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PLANAR LOOPS WITH PRESCRIBED CURVATURE: EXISTENCE, MULTIPLICITY AND UNIQUENESS RESULTS

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ABSTRACT. Let $k : \mathbb{C} \to \mathbb{R}$ be a smooth given function. A k-loop is a closed curve u in $\mathbb{C}$ having prescribed curvature $k(p)$ at every point $p \in u$. We use variational methods to provide sufficient conditions for the existence of k-loops. Then we show that a breaking symmetry phenomenon may produce multiple k-loops, in particular when k is radially symmetric and somewhere increasing. If $k > 0$ is radially symmetric and non-increasing, we prove that any embedded k-loop is a circle; that is, round circles are the only convex loops in $\mathbb{C}$ whose curvature is a non-increasing function of the Euclidean distance from a fixed point. The result is sharp, as there exist radially increasing curvatures $k > 0$ which have embedded k-loops that are not circles.

INTRODUCTION

In this paper we prove existence, multiplicity and uniqueness results for the following *k-loop problem*: given a non constant function $k : \mathbb{C} \to \mathbb{R}$, find a k-loop, that is, a closed curve u in $\mathbb{C}$ having prescribed curvature $k(p)$ at every point $p \in u$.

Several papers deal with loops with prescribed curvature and with related questions; see for example [4], [5], [8], [9], [12], [13]. For similar problems in higher dimensions we quote [3], [6], [7], [13] and references therein.

In Section 2 we provide global sufficient conditions for the existence of k-loops by studying the ordinary differential system

$$(0.1) \qquad \begin{cases} u'' = \left(\int_0^1 |u'|^2 \right)^{1/2} k(u)(iu') & \text{on } (0,1), \\ u \in H_{per}, \end{cases}$$

where H_{per} is the Sobolev space of 1-periodic functions in $H^1_{\text{loc}}(\mathbb{R}, \mathbb{C})$. Any non constant weak solution u to (0.1) is smooth by regularity theory. As u'' is orthogonal to u', we have that $|u'|$ is a constant. Thus $u'' \cdot (iu') = |u'|^3 k(u)$; that is, u is a k-loop.

If the prescribed curvature $k \in C^2(\mathbb{C})$ satisfies

(k_1) $M_k := \sup_{z \in \mathbb{C}} |(\nabla k(z) \cdot z)\, z| < 1$,

(k_2) there exists a constant $k_\infty > 0$ such that $k(z) = k_\infty + o(|z|^{-1})$ as $|z| \to \infty$,

then problem (0.1) can be plainly studied via variational methods. Indeed solutions to (0.1) turn out to be critical points of an energy functional $\mathcal{E}_k : H_{per} \to \mathbb{R}$ that

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enjoys good regularity and geometrical properties thanks to assumptions (k_1) and (k_2) (see Section 1 for details). In particular the Nehari manifold

$$\Sigma = \{u \in H_{per} \mid u \text{ is non-constant, } \mathcal{E}_k'(u) \cdot u = 0\}$$

is a smooth submanifold of H_{per} and the functional $\mathcal{E}_k$ is positively bounded from below on Σ. Notice that every k-loop admits a parametrization $u \in \Sigma$. In spite of the non-homogeneous nature of problem (0.1), it turns out that Σ is a natural constraint for $\mathcal{E}_k$; that is, any stationary point for $\mathcal{E}_k(u)$ on Σ is a k-loop. In particular, if $u \in \Sigma$ achieves the infimum

$$\underline{c} := \inf_{u \in \Sigma} \mathcal{E}_k(u), \tag{0.2}$$

then u is a *minimal k-loop*, that means that u has the minimal energy among all k-loops.

The infimum $\underline{c}$ might be not achieved, due to the lack of compactness produced by the group of translations $u \mapsto u + p$. On the other hand, in Lemma 1.4 we show that $\underline{c} \le \pi/k_\infty$. Then we prove in Theorem 2.1 that a minimal k-loop $u \in \Sigma$ exists, provided that $\underline{c} < \pi/k_\infty$. In particular, it turns out that $\underline{c} < \pi/k_\infty$ if $k(z) > k_\infty$ for $|z|$ large; see Theorem 2.5.

The main results in the present paper highlight the connection between the monotonicity properties of the prescribed curvature k along radial directions and the existence of geometrically distinct k-loops.

For the sake of clarity in this introduction we restrict ourselves to radially symmetric curvatures. It is straightforward to notice that if $k(z) = k(|z|)$, then $\mathcal{R} \circ u$ is a k-loop for any k-loop u and for any rotation $\mathcal{R}$ of the complex plane. Thus, any non-round k-loop generates a 1-dimensional manifold of distinct k-loops.

Multiple k-loops may be produced by a breaking symmetry phenomenon for problem (0.2). For instance, one can construct a curvature $k(z) = k(|z|)$ satisfying (k_1), (k_2) and such that there exist a k-loop which is a circle C_R about the origin and a minimal k-loop which is not a circle (see Corollary 3.2). Here we have parametrized C_R by the map $t \mapsto Re^{2\pi i t}$, so that C_R can be regarded as a k-loop in H_{per}. More precisely, it turns out that C_R is not a minimal k-loop if k increases along the normal directions to C_R.

Theorem 0.1. *Assume that $k \in C^2(\mathbb{C})$ is a radially symmetric curvature satisfying (k_1) and (k_2). If there is a radius $R > 0$ with $Rk(R) = 1$ and $k'(R) > 0$, then the circle C_R is not a minimal k-loop.*

Theorem 0.1 is a consequence of a more general result that will be stated in Section 3 (see Theorem 3.1).

Roughly speaking, if the prescribed curvature is somewhere increasing, then one can expect breaking symmetry phenomena and multiple k-loops. This might happen also for curvatures that do not satisfy the assumptions (k_1) and (k_2). In Example 4.2 we notice that the curvature of any ellipse increases with the distance from the center. Conversely, non-increasing curvatures may have only round k-loops.

Theorem 0.2. *Let $k \in C^0(\mathbb{C})$ be a positive and radially symmetric function. Assume that k is non-increasing as a function of the distance from the origin. Then any embedded k-loop is a circle.*

The result is sharp, in view of Example 4.2 in Section 4.

Our proof of Theorem 0.2 is based on Osserman's construction for the four-vertex theorem. One of the main tools is the *Touching Lemma* for the mean curvature operator, that follows from Hopf's maximum principle.

Several uniqueness (up to homothety) results are available for similar geometrical problems. A complete list of references would lead us far from our purpose. We quote the pioneering papers [2] by Alexandrov and [1] by Aeppli, where the prescribed curvature is assumed to be homogeneous of degree -1. Treibergs and Wei [13] proved the uniqueness of embedded radial graphs over the unit sphere $\mathbb{S}^n$ having positive prescribed mean curvature $H \in C^1(\mathbb{R}^{n+1})$, such that $H(p) + \nabla H(p) \cdot p \le 0$ for any $p \in \mathbb{R}^{n+1}$.

