjoint pair (see [1] Section 4, or [1] Proposition 1.6], or [23] Propositions 4.9.1, 4.11.1]).

We collect in the following theorem results appearing in [15] and [23, Section 4.9]. For the sake of completeness we include the proof.

**Theorem 2.3.** Let $(\Phi, \Psi)$ be an adjoint pair of covariant functors between triangulated categories

$$C \xleftarrow{\Phi} \xrightarrow{\Psi} D.$$

Denote by $\phi : 1_C \to \Psi \circ \Phi$ and $\psi : \Phi \circ \Psi \to 1_D$ the corresponding unit and counit. If $\psi$ is a natural isomorphism, then the functor $L := \Psi \circ \Phi$ is a localization functor with kernel $L = \text{Ker } \Phi$. The functor $\Psi$ factorizes through $L_{\perp}$ as $\Psi = j \circ \Psi$, where $j$ is the inclusion $L_{\perp} \xrightarrow{j} C$. Finally we have a triangle equivalence

$$L_{\perp} \xleftarrow{\Phi \circ j} \Psi \xrightarrow{\Psi \circ j} C$$

where $\Phi \circ j$ is the restriction of $\Phi$ to $L_{\perp}$ and $\Psi$ is the corestriction of $\Psi$ to $L_{\perp}$.

**Proof.** Since $(\Phi, \Psi)$ is an adjoint pair, we have

$$\psi_{\Phi(X)} \circ \Phi(\phi_X) = 1_{\Phi(X)};$$

applying the functor $\Psi$ we get

$$\Psi(\psi_{\Phi(X)}) \circ L(\phi_X) = 1_{L(X)}.$$

On the other hand, again by the adjunction, we have

$$\Psi(\psi_{\Phi(X)}) \circ \phi_{\Phi(\Phi(X))} = 1_{\Phi(\Phi(X))}, \text{ i.e. } \Psi(\psi_{\Phi(X)}) \circ L(\phi_X) = 1_{L(X)}.$$

Since $\psi_{\Phi(X)}$ is an isomorphism by assumption, we have that for each $X$ in $C$,

$$L(\phi_X) = \phi_{L(X)} = (\Psi(\psi_{\Phi(X)}))^{-1}$$

is an isomorphism. Hence $L$ is a localization functor.

An object $X$ belongs to $L = \text{Ker } L$ if and only if we have $0 = \Phi(0) = \Phi(\Psi(\Phi(X))) \cong \Phi(X)$.

Next, since $L = \Psi \circ \Phi$ factorizes through $L_{\perp}$ and $\Phi(\Psi(Y)) \cong Y$ for each $Y$ in $D$, $\Psi$ also factorizes through $L_{\perp}$. Therefore we have the following commutative diagram:

$$\begin{align*}
L_{\perp} & \xrightarrow{j} C \xrightarrow{\rho} C/\text{Ker } \Phi \\
& \xrightarrow{\Psi \circ j} D
\end{align*}$$

Finally $\Phi \circ j \circ \Psi = \Phi \circ \Psi \cong 1_D$, and $\Psi \circ \Phi \circ j = \rho \circ q \circ j$, being a composition of two equivalences, is naturally isomorphic to $1_{L_{\perp}}$. □

Applying Theorem 2.3 to our context we obtain the following result.
Corollary 2.4. Let $T_R$ be a good $n$-tilting $R$-module and $S = \text{End}(T)$. Denoting by $\mathcal{E}$ the kernel of $L_G$, and denoting by $R^i H$ and $L_G$ also their restriction and corestriction, we have a triangulated equivalence

$$D(R) \xrightarrow{R^i H \ L_G} \mathcal{E}_\perp.$$ 

Embedding right $R$-modules and $S$-modules in $D(R)$ and $D(S)$ via the canonical functor, we obtain the following generalization of Miyashita’s results [27, Theorem 1.16]:

Corollary 2.5. Let $T_R$ be a good $n$-tilting $R$-module and $S = \text{End}(T)$. Then for each $0 \leq i \leq n$ there is an equivalence

$$KE_i \xleftarrow{\text{Ext}^i_R(T, -)} \xrightarrow{\text{Tor}^S_i(-, T)} KT_i \cap \mathcal{E}_\perp.$$

Proof. Let $M \in KE_i$. Then by Corollary 2.4, $R^i H(M) = R^i H(M)[-i] = \text{Ext}^i_R(T, M)[-i]$ belongs to $\mathcal{E}_\perp$. Since $\mathcal{E}_\perp$ is closed under shift, $\text{Ext}^i_R(T, M) \in \mathcal{E}_\perp$. In $D(R)$, by Theorem 2.2 (1), we have

$$M \cong L_G R^i H(M) = L_G (\text{Ext}^i_R(T, M)[-i]).$$

Then for each $j \neq 0$,

$$0 = H^j L_G (\text{Ext}^i_R(T, M)[-i]) = H^j L_G (\text{Ext}^i_R(T, M)) = \text{Tor}^S_j (\text{Ext}^i_R(T, M), T).$$

Therefore $\text{Ext}^i_R(T, M)$ belongs to $KT_i \cap \mathcal{E}_\perp$ and $M \cong \text{Tor}^S_i (\text{Ext}^i_R(T, M), T)$. Analogously if $N \in KT_i \cap \mathcal{E}_\perp$, then

$$L_G (N) = L^{-i} G(N)[i] = \text{Tor}^S_i (N, T)[i];$$

and since $R^i H L_G (N) = N$ in $D(S)$, necessarily $\text{Tor}^S_i (N, T)$ belongs to $KE_i$ and $N \cong \text{Ext}^i_R(T, \text{Tor}^S_i (N, T))$. □

Proposition 2.6. In the notation of Corollary 2.4, the following are equivalent:

1. $T_R$ is a classical $n$-tilting;
2. $\mathcal{E} = 0$ or equivalently $\mathcal{E}_\perp = D(S)$;
3. the class $\mathcal{E}$ is smashing.

Proof. (1 $\Rightarrow$ 2). Let $N^\bullet$ be a complex in $\mathcal{E}$ and $pN^\bullet$ a $K$-projective resolution of $N^\bullet$. By Lemma 1.5 (3) and (4), we have

$$0 = R^j H (L_G N^\bullet) = R^j H (L_G pN^\bullet) = R^j H (pN^\bullet \otimes_S T)$$

$$= \text{Hom}_R (T, pN^\bullet \otimes_S T) \cong pN^\bullet = N^\bullet.$$ 

We conclude that $\mathcal{E} = 0$ by Corollary 2.4.

(2 $\Rightarrow$ 3) is obvious.

(3 $\Rightarrow$ 2). Since $S = R^j H (T_R)$, $\mathcal{E}_\perp$ contains the bounded complexes of finitely generated projective $S$-modules; that is, $\mathcal{E}_\perp$ contains the set $T^c$ of the compact objects of $D(S)$.

Since $D(S)$ is compactly generated by $T^c$, $D(S)$ is the smallest triangulated category closed under coproducts and containing $T^c$. Thus, if $\mathcal{E}_\perp$ is closed under coproducts, we get that $\mathcal{E}_\perp = D(S)$; hence $\mathcal{E} = 0$.

(2 $\Rightarrow$ 1). By Corollary 2.4, condition (2) implies that $L_G$ induces an equivalence between $D(S)$ and $D(R)$. Hence by [18] or [22] Section 4.1, $T_R$ is a classical $n$-tilting module. □
References


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