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EIGENFUNCTION EXPANSIONS IN $\mathbb{R}^n$

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Abstract. The main goal of this paper is to extend in $\mathbb{R}^n$ a result of Seeley on eigenfunction expansions of real analytic functions on compact manifolds. As a counterpart of an elliptic operator in a compact manifold, we consider in $\mathbb{R}^n$ a selfadjoint, globally elliptic Shubin type differential operator with spectrum consisting of a sequence of eigenvalues $\lambda_j, j \in \mathbb{N}$, and a corresponding sequence of eigenfunctions $u_j, j \in \mathbb{N}$, forming an orthonormal basis of $L^2(\mathbb{R}^n)$. Elements of Schwartz $\mathcal{S}(\mathbb{R}^n)$, resp. Gelfand-Shilov $S^{1/2}_{1/2}$ spaces, are characterized through expansions $\sum_j a_j u_j$ and the estimates of coefficients a_j by the power function, resp. exponential function of λ_j.

1. Introduction and statement of the result

We shall give a version in $\mathbb{R}^n$ of some results, already known on compact manifolds, concerning eigenfunction expansions. Broadly speaking, the aim is to relate the regularity of a function with the decay properties of the sequence of the Fourier coefficients. More precisely, we want to reproduce in $\mathbb{R}^n$ the classical results of [15], Section 10, for Sobolev regularity and [16] for analytic functions on a compact manifold taking into account Weyl asymptotics for eigenvalues.

Our basic example of an operator will be the harmonic oscillator appearing in Quantum Mechanics,

$$H = -\triangle + |x|^2, \tag{1.1}$$

whose eigenfunctions are the Hermite functions

$$h_\alpha(x) = H_\alpha(x)e^{-|x|^2/2}, \quad \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n, \tag{1.2}$$

where $H_\alpha(x)$ is the α-th Hermite polynomial; cf. [5]. See for example [14], [13], [11], for related Hermite expansions as well as [7], [18] for connections with a degenerate harmonic oscillator.

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Here we shall consider a more general class of operators with polynomial coefficients in $\mathbb{R}^n$:

$$P = \sum_{|\alpha|+|\beta|\le m} c_{\alpha\beta} x^\beta D_x^\alpha, \quad D^\alpha = (-i)^{|\alpha|}\partial_x^\alpha, \tag{1.3}$$

studied by Shubin [17] in the frame of a global pseudo-differential calculus; see also [8], [1], [12]. Let us recall, in short, some definitions and results from [17], Chapter IV.

First, global ellipticity for P in (1.3) is defined by imposing

$$p_m(x,\xi) = \sum_{|\alpha|+|\beta|=m} c_{\alpha\beta} x^\beta \xi^\alpha \neq 0 \quad \text{for} \quad (x,\xi) \neq (0,0). \tag{1.4}$$

This condition is obviously satisfied by H in (1.1). For these operators, the counterpart of the standard Sobolev spaces comprises the spaces

$$Q^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n), \ \ \|u\|_{Q^s} := \sum_{|\alpha|+|\beta|\le s} \|x^\beta \partial_x^\alpha u\|_{L^2(\mathbb{R}^n)} < +\infty\}, \tag{1.5}$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the class of the tempered distributions of Schwartz and $s \in \mathbb{N}$. Under the global ellipticity assumption (1.4), $P : Q^m(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Fredholm operator. The finite-dimensional null-space $\operatorname{Ker} P$ is given by functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Following Gelfand and Shilov, it is natural to consider as a global counterpart in $\mathbb{R}^n$ of the real analytic class the inductive (respectively, projective) Gelfand-Shilov classes $S_\nu^\mu(\mathbb{R}^n)$ (respectively, $\Sigma_\nu^\mu(\mathbb{R}^n)$), $\mu > 0$, $\nu > 0$, $\mu + \nu \ge 1$ (respectively $\mu + \nu > 1$), defined as the set of all $u \in S(\mathbb{R}^n)$ for which there exist $A > 0, C > 0$ (respectively, for every $A > 0$ there exists $C > 0$) such that

$$|x^\beta \partial_x^\alpha u(x)| \le C A^{-|\alpha|-|\beta|} (\alpha!)^\mu (\beta!)^\nu, \ \ \alpha \in \mathbb{N}^n, \ \beta \in \mathbb{N}^n;$$

see [6], [11] and Chapter 6 in [12]. In the sequel we shall limit our attention to $S_\mu^\mu(\mathbb{R}^n), \mu \ge 1/2$ (respectively, $\Sigma_\mu^\mu(\mathbb{R}^n), \mu > \frac{1}{2}$). We recall that $u \in S_\mu^\mu(\mathbb{R}^n)$ iff there exist $A > 0$, $C > 0$ (respectively, for every $A > 0$ one can find $C > 0$) such that

$$\sum_{|\alpha|+|\beta|=s} \|x^\beta \partial_x^\alpha u\|_{L^2(\mathbb{R}^n)} \le C A^{-s} (s!)^\mu, \ \ s \in \mathbb{N}. \tag{1.6}$$