A result similar to Theorem 0.2 is missing for the H-bubble problem. It would be of interest to know if spheres are the only convex surfaces in $\mathbb{R}^3$ whose mean curvature H is a non-increasing function of the distance from the origin.

Notation. Let z_1, z_2 be two points in the complex plane $\mathbb{C}$. We denote by $z_1 z_2 \in \mathbb{C}$ and by $z_1 \cdot z_2 \in \mathbb{R}$ their complex and scalar products, respectively. Let $R > 0$ and $z \in \mathbb{C}$. The open disk of center z and radius R is denoted by $D_R(z)$. If $z = 0$ we simply write D_R instead of $D_R(0)$. The unit circle $\mathbb{S}^1 = \partial D_1$ is parametrized by the function

$$\omega(t) := e^{2\pi i t}, \quad \omega : (0,1) \to \mathbb{C}\,. \tag{0.3}$$

A *loop* is a closed curve parametrized by a periodic function $g : \mathbb{R} \to \mathbb{C}$ of class C^2 and such that $g'(t) \neq 0$ for any $t \in I$. We will often identify the curve g with its image. It is well known that the only loops having constant curvature are circles. An *embedded loop* is a closed curve without self-intersections. Any compact, connected, 1-dimensional submanifold of $\mathbb{C}$ without boundary is an embedded loop.

We let

$$H_{per} = \left\{ \overline{u} \circ \omega \mid \overline{u} \in H^1(\mathbb{S}^1, \mathbb{C}) \right\},$$

where ω is defined in (0.3). We define in H_{per} the Hilbertian scalar product

$$\langle u, v \rangle = \int_0^1 u' v' + \left(\int_0^1 u \right) \cdot \left(\int_0^1 v \right).$$

Notice that H_{per} contains $\mathbb{C}$ as a closed subspace. We denote by H^{-1}_{per} the topological dual space to H_{per}.

1. Preliminaries

We start by recalling the main features of the variational approach to problem (0.1). For details we refer to [5] and to [9]. Let $k \in C^0(\mathbb{C})$ be a given bounded function, and set

$$m(z) = \int_0^1 k(sz) s \, ds, \quad m : \mathbb{C} \to \mathbb{R}.$$

For $u \in H_{per}$ we put

$$\mathcal{L}(u) = \left(\int_0^1 |u'|^2 \right)^{1/2}, \quad \mathcal{A}_k(u) = \int_0^1 m(u) u \cdot (iu').$$

The functional $\mathcal{A}_k(u)$ is well defined on H_{per} as $u \in L^\infty$ for any $u \in H_{per}$, by the Sobolev embedding theorem. The real number $\mathcal{A}_k(u)$ measures the algebraic area enclosed by the curve u with respect to the weight k. Moreover, the following isoperimetric inequality holds:

$$4\pi|\mathcal{A}_k(u)| \le \|k\|_\infty \mathcal{L}(u)^2 \quad \text{for any } u \in H_{per}\,. \tag{1.1}$$

For any constant $k_\infty \neq 0$ it turns out that

$$\mathcal{A}_{k_\infty}(u) = \frac{k_\infty}{2}\int_0^1 u\cdot(iu')\,. \tag{1.2}$$

In the next lemma we state some remarks on the area functional. We omit the simple proofs.

Lemma 1.1. *Let $k \in C^1(\mathbb{C})$. Assume that $M_k = \sup_{z\in\mathbb{C}}|(\nabla k(z)\cdot z)z|$ is finite and that there exists $\lim_{|z|\to\infty} k(z) = k_\infty \in \mathbb{R}$. Then*

$$|(k(z)-k_\infty)z| \le M_k, \tag{1.3}$$

$$|(2m(z)-k(z))z| \le M_k, \tag{1.4}$$

$$2\left|\mathcal{A}_{k-k_\infty}(u)\right| \le M_k\ \mathcal{L}(u) \quad \textit{for any } u \in H_{per}. \tag{1.5}$$

The energy $\mathcal{E}_k : H_{per} \to \mathbb{R}$ is defined by

$$\mathcal{E}_k(u) = \mathcal{L}(u) + \mathcal{A}_k(u) = \left(\int_0^1 |u'|^2\right)^{1/2} + \int_0^1 m(u)u\cdot(iu')\ .$$

If k is of class C^1, then the functional $\mathcal{E}_k$ is Frechét differentiable on $H_{per}\backslash\mathbb{C}$ and

$$\mathcal{E}_k'(u)\cdot\varphi = \frac{1}{\mathcal{L}(u)}\int_0^1 u'\cdot\varphi' + \int_0^1 k(u)\varphi\cdot(iu') \quad \text{for any } u\in H_{per}\backslash\mathbb{C},\ \varphi\in H_{per} \tag{1.6}$$

(see for example [5]). In particular, any critical point for $\mathcal{E}_k$ on $H_{per}\backslash\mathbb{C}$ parametrizes a smooth k-loop.

Remark 1.2. Let k be as in Lemma 1.1, and assume in addition that $k_\infty \neq 0$. Then the energy $\mathcal{E}_k$ is unbounded from below. Fix any map $u \in H_{per}$ such that $\mathcal{A}_{k_\infty}(u) < 0$. Then, using (1.5), we get

$$\begin{aligned} \mathcal{E}_k(su) &= \mathcal{L}(su) + \mathcal{A}_{k-k_\infty}(su) + \mathcal{A}_{k_\infty}(u) \\ &\le (1+M_k)s\mathcal{L}(u) + s^2\mathcal{A}_{k_\infty}(u) \to -\infty \quad \text{as } s\to\infty. \end{aligned}$$

Next we introduce the curvature function

$$f(z) := 2k(z) + (\nabla k(z)\cdot z), \quad f:\mathbb{C}\to\mathbb{R}\,.$$

Since $\displaystyle\int_0^1 f(sz)s\ ds = k(z)$, then from (1.6) we infer

$$\mathcal{E}_k'(u)\cdot u = \mathcal{L}(u) + \int_0^1 k(u)u\cdot(iu') = \mathcal{E}_f(u)\,. \tag{1.7}$$

Thus, $\mathcal{E}_k'(u)\cdot u$ equals the energy of u with respect to the curvature f. If k is of class C^2, then, coherently with (1.6), the functional $\mathcal{E}_f$ is Frechét differentiable on $H_{per}\backslash\mathbb{C}$, and

$$\mathcal{E}_f'(u)\cdot\varphi = \frac{1}{\mathcal{L}(u)}\int_0^1 u'\cdot\varphi' + \int_0^1 \left(2k(u) + (\nabla k(u)\cdot u)\right)\varphi\cdot(iu') \tag{1.8}$$

for any $u\in H_{per}\backslash\mathbb{C}$, $\varphi\in H_{per}$. Lastly we introduce the Nehari manifold

$$\Sigma = \{u\in H_{per} \mid \mathcal{L}(u) > 0,\ \mathcal{E}_k'(u)\cdot u = \mathcal{E}_f(u) = 0\}$$

and the infimum

$$\underline{c} := \inf_{\Sigma} \mathcal{E}_k\,. \tag{1.9}$$

Remark 1.3. Assume that $k \in C^2(\mathbb{C})$ satisfies $M_k < 1$ and $k(z) \to k_\infty = 0$ as $|z| \to \infty$. Then from (1.5), (1.7) and (1.3) it easily follows that the energy is coercive with respect to the seminorm $\mathcal{L}(u)$ and that the manifold Σ is empty.

From now on we assume that $k \in C^2(\mathbb{C})$ satisfies the assumptions (k_1) and (k_2) in the introduction. In the next lemma we show that the set Σ is not empty and smooth, and it is a natural constraint for $\mathcal{E}_k$.

Lemma 1.4. *The following facts hold.*

1. *Let $u \in H_{per} \setminus \mathbb{C}$. Then $u \in \Sigma$ if and only if $\mathcal{E}_k(u) = \sup_{s>0} \mathcal{E}_k(su)$.*
2. *Σ is a non-empty submanifold of H_{per} of class C^1.*
3. *$0 < \underline{c} \le \pi/k_\infty$.*
4. *Every critical point u for $\mathcal{E}_k$ on Σ solves $\mathcal{E}_k'(u) = 0$.*

Proof. We start by noticing that

$$\mathcal{E}_f'(u)\cdot u = -\mathcal{L}(u) + \int_0^1 (\nabla k(u)\cdot u) u \cdot (iu') \quad \text{for any } u \in \Sigma \tag{1.10}$$

by (1.8). Thus from (k_1) we readily get

$$-(1+M_k)\mathcal{L}(u) \le \mathcal{E}_f'(u)\cdot u \le (-1+M_k)\mathcal{L}(u) < 0 \quad \text{for any } u \in \Sigma. \tag{1.11}$$

To prove (1) we fix any function u in $H_{per}\setminus\mathbb{C}$ such that $\mathcal{E}_k(u) = \sup_{s>0}\mathcal{E}_k(su)$. Then the derivative of the function $s \mapsto \mathcal{E}_k(su)$ vanishes at $s=1$; that is, $u \in \Sigma$. Conversely, if $u \in \Sigma$ we put $F(s) := \mathcal{E}_k(su)$ for $s > 0$. We use (1.7), (1.10) and (1.11) to get

$$\begin{aligned} sF'(s) &= \mathcal{E}_f(su), \\ sF'(s) + s^2F''(s) &= \mathcal{E}_f'(su)\cdot(su) \le (-1+M_k)s^2\mathcal{L}(u). \end{aligned}$$

In particular, $s=1$ is a critical point for F on $\{s>0\}$ and every critical point for F is a local maximum for F. Thus the function F achieves its absolute maximum at $s=1$; that is, $\mathcal{E}_k(u) = \sup_{s>0}\mathcal{E}_k(su)$. This completes the proof of (1).

We point out that from (1) and from Remark 1.2 it follows that Σ is non-empty and

$$\underline{c} = \inf_{\substack{u\in H_{per} \\ \mathcal{A}_{k_\infty}(u)<0}} \sup_{s>0} \mathcal{E}_k(su)\,. \tag{1.12}$$

Since $k \in C^2(\mathbb{C})$ we have that $\mathcal{E}_f$ is of class C^1 on $H_{per}\setminus\mathbb{C}$. In addition, from (1.11) it follows that $\mathcal{E}_f'(u) \ne 0$ for any $u \in \Sigma$. Thus the constraint Σ has a normal direction at every point, and claim (2) is proved.

To check that $\underline{c}$ is positive we use the isoperimetric inequality (1.1). For any $u \in \Sigma$ we get $0 = \mathcal{E}_f(u) = \mathcal{L}(u) + \mathcal{A}_f(u) \ge \mathcal{L}(u) - \|f\|_\infty (4\pi)^{-1}\mathcal{L}(u)^2$. In particular,

$$\|f\|_\infty \mathcal{L}(u) \ge 4\pi \quad \text{for any } u \in \Sigma\,. \tag{1.13}$$

As $\mathcal{E}_f(u) = 0$ for any $u \in \Sigma$, then from (1.7) we get

$$2\mathcal{E}_k(u) = \mathcal{L}(u) + \int_0^1 (2m(u) - k(u))\, u\cdot(iu') \quad \text{for any } u \in \Sigma\,. \tag{1.14}$$

Thus from (1.4) we infer

$$2\mathcal{E}_k(u) \geq (1 - M_k)\mathcal{L}(u) \quad \text{for any } u \in \Sigma, \tag{1.15}$$

that compared with (1.13) gives $\underline{c} > 0$, since $M_k < 1$. To prove that $\underline{c} \leq \pi/k_\infty$ we parametrize the unit circle around 0 with the curve $\omega(t) = e^{2\pi i t}$ as in (0.3). By (1.12) it turns out that

$$\underline{c} \leq \sup_{s>0} \mathcal{E}_k(s(\omega + p)) = \sup_{s>0} \left[\mathcal{E}_{k_\infty}(s(\omega + p)) + \mathcal{A}_{k-k_\infty}(s(\omega + p))\right]$$

for any $p \in \mathbb{C}$. It is easy to compute

$$\mathcal{E}_{k_\infty}(s(\omega + p)) = 2\pi s - \pi k_\infty s^2 \,.$$

Since $s(\omega + p)$ parametrizes the circle of radius s around sp, then

$$|\mathcal{A}_{k-k_\infty}(s(\omega + p))| = \left| \int_{D_s(sp)} (k(z) - k_\infty)\, dz \right| \leq M_k \int_{D_s(sp)} \frac{1}{|z|}\, dz$$

by the divergence theorem and by (1.3). Thus for $|p|$ large it holds that

$$|\mathcal{A}_{k-k_\infty}(s(\omega + p))| \leq M_k \int_{D_s(sp)} \frac{1}{s(|p| - 1)}\, dz = M_k \frac{\pi s}{|p| - 1}\,,$$

and therefore

$$\sup_{s>0} \mathcal{E}_k(s(\omega + p)) \leq \sup_{s>0}\; \pi s \left(2 + \frac{M_k}{|p| - 1} - k_\infty s\right) = \frac{\pi}{k_\infty} + O(|p|^{-1})\,. \tag{1.16}$$

Hence $\underline{c} \leq \pi/k_\infty$, as desired.

It remains to prove (4). If $u \in \Sigma$ is a critical point for $\mathcal{E}_k$ on Σ, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $\mathcal{E}'_k(u) \cdot \varphi = \lambda \mathcal{E}'_f(u) \cdot \varphi$ for any $\varphi \in H_{per}$. In particular $\lambda \mathcal{E}'_f(u) \cdot u = \mathcal{E}'_k(u) \cdot u = 0$ as $u \in \Sigma$. Thus $\lambda = 0$ by (1.11), and $\mathcal{E}'_k(u) = 0$. □

The next lemma will be crucial later on.