It was shown recently that every solution $u \in \mathcal{S}'(\mathbb{R}^n)$ of $Pu = 0$ belongs to $S_{1/2}^{1/2}(\mathbb{R}^n)$ provided (1.4) holds; see [3], [4] for details and more general results.

We assume, as in Seeley [16], that P is a normal operator (i.e., $P^*P = PP^*$) satisfying the global ellipticity condition (1.4). This guarantees the existence of a basis of orthonormal eigenfunctions u_j, $j \in \mathbb{N}$, with eigenvalues λ_j, $\lim_{j\to\infty} |\lambda_j| = +\infty$ (see Seeley [16] and Shubin [17]). Hence, given $u \in L^2(\mathbb{R}^n)$ or $u \in \mathcal{S}'(\mathbb{R}^n)$, we can expand

$$u = \sum_{j=1}^\infty a_j u_j, \tag{1.7}$$

where the Fourier coefficient $a_j \in \mathbb{C}$ is defined by

$$a_j = (u, u_j)_{L^2(\mathbb{R}^n)}, \ \ j = 1, 2, ..., \tag{1.8}$$

with convergence in $L^2(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$ for (1.7).

In view of [3] the eigenfunctions u_j belong to $S_{1/2}^{1/2}(\mathbb{R}^n)$.

We state the first main result.

Theorem 1.1. *Suppose that P is globally elliptic (cf. (1.3), (1.4)) and normal. Then:*

(i) $u \in Q^s(\mathbb{R}^n) \Longleftrightarrow \sum_{j=1}^{\infty} |a_j|^2 |\lambda_j|^{2s/m} < \infty \Leftrightarrow \sum_{j=1}^{\infty} |a_j|^2 j^{s/n} < \infty$, $s \in \mathbb{N}$.

(ii) $u \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow |a_j| = O(|\lambda_j|^{-s})$, $j \to \infty \Longleftrightarrow |a_j| = O(j^{-s})$, $j \to \infty$ *for all* $s \in \mathbb{N}$.

Next, we show the global analogue to Seeley's theorem in [16].

Theorem 1.2. *Let P be as before and let $\mu \geq 1/2$ (respectively, $\mu > 1/2$). Then we have:*

$$u \in S_\mu^\mu(\mathbb{R}^n) \Longleftrightarrow \sum_{j=1}^{\infty} |a_j|^2 e^{\epsilon |\lambda_j|^{1/(m\mu)}} < \infty$$

for some $\epsilon > 0 \Leftrightarrow \sum_{j=1}^{\infty} |a_j|^2 e^{\epsilon j^{1/(m\mu)}} < \infty$ *for some* $\epsilon > 0 \Leftrightarrow$ *there exist* $C > 0$, $\epsilon > 0$ *such that*

$$|a_j| \leq C\, e^{-\epsilon j^{1/(2n\mu)}}, \qquad j \in \mathbb{N} \tag{1.9}$$

(respectively, $u \in \Sigma_\mu^\mu(\mathbb{R}^n) \Leftrightarrow \sum_{j=1}^{\infty} |a_j|^2 e^{\epsilon |\lambda_j|^{1/(m\mu)}} < \infty$ *for all* $\epsilon > 0 \Leftrightarrow \sum_{j=1}^{\infty} |a_j|^2 e^{\epsilon j^{1/(m\mu)}}$ $< \infty$ *for all* $\epsilon > 0 \Leftrightarrow$ *for every* $\epsilon > 0$ *there exist* $C > 0$ *such that*

$$|a_j| \leq C\, e^{-\epsilon j^{1/(2n\mu)}}, \qquad j \in \mathbb{N}). \tag{1.10}$$

Remark 1.3. Choosing as P the harmonic oscillator H in (1.1), with eigenfunctions $h_\alpha(x)$ as in (1.2), we recapture the results on the Hermite expansions related to Gelfand-Shilov type spaces for $n = 1$ whereas for $n \geq 2$, taking into account the multiplicity of the eigenvalue $\lambda_\alpha = \sum_{j=1}^{n}(2\alpha_j + 1)$ for h_α, $\alpha \in \mathbb{N}^n$, we obtain as a particular case of Theorem 1.2 the characterization: $u \in S_{1/2}^{1/2}(\mathbb{R}^n)$ iff

$$|a_\alpha| \leq C\, e^{-\epsilon|\alpha|},\ \alpha \in \mathbb{N}^n,$$

for positive constants C and ϵ, where $u = \sum_{\alpha \in \mathbb{N}^n} a_\alpha h_\alpha$; cf. [11], [13] and the references therein.