Lemma 1.5. *For any $p \in \mathbb{C}$ there exists a unique $s_p > 0$ such that $s_p(\omega + p) \in \Sigma$. Moreover, the function $p \mapsto s_p$ is of class C^1 on $\mathbb{C}$.*

Proof. Argue as in the proof of claim (1) to check that for any $p \in \mathbb{C}$ the map $s \mapsto \mathcal{E}_k(s(\omega + p))$ achieves its maximum on $(0, \infty)$ at its unique critical point $s_p \in (0, \infty)$. Thus $s_p(\omega + p) \in \Sigma$. As s_p is uniquely and implicitly defined by the equation $\mathcal{E}_f(s_p(\omega+p)) = 0$, and since the functional $\mathcal{E}_f$ is continuously differentiable on $H_{per} \setminus \mathbb{C}$, then the function $p \mapsto s_p$ is of class C^1 on $\mathbb{C}$ by (1.11). □

Remark 1.6. Lemma 1.4 holds whenever $M_k < 1$ and $k(z) \to k_\infty > 0$ as $|z| \to \infty$.

2. Existence

We are now in a position to prove our existence result.

Theorem 2.1. *Assume that $k \in C^2(\mathbb{C})$ satisfies* (k_1), (k_2) *and let $\underline{c}$ be the infimum in* (1.9). *If $\underline{c} < \pi/k_\infty$, then there exists a minimal k-loop.*

Proof. Let $u_n \in \Sigma$ be a minimizing sequence for $\mathcal{E}_k$ on Σ, that is,

$$\mathcal{E}_k(u_n) \to \underline{c} < \frac{\pi}{k_\infty}\,. \tag{2.1}$$

By Ekeland's variational principle we can assume that u_n is a sequence of approximate critical points for $\mathcal{E}_k$ on the constraint Σ. This means that

$$\mathcal{E}_k'(u_n) - \lambda_n \mathcal{E}_f'(u_n) \to 0 \quad \text{in } H^{-1}_{per} \tag{2.2}$$

for some $\lambda_n \in \mathbb{R}$. The seminorms $\mathcal{L}(u_n)$ are uniformly bounded by (1.15). Since the L^∞ norm of $f(u) = 2k(u) + (\nabla k(u)\cdot u)$ is bounded by a constant that depends only on k, then the sequence $\mathcal{E}_f'(u_n)$ is bounded in H^{-1}_{per} by (1.8). Finally, we notice that the sequence $\mathcal{E}_f'(u_n)\cdot u_n$ is bounded and bounded away from zero by (1.11).

To conclude, we only have to prove that the sequence

$$p_n := \int_0^1 u_n$$

is bounded in $\mathbb{C}$. Indeed, in this case we can assume that $u_n \rightharpoonup u$ weakly in H_{per}, for some $u \in H_{per}$. In particular, $u_n \to u$ uniformly on $[0,1]$ by the Sobolev embedding theorem. Test (2.2) with u_n to get

$$0 = \mathcal{E}_k'(u_n)\cdot u_n = \lambda_n \mathcal{E}_f'(u_n)\cdot u_n + o(1)\,.$$

Since the sequence $\mathcal{E}_f'(u_n)\cdot u_n$ is bounded away from zero, then $\lambda_n \to 0$, which implies $\mathcal{E}_k'(u_n) \to 0$ by (2.2). Since $|k(u_n)(u_n-u)\cdot(iu_n')| \le \|k\|_\infty \|u_n-u\|_\infty\, |u_n'| \to 0$ in L^1, then, in particular,

$$o(1) = \mathcal{E}_k'(u_n)\cdot(u_n-u) = \frac{1}{\mathcal{L}(u_n)}\int_0^1 u_n'\cdot(u_n'-u') + o(1) = \frac{1}{\mathcal{L}(u_n)}\int_0^1 |u_n'-u'|^2 + o(1).$$

Taking (1.13) into account we infer that $u_n' \to u'$ strongly in L^2. Therefore $u_n \to u$ in the H_{per}-norm. Then $\mathcal{E}'(u) = 0$ and $\mathcal{E}_k(u) = \underline{c}$ easily follow by continuity.

Assume by contradiction that the averages p_n are not bounded. Then, for a subsequence, it turns out that $u_n - p_n \rightharpoonup U$ weakly in H_{per} and uniformly on $[0,1]$, where $U \in H_{per}$ has zero mean value on $(0,1)$. We can also assume that there exists

$$\alpha := \lim_{n\to\infty} \mathcal{L}(u_n) \in \mathbb{R}\,.$$

Notice that $\alpha > 0$ by (1.13). Since $u_n - p_n \to U$ and $|u_n| \to \infty$ uniformly, then from (k_1) and (k_2) it follows that

$$(k(u_n) - k_\infty)u_n \to 0 \quad \text{uniformly}, \tag{2.3}$$

$$k(u_n)(u_n - p_n) \to k_\infty U \quad \text{uniformly}, \tag{2.4}$$

$$\nabla k(u_n)\cdot u_n \to 0 \quad \text{uniformly}\,. \tag{2.5}$$

Using (2.3) we get

$$\int_0^1 k(u_n)u_n\cdot(iu_n') = k_\infty \int_0^1 u_n\cdot(iu_n') + o(1) = k_\infty \int_0^1 U\cdot(iU') + o(1)\,,$$

as $\int_0^1 (iu_n') = 0$. In particular, from $\mathcal{E}_k'(u_n)\cdot u_n = 0$ and from (1.7) we infer that $U \ne 0$ and

$$\alpha = -k_\infty \int_0^1 U\cdot(iU')\,. \tag{2.6}$$

Next we notice that $(k(u_n)-k_\infty)p_n = (k(u_n)-k_\infty)(p_n-u_n)+(k(u_n)-k_\infty)u_n \to 0$ uniformly, since $k(u_n)\to k_\infty$ uniformly, $\sup_n \|u_n-p_n\|<\infty$, and by (2.3). This implies

$$\int_0^1 k(u_n)p_n\cdot(iu_n') = k_\infty p_n \cdot \int_0^1 (iu_n') + o(1) = o(1)\,.$$

Using (1.6) once again we infer

$$\mathcal{E}_k'(u_n)\cdot(u_n-p_n) = -\mathcal{E}_k'(u_n)\cdot p_n = \int_0^1 k(u_n)p_n\cdot(iu_n') = o(1)\,,$$

as $u_n\in\Sigma$. By similar arguments, from (1.8) we also obtain

$$\mathcal{E}_f'(u_n)\cdot(u_n-p_n) = \alpha + 2k_\infty \int_0^1 U\cdot(iU') + o(1) = -\alpha + o(1) \tag{2.7}$$

by (2.4), (2.5) and (2.6). Consequently, since by (2.2) we have that

$$\mathcal{E}_k'(u_n)\cdot(u_n-p_n) = \lambda_n \mathcal{E}_f'(u_n)\cdot(u_n-p_n) + o(1),$$

then $o(1)=\lambda_n(-\alpha+o(1))$. Since $\alpha>0$, then $\lambda_n\to 0$. Thus $\mathcal{E}_k'(u_n)\cdot\varphi\to 0$ for any $\varphi\in H_{per}$ by (2.2), as the sequence $\mathcal{E}_f'(u_n)$ is bounded in H^{-1}_{per}. Therefore

$$o(1) = \mathcal{E}_k'(u_n)\cdot\varphi = \frac{1}{\alpha}\int_0^1 U'\varphi' + 2k_\infty\int_0^1 \varphi\cdot(iU') + o(1) \quad \text{for any } \varphi\in H_{per}.$$

Thus, U solves $U''=\alpha k_\infty(iU')$, and, in particular, $\mathcal{L}(U)=\alpha=\lim_n \mathcal{L}(u_n)$ by (2.6). This is sufficient to conclude that $u_n-p_n\to U$ strongly in H_{per}. In addition, U is a non-constant solution to the linear ordinary differential system $U''=\mathcal{L}(U)k_\infty(iU')$. Thus U parametrizes a circle of radius k_∞^{-1}, and

$$\mathcal{L}(U) = -k_\infty\int_0^1 U\cdot(iU') \geq \frac{2\pi}{k_\infty}\ , \quad \mathcal{E}_{k_\infty}(U) = \frac{L(U)}{2}\,. \tag{2.8}$$

Next we claim that $\mathcal{E}_k(u_n)=\mathcal{E}_{k_\infty}(U)+o(1)$. Notice that

$$(2m(u_n)-k_\infty)\,(u_n) = 2\int_0^1 (k(su_n)-k_\infty)(su_n)\;ds \to 0$$

pointwise in $(0,1)$ by (k_2), (1.3) and Lebesgue's Theorem. Since, in addition, $iu_n'\to iU'$ in L^2, then

$$2\mathcal{A}_{k-k_\infty}(u_n) = \int_0^1 (2m(u_n)-k_\infty)\,(u_n)\cdot(iu_n') = o(1)$$

by (1.4) and again by Lebesgue's Theorem. In conclusion, we get

$$\mathcal{E}_k(u_n) = \mathcal{L}(u_n) + \frac{k_\infty}{2}\int_0^1 (u_n-p_n)\cdot(iu_n') + \mathcal{A}_{k-k_\infty}(u_n) = \mathcal{E}_{k_\infty}(U)+o(1) = \frac{\pi}{k_\infty}+o(1)\,.$$

Taking (2.1) and (2.8) into account we get

$$\frac{\pi}{k_\infty} > \underline{c} = \mathcal{E}_k(u_n)+o(1) = \mathcal{E}_{k_\infty}(U)+o(1) \geq \frac{\pi}{k_\infty}+o(1)\,,$$

a contradiction for n sufficiently large. This proves that the sequence p_n is bounded in $\mathbb{C}$ and completes the proof of the theorem. □

Remark 2.2. We notice that the assumption on the sign of k_∞ is not restrictive. Indeed, $t\mapsto u(t)$ is a k-loop if and only if $t\mapsto u(1-t)$ has curvature $-k$.

Remark 2.3. By suitably modifying the compactness argument in the proof of Theorem 2.1 one can completely describe the behavior of any sequence of approximate solutions. More precisely, assume that $u_n \in \Sigma$ satisfies $\mathcal{E}_k(u_n) \to c$ and $\mathcal{E}'_k(u_n) - \lambda_n \mathcal{E}'_f(u_n) \to 0$ for some $c \in \mathbb{R}$ and for a sequence $\lambda_n \in \mathbb{R}$. Then, up to a subsequence, either u_n converges in H_{per} to a k-loop u such that $\mathcal{E}_k(u) = c$, or $\mu := \pi c k_\infty^{-1}$ is a positive integer, and

$$\left| \int_0^1 u_n \right| \to \infty\ , \quad u_n - \int_0^1 u_n \to\ U \quad \text{in } H_{per},$$

where $U(t) = k_\infty^{-1} \sigma e^{2\pi\mu i t}$ for some $\sigma \in \mathbb{S}^1$.

Remark 2.4. Assume that $\underline{c} = \pi/k_\infty$ and that no minimal k-loop exists. Then using Remark 2.3 and arguing as in [7] one can find sufficient conditions for the existence of highly unstable k-loops.

Some conditions to ensure that $\underline{c} < \pi/k_\infty$ can be easily given. For instance, notice that

$$\underline{c} \le \sup_{s>0}\ \mathcal{E}_k(s(\omega + p))$$

for any $p \in \mathbb{C}$, by (1.12), where ω is defined in (0.3). Since $s(\omega+p)$ parametrizes the circle of radius s around sp, then $\mathcal{A}_k(s(\omega+p)) = -\int_{D_s(sp)} k(z)\ dz$ by the divergence theorem. Thus

$$\mathcal{E}_k(s(\omega + p)) =\ 2\pi s - \int_{D_s(sp)} k(z)\ dz\,. \tag{2.9}$$

In particular, $\underline{c} < \pi/k_\infty$ if there exists a point $p \in \mathbb{C}$ such that

$$\sup_{s>0} \left(2\pi s - \int_{D_s(sp)} k(z)\ dz \right) < \frac{\pi}{k_\infty}\,.$$

This happens, for instance, if $k(z) > k_\infty$ on $\mathbb{C}$ (take $p = 0$). Actually, weaker sufficient conditions can be given.

Theorem 2.5. *Assume that $k \in C^2(\mathbb{C})$ satisfies* (k_1), (k_2). *If $k(z) > k_\infty$ for $|z|$ large, then there exists a minimal k-loop.*

Proof. We only have to show that $\underline{c} < \pi/k_\infty$. Put as before $\omega(t) = e^{2\pi i t}$. For any $p \in \mathbb{C}$ let $s_p > 0$ be the unique positive number defined in Lemma 1.5. Then $\underline{c} \le \mathcal{E}_k(s_p(\omega + p))$, since $s_p(\omega + p) \in \Sigma$.

Recall from (1.7) that $\mathcal{E}_f(u) = \mathcal{L}(u) + k_\infty \int_0^1 u \cdot (iu') + \int_0^1 (k(u) - k_\infty) u \cdot (iu')$ for any $u \in H_{per}$, and therefore

$$0 = \mathcal{E}_f(s_p(\omega + p)) = 2\pi s_p - 2\pi k_\infty s_p^2 + s_p \int_0^1 (k(s_p(\omega + p)) - k_\infty) s_p(\omega + p) \cdot (i\omega')\,.$$

In particular we infer

$$k_\infty s_p = 1 - \int_0^1 (k(s_p(\omega + p)) - k_\infty) s_p(\omega + p) \cdot \omega \tag{2.10}$$

since $i\omega' = -2\pi\omega$. From (1.3) and (2.10) we get

$$1 - M_k \le k_\infty s_p \le 1 + M_k \quad \text{for any } p \in \mathbb{C}. \tag{2.11}$$

Next we take a sequence of points $p_n \in \mathbb{C}$ such that $|p_n| \to \infty$. By (2.9), (2.11), and since $k(z) > k_\infty$ for $|z|$ large enough, we get

$$\underline{c} \le \mathcal{E}_k(s_{p_n}(\omega + p_n)) = 2\pi s_{p_n} - \int_{D_{s_{p_n}}(s_{p_n} p_n)} k(z)\, dz < 2\pi\, s_{p_n} - \pi k_\infty s_{p_n}^2 .$$

Thus $\underline{c} < \sup_{s>0} \left(2\pi s - \pi k_\infty s^2\right) = \pi / k_\infty$, and the theorem is completely proved. □

Remark 2.6. A slightly different proof can be obtained by following the arguments in [6], proof of Corollary 2.13. As in [6] the assumption on the sign of $k(z) - k_\infty$ can be weakened. It is sufficient to ask that there exist $\sigma \in \mathbb{S}^1$, $R, \delta > 0$ such that $k(z) > k_\infty$ for any $z \in \mathbb{C} \setminus D_R$ with $\left|\sigma - z|z|^{-1}\right| < \delta$.