2. Proof of Theorem 1.1

It is not restrictive to assume that P is positive, with $\lambda_j > 0$; cf. [16]. We need some preliminary results from [17]. Namely, concerning asymptotics of eigenvalues, Theorem 30.1 in [17] and Proposition 4.6.4 in [12] give the following lemma.

Lemma 2.1. *Let P be globally elliptic of order $m > 0$, cf. (1.3), (1.4), and strictly positive. Then for the eigenvalues λ_j, $j = 1, 2, \ldots$, we have*

$$\lambda_j \sim C\, j^{m/(2n)} \quad \text{as } j \to +\infty,$$

for a positive constant C.

Now, for P as before and $r \in \mathbb{R}$, $r \neq 0$, introduce the r-th power

$$P^r u = \sum_{j=1}^{\infty} \lambda_j^r a_j u_j, \tag{2.1}$$

with a_j, u_j as in (1.7), (1.8). The operator P^r is well defined as a map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Lemma 2.2. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $u \in Q^s(\mathbb{R}^n)$ if and only if $P^{s/m}u \in L^2(\mathbb{R}^n)$, $s \in \mathbb{N}$. The norms $\|u\|_{Q^s}$ and $\|P^{s/m}u\|_{L^2(\mathbb{R}^n)}$ are equivalent.*

In fact, $P^{s/m}$ is an elliptic operator of order s in the pseudo-differential calculus of [17], cf. Section 4.3 in [12], and consequently $P^{s/m}u \in L^2(\mathbb{R}^n)$ corresponds to $u \in Q^s(\mathbb{R}^n)$ with equivalence of norms; cf. Proposition 2.1.9 and Theorem 2.1.12 in [12].

We may now prove (i) in Theorem 1.1. From (2.1) and the Parseval identity:

$$\|P^{s/m}u\|^2_{L^2(\mathbb{R}^n)} = \|\sum_{j=1}^{\infty} \lambda_j^{s/m} a_j u_j\|^2_{L^2(\mathbb{R}^n)} = \sum_{j=1}^{\infty} \lambda_j^{2s/m} |a_j|^2.$$

In view of Lemma 2.1 we have $\lambda_j^{2s/m} \sim C' j^{s/n}$, and therefore from Lemma 2.2:

$$c_1\|u\|^2_{Q^s} \le \sum_{j=1}^{\infty} j^{s/n}|a_j|^2 \le c_2\|u\|^2_{Q^s}$$

for suitable positive constants c_1, c_2. This gives (i). On the other hand, $\mathcal{S}(\mathbb{R}^n) = \bigcap_{s\in\mathbb{N}} Q^s(\mathbb{R}^n)$; hence (ii) follows from (i). Note also that by Lemma 2.2 we may generalize the definition of $Q^s(\mathbb{R}^n)$ to all $s \in \mathbb{R}$, and (i) extends obviously to these spaces. Finally, we observe that the preceding arguments and the statement of Theorem 1.1 remain valid for any globally elliptic normal pseudo-differential operator in [17].

3. Proof of Theorem 1.2

We shall follow the argument of [16], pages 737-738. Namely, we shall use the following adapted version of the celebrated theorem of the iterates of [10].

Lemma 3.1. *Let P be globally elliptic (cf. (1.3), (1.4)) of order m. Let $\mu \ge 1/2$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $u \in S^\mu_\mu(\mathbb{R}^n)$ if and only if for some $C > 0$,*

$$\|P^M u\|_{L^2(\mathbb{R}^n)} \le C^{M+1}(M!)^{\mu m} \quad \text{for all } M \in \mathbb{N}. \tag{3.1}$$