3. Breaking symmetry and multiplicity

In this section we identify the circle $C_R = \{|z| = R\}$ with its parametrization $t \mapsto Re^{2\pi i t}$. With this notation we have that C_R is a k-loop if and only if $Rk(\cdot) \equiv 1$ on C_R. Theorem 0.1 is an immediate consequence of the next result.

Theorem 3.1. *Let $k \in C^2(\mathbb{C})$ be a curvature satisfying* (k_1) *and* (k_2)*. Assume that there exists $R > 0$ such that $Rk(z) \equiv 1$ and $\nabla k(z) \cdot z \ge 0$ for any $z \in C_R$. If the circle C_R is a minimal k-loop, then $\nabla k(z) \cdot z \equiv 0$ on C_R.*

Proof. We start by noticing that $u \in H_{per}$ is a k-loop if and only if $u_R := Ru$ is a k_R-loop, where $k_R(z) = R^{-1}k(R^{-1}z)$. Therefore we are allowed to assume $R = 1$ and $C_R = \mathbb{S}^1$. Hence $k \equiv 1$ on the unit sphere, $\mathbb{S}^1$ is a minimal k-loop and

$$A(z) := \nabla k(z) \cdot z \ge 0 \quad \text{on } \mathbb{S}^1. \tag{3.1}$$

Since $k \in C^2(\mathbb{C})$, then the energy $\mathcal{E}_k$ is twice differentiable on $H_{per} \setminus \mathbb{C}$, and

$$\begin{aligned}\mathcal{E}_k''(u)[\varphi, \psi] = \frac{1}{\mathcal{L}(u)} \int_0^1 \varphi' \cdot \psi' \quad &- \quad \frac{1}{\mathcal{L}(u)^3} \left(\int_0^1 u' \cdot \varphi'\right) \left(\int_0^1 u' \cdot \psi'\right) \\ &+ \quad \int_0^1 \varphi \cdot [k(u)(i\psi') + (\nabla k(u) \cdot \psi)(iu')]\end{aligned}$$

for all $u \in H_{per} \setminus \mathbb{C}$, and for any $\varphi, \psi \in H_{per}$ (see [5]). To compute $\mathcal{E}_k''(\omega)$ we notice that $\mathcal{L}(\omega) = 2\pi$ and $\omega'' = 2\pi(i\omega') = -(2\pi)^2\omega$. In particular $\int \omega' \cdot \varphi' = (2\pi)^2 \int \omega \cdot \varphi$ for any $\varphi \in H_{per}$. Since k is constant on $\mathbb{S}^1$, then $\nabla k(\omega) = A(\omega)\omega$, where A is defined in (3.1). Thus we get

$$\begin{aligned}\mathcal{E}_k''(\omega)[\varphi, \psi] = \frac{1}{2\pi} \int_0^1 \varphi' \cdot \psi' \quad &- \quad 2\pi \left(\int_0^1 \omega \cdot \varphi\right) \left(\int_0^1 \omega \cdot \psi\right) \\ &+ \quad \int_0^1 \varphi \cdot [(i\psi') - 2\pi A(\omega)(\omega \cdot \psi)\omega]\end{aligned} \tag{3.2}$$

for any $\varphi, \psi \in H_{per}$. From (3.2) we first get $\mathcal{E}_k''(\omega)[\varphi, \omega] = -2\pi \int_0^1 (1 + A(\omega))\, \omega \cdot \varphi$ for any $\varphi \in H_{per}$, and in particular

$$\mathcal{E}_k''(\omega)[\omega, \omega] = -2\pi \left(1 + \int_0^1 A(\omega)\right) < 0 \,. \tag{3.3}$$

Taking instead $\varphi \equiv p$ to be a constant function we get

$$\mathcal{E}_k''(\omega)[p, \omega] = -2\pi \int_0^1 A(\omega)\omega \cdot p \,, \tag{3.4}$$

since ω has zero mean value on $(0,1)$. From (3.2) we infer also

$$\mathcal{E}_k''(\omega)[p,p] = -2\pi \int_0^1 A(\omega)(\omega\cdot p)^2 \,. \tag{3.5}$$

For any point $p\in\mathbb{C}$ we put $s(t):=s_{tp}$, where $s_{tp}>0$ is defined by the condition

$$s_{tp}(\omega+tp) = s(t)(\omega+tp)\in\Sigma\,,$$

as in Lemma 1.5. Since ω is a minimal k-loop, the function $g(t):=\mathcal{E}_k(s(t)(\omega+tp))$ attains its minimum at $t=0$. Notice that $s(0)=1$, since ω is a k-loop. Moreover the map $s(t)$ is of class C^1 on $(0,\infty)$ by Lemma 1.5 and

$$\mathcal{E}_k'(s(t)(\omega+tp))\cdot(\omega+tp) = 0 \quad \text{for any } t\in\mathbb{R}. \tag{3.6}$$

We compute

$$g'(t) = \mathcal{E}_k'(s(t)(\omega+tp))\cdot(s'(t)(\omega+tp)+s(t)p) = s(t)\ \mathcal{E}_k'(s(t)(\omega+tp))\cdot p,$$

by (3.6). Thus g is twice differentiable, and

$$g''(t) = s'(t)\mathcal{E}_k'(s(t)(\omega+tp))\cdot p + s(t)\mathcal{E}_k''(s(t)(\omega+tp))\,[p, s'(t)(\omega+tp)+p]\,.$$

From $s(0)=1$ and $\mathcal{E}_k'(\omega)=0$ we get

$$g''(0) = \beta\mathcal{E}_k''(\omega)[p,\omega] + \mathcal{E}_k''(\omega)[p,p]\,, \tag{3.7}$$

where we have set $\beta = s'(0)$. To compute β we differentiate (3.6) with respect to t:

$$\mathcal{E}_k'(s(t)(\omega+tp))\cdot p + \mathcal{E}_k''(s(t)(\omega+tp))[\omega+tp, s'(t)(\omega+tp)+s(t)p] = 0\,.$$

Thus at $t=0$ it holds that $\mathcal{E}_k''(\omega)[\omega,\beta\omega+p]=0$, compared with (3.7), gives

$$\begin{aligned} g''(0) &= -\frac{(\mathcal{E}_k''(\omega)[p,\omega])^2}{\mathcal{E}_k''(\omega)[\omega,\omega]} + \mathcal{E}_k''(\omega)[p,p] \\ &= 2\pi\left[\frac{\left(\int_0^1 A(\omega)(\omega\cdot p)\right)^2}{1+\int_0^1 A(\omega)} - \int_0^1 A(\omega)(\omega\cdot p)^2\right] \end{aligned}$$

by (3.3), (3.4) and (3.5). Since 0 is a minimum point for g, then $g''(0)\geq 0$; that is,

$$\begin{aligned} \left(\int_0^1 A(\omega)(\omega\cdot p)^2\right)\left(1+\int_0^1 A(\omega)\right) &\leq \left(\int_0^1 A(\omega)(\omega\cdot p)\right)^2 \\ &\leq \left(\int_0^1 A(\omega)(\omega\cdot p)^2\right)\left(\int_0^1 A(\omega)\right) \end{aligned}$$

by the Hölder inequality. We infer that $\int A(\omega)(\omega\cdot p)^2 = 0$ for any $p\in\mathbb{C}$, since $A(\omega)\geq 0$ by assumption. Take $p=1\in\mathbb{C}$ and then $p=i$ to get

$$0 = \int_0^1 A(\omega)[(\omega\cdot 1)^2 + (\omega\cdot i)^2] = \int_0^1 A(\omega)\,.$$

Therefore $A(\omega)\equiv 0$, and the theorem is proved. □

From now on we restrict our attention to radially symmetric curvatures $k(P)\equiv k(|P|)$. Quite trivially, if k is any continuous radially symmetric map on $\mathbb{C}$ such that $k(z)\to k_\infty>0$ as $|z|\to\infty$, then there exists $R>0$ such that $Rk(R)=1$. In this case the circle C_R is a k-loop. In addition, if u is a k-loop, then $\mathcal{R}\circ u$ is a k-loop for any rotation $\mathcal{R}$ of the complex plane. Therefore, the next multiplicity result immediately follows from Theorem 0.1.

Corollary 3.2. *Let $k \in C^2(\mathbb{C})$ be a radially symmetric function satisfying* (k_1), (k_2). *Assume that $k'(R) > 0$ for any R such that $Rk(R) = 1$. If $\underline{c}$ is achieved, then no minimal k-loop is a circle around the origin. Hence there exist at least one round k-loop and a rotationally-invariant family of non-round k-loops.*

It has to be noticed that the class of curvatures described in Corollary 3.2 is non-empty. For instance, let $\lambda > 0$ and put

$$k(r) = 1 + \lambda \, \frac{r^2 - 1}{r^4 + 1} \, .$$

If λ is sufficiently close to 0, then k satisfies (k_1) and (k_2). In addition $Rk(R) = 1$ if and only if $R = 1$, and $k'(1) > 0$. The existence of a minimal k-loop is given by Theorem 2.5.

Remark 3.3. Let k be any continuous radially symmetric curvature. Then the existence of a round k-loop centered at 0 is a necessary condition for the existence of k-loops. Assume that $g : \mathbb{R} \to \mathbb{C}$ is a periodic parametrization by arclength of a k-loop. Then g solves

$$\begin{cases} g'' = k(|g|)(ig'), \\ |g'| = 1\, . \end{cases} \tag{3.8}$$

Let $P_0 = g(t_0)$ be a point on g such that $|P_0| = \bar{R} := \max\{|P| \mid P \in g\}$. Set $v(t) := |g(t)|^2$ and compute $v'' = 2(|g'|^2 + g \cdot g'') \geq 2(1 - |k(|g|)g|)$. Since v takes its maximum value at t_0, then $0 \geq v''(t_0) \geq 2(1 - |k(\bar{R})|\bar{R})$. Hence $\bar{R}|k(\bar{R})| \geq 1$. By the continuity of the function $r \mapsto r|k(r)|$, there exists a radius $R \in (0, \bar{R}]$ such that $R|k(R)| = 1$. Hence the circle C_R is a k-loop.

Remark 3.4. As an immediate consequence of Remark 3.3 we notice that no k-loops may exist if $k : \mathbb{C} \to \mathbb{R}$ is a continuous radially symmetric function such that $R|k(R)| < 1$ for any $R > 0$.

4. Uniqueness up to similarity

In this section we prove Theorem 0.2. We start by fixing some notation. Let g be an embedded loop in $\mathbb{C}$, and let $k : g \to \mathbb{R}$ be its curvature. We assume that g is positively oriented, so that the interior is to the left. Let P_1, P_2 be two distinct points in g. We denote by $A_g(P_1, P_2)$ the closed arc of g having endpoints P_1 and P_2, oriented accordingly to the orientation of g. The open arc will be denoted by $\mathring{A}_g\,(P_1, P_2)$.

Following Osserman's definition in [10], we let $C_R^g(X) = \{P \in \mathbb{C} \mid |P - X| = R\}$ be the circumscribed circle about g. Thus, R is the minimum positive number r such that there exists a circle of radius r including g.

To prove Theorem 0.2 we need a preliminary result, which is based on Hopf's maximum principle. More precisely, the next lemma is a consequence of the *Touching Lemma* for the mean curvature operator. Although it is essentially well known, we state it here for the sake of completeness.

Lemma 4.1. *Let $P \in g \cap C_R^g(X)$. If $k(M) \leq 1/R$ for every $M \in g$ close to P, then $g \cap C_R^g(X)$ contains an arc around P.*

Proof. Up to a rotation and a translation we can assume that $X = 0$ and $P = iR$. An arc of $C_R(0)$ around P is the graph of the function $\gamma(t) = \sqrt{R^2 - t^2}$. Since g is an immersion, then g is locally the graph of a function $f : I = (-\delta, \delta) \to \mathbb{R}$ such that $f(0) = R$. By assumption $k \le 1/R$ in a neighborhood of P, and therefore

$$\left(\frac{\gamma'}{\sqrt{1 + |\gamma'|^2}}\right)' + \frac{1}{R} = 0 \le \left(\frac{f'}{\sqrt{1 + |f'|^2}}\right)' + \frac{1}{R}$$

for any t close to 0. In addition we have that $f \le \gamma$ and $f(0) = \gamma(0) = R$. Thus $f \equiv \gamma$ in a neighborhood of 0 by Theorem 2.3 in [11]. □

Proof of Theorem 0.2. Let k and g be as in the statement of the theorem. Then g is convex, since its curvature is positive. Let $C^g_R(X)$ be the circumscribed circle about g. By Lemma 3 in [10] it turns out that $k(P) \ge 1/R$ for any $P \in g \cap C^g_R(X)$.

We claim that g coincides with its circumscribed circle. By contradiction, we assume that $g \cap C^g_R(X)$ is strictly contained in $C^g_R(X)$. We distinguish two cases, depending on the number of connected components in $C^g_R(X) \setminus g$.

Case 1. The set $C^g_R(X) \setminus g$ is not connected. Then we can fix two distinct points P_1 and P_2 on $g \cap C^g_R(X)$ such that the arcs $A_g(P_1, P_2)$ and $A_g(P_2, P_1)$ are not contained in the circle $C^g_R(X)$. By Lemma 4 in [10], there exist two points, $Q_1 \in \mathring{A}_g\,(P_1, P_2)$ and $Q_2 \in \mathring{A}_g\,(P_2, P_1)$, such that

$$|Q_i - X| < R = |P_j - X|, \quad k(Q_i) < 1/R \le k(P_j),$$

for $i, j = 1, 2$. Since k is a non-increasing function of the distance from the origin, then $|Q_i| > |P_j|$. Thus $X \ne 0$ and

$$0 < |Q_i|^2 - |P_j|^2 = |Q_i - X|^2 - |P_j - X|^2 + 2X \cdot (Q_i - P_j) < 2X \cdot (Q_i - P_j);$$

that is,

$$\min\{X \cdot Q_1, X \cdot Q_2\} > \max\{X \cdot P_1, X \cdot P_2\}\,. \tag{4.1}$$

Denote by $[Q_1, Q_2]$ the segment joining Q_1 and Q_2, and by $[P_1, P_2]$ the segment joining P_1 and P_2. By the convexity of the curve g the segments $[Q_1, Q_2]$ and $[P_1, P_2]$ intersect at a point Q. Since $Q \in [Q_1, Q_2]$, then $X \cdot Q > \max\{X \cdot P_1, X \cdot P_2\}$ by (4.1). On the other hand, $X \cdot Q \le \max\{X \cdot P_1, X \cdot P_2\}$ as $Q \in [P_1, P_2]$, a contradiction. Thus Case 1 is excluded.

Case 2. The set $C^g_R(X) \setminus g$ is connected. We find $P_1, P_2 \in g$ such that

$$C^g_R(X) \setminus g = \mathring{A}_g\,(P_2, P_1)$$

and such that the arc $A_g(P_1, P_2)$ is contained in $C^g_R(X)$. In particular $k(P) = 1/R$ for any $P \in A_g(P_1, P_2)$. Since $C^g_R(X)$ is the smallest circle circumscribing g, then $A_g(P_1, P_2)$ is larger than a semicircle ([10], Lemma 2). In particular the antipodal $\tilde{P}_1 = 2X - P_1$ to P_1 belongs to $A_g(P_1, P_2)$. Up to a rotation and up to a change of indexes we can assume that P_1, $\tilde{P}_1$ lie on the same vertical line $\mathcal{V}$, with P_1 above $\tilde{P}_1$ and with $A_g(P_1, \tilde{P}_1)$ on the left of $\mathcal{V}$.

By Lemma 4 in [10], there exists a point $Q \in \mathring{A}_g\,(P_2, P_1)$ such that $k(Q) < 1/R$. Since g is convex, then Q lies on the right of the vertical line $\mathcal{V}$. By Lemma 4.1 there exists $M \in A_g(Q, P_1)$ on the right of $\mathcal{V}$ such that $k(M) > 1/R$. Since $k \equiv 1/R$

on $A_g(P_1,P_2)$ and since k is radially symmetric and non-increasing by assumption, then

$$|M| < |P| < |Q| \quad \text{for any } P \in A_g(P_1,P_2) \subset C_R^g(X)\,. \tag{4.2}$$

In particular, $X \neq 0$, as $|X-Q| < R$. We put $P_0 := X(1-R|X|^{-1}) \in C_R^g(X)$ and $\tilde{P}_0 := 2X - P_0 \in C_R^g(X)$.

We claim that 0 cannot belong to $\mathbb{C}\backslash\overline{D_R(X)}$. Indeed, if $|X| \geq R$, then the point P_0 is the minimal distance projection of 0 on $\overline{D_R(X)}$. Since $|M| < |P|$ for any $P \in A_g(P_1,P_2)$ and since M is in the interior of $D_R(X)$, then

$$P_0 \in \mathring{A}_g\,(P_2,P_1) \subset \mathring{A}_g\,(\tilde{P}_1,P_1).$$

But then, $\tilde{P}_0 \in A_g(P_1,\tilde{P}_1)$. Thus $|\tilde{P}_0| < |Q|$ by (4.2). However, this is impossible, as $|\tilde{P}_0| = |X| + R$ and $|Q| \leq |Q-X| + |X| < R + |X|$.

Thus $0 \in D_R(X)$ and $|P_0| = R - |X| > 0$. Since $|P_1|, |\tilde{P}_1| < |Q|$ and since Q lies on the right of the axes $\mathcal{V}$ joining $P_1, \tilde{P}_1$, then 0 is on the left of $\mathcal{V}$. Therefore $P_0 \in A_g(P_1,\tilde{P}_1) \subset C_R^g(X)$ and $k(P_0) = 1/R$. Thus $k \equiv 1/R$ on $C_{|P_0|}(0)$ and $k \leq 1/R$ outside $D_{|P_0|}(0)$, as k is radially symmetric and non increasing. Finally, we notice that the circle $C_{|P_0|}(0)$ is tangent to $C_R^g(X)$ at P_0 from the interior. Thus $k \leq 1/R$ in a neighborhood of P_1, and therefore P_1 is in the interior of $g \cap C_R^g(X)$ by Lemma 4.1, a contradiction. The theorem is completely proved. $\square$

Theorem 0.2 is sharp in view of the next example.

Example 4.2. There exist a positive, radially symmetric and increasing curvature $k : \mathbb{C} \to \mathbb{R}$ that has a round k-loop and a non-round embedded k-loop.

To exhibit such a curvature we fix a pair of positive numbers $a < b$. Let $k_{a,b}$ be any smooth function such that

$$k_{a,b}(z) := \frac{ab}{(a^2+b^2-|z|^2)^{3/2}} \tag{4.3}$$

on the disk $\{|z| \leq b\ \}$. Notice that $k_{a,b}$ can be taken to be radially symmetric and strictly increasing as a function of the distance from the origin.

The ellipse $E_{a,b}$ of equation $a^{-2}x^2 + b^{-2}y^2 = 1$ has curvature k at any point. For the proof, it is convenient to parametrize $E_{a,b}$ by $g(t) = (a\cos t, b\sin t)$.

By elementary continuity arguments there exists a radius $R \in (a,b)$ such that $Rk_{a,b}(z) \equiv 1$ on the circle C_R, coherently with Remark 3.3. Thus the circle C_R and the ellipse $E_{a,b}$ are distinct k-loops.

Theorem 0.2 provides a new characterization of circles.

Corollary 4.3. *Round circles are the only convex loops in $\mathbb{C}$ whose curvature is a non increasing function of the Euclidean distance from a fixed point.*

We conclude the paper by pointing out the following uniqueness (up to homothety) result, which is an immediate consequence of Theorem 0.2.

Corollary 4.4. *Let k be a continuous, positive and radially symmetric function. If k is radially decreasing, then any embedded k-loop is a circle around the origin.*

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