A short proof of Lemma 3.1 will be given in Section 4; for more details, we refer to the forthcoming paper [2]. By applying Lemma 3.1, in the sequel we may then take the estimates (3.1) as an equivalent definition of the class $S^\mu_\mu(\mathbb{R}^n)$, fixing as P the operator in Theorem 1.1. On the other hand, assuming without loss of generality that $u \in \mathcal{S}(\mathbb{R}^n)$, we have:

$$\|P^M u\|^2_{L^2(\mathbb{R}^n)} = \|\sum_{j=1}^{\infty} a_j P^M u_j\|^2_{L^2(\mathbb{R}^n)} = \|\sum_{j=1}^{\infty} \lambda_j^M a_j u_j\|^2_{L^2(\mathbb{R}^n)} = \sum_{j=1}^{\infty} \lambda_j^{2M}|a_j|^2,$$

in view of (1.7), (1.8) and Parseval's identity. It follows from Lemma 2.1 that

$$C_1\|P^M u\|^2_{L^2(\mathbb{R}^n)} \le \sum_{j=1}^{\infty} j^{mM/n}|a_j|^2 \le C_2\|P^M u\|^2_{L^2(\mathbb{R}^n)} \tag{3.2}$$

for suitable positive constants C_1, C_2. Assume now that the estimate (1.9) is satisfied. Then from the first estimate in (3.2) we have for some $C > 0$, $\epsilon > 0$,

$$\|P^M u\|^2_{L^2(\mathbb{R}^n)} \leq C \sum_{j=1}^{\infty} j^{mM/n} e^{-2\epsilon j^{1/(2n\mu)}} \leq \tilde{C} \sup_j j^{mM/n} e^{-\epsilon j^{1/(2n\mu)}} \tag{3.3}$$

with

$$\tilde{C} = C \sum_{j=1}^{\infty} e^{-\epsilon j^{1/(2n\mu)}}.$$

Now observe that the identity

$$e^{\omega j^{1/(2n\mu)}} = \sum_{M=0}^{\infty} \frac{\omega^M j^{M/(2n\mu)}}{M!}$$

implies that for any $\omega > 0$ and $M \in \mathbb{N}$:

$$j^{M/(2n\mu)} e^{-\omega j^{1/(2n\mu)}} \leq \omega^{-M} M!. \tag{3.4}$$

Taking the $2\mu m$-th power of both sides of (3.4) and applying it in the last estimate in (3.3) with $\epsilon = 2\mu m\omega$, we obtain

$$\|P^M u\|^2_{L^2(\mathbb{R}^n)} \leq \tilde{C}(\omega^{-M} M!)^{2\mu m},$$

which gives (3.1) for some $C > 0$. Similarly, assuming (3.1) and using the second estimate in (3.2), we deduce (1.9). The same computations give the other equivalences in Theorem 1.2.

4. Proof of Lemma 3.1

We shall use the estimates (1.6) as the definition of $S^\mu_\mu(\mathbb{R}^n)$. It is then easy to show that $u \in S^\mu_\mu(\mathbb{R}^n)$ implies (3.1). In the opposite direction, we assume (3.1) and prove (1.6). Write for short

$$|u|_s = \sum_{|\alpha|+|\beta|=s} \|x^\beta \partial_x^\alpha u\|_{L^2(\mathbb{R}^n)}. \tag{4.1}$$

The following interpolation result for the semi-norms $|u|_s$ is needed in the case when $m \geq 2$, the integer m being the order of P.

Proposition 4.1. *There exists a constant $C > 0$ such that for any $s \in \mathbb{N}$, with $s = pm + r$, $p \in \mathbb{N}$, $0 < r < m$, and for all $\epsilon > 0$,*

$$|u|_s \leq \epsilon |u|_{(p+1)m} + C\epsilon^{-\frac{r}{m-r}} |u|_{pm} + C^s (s!)^{1/2} \|u\|_{L^2(\mathbb{R}^n)}. \tag{4.2}$$

The proof of Proposition 4.1 is omitted for brevity. A corresponding result for the homogeneous Sobolev spaces is well known; see for example [10], Lemma 3.3, and subsequent remarks. A novelty with respect to Sobolev spaces is the last term on the right-hand side of (4.2): the factor $(s!)^{1/2}$ comes from the symbolic calculus of [17], Section 24; see also [12], Sections 1.7, 1.8.

Since $\mu \geq 1/2$, Proposition 4.1, with $\epsilon = 1$, say, implies that we may limit ourselves to checking (1.6) for $s = pm$, $p = 0, 1, ...$ Namely, we shall prove that the sequence

$$\sigma_p(u, \lambda) = (pm)!^{-\mu} \lambda^{-p} |u|_{pm}, \quad p = 0, 1, ..., \tag{4.3}$$

is bounded if λ is sufficiently large. To this end, we use the following proposition.

Proposition 4.2. *Let P be as in Lemma* 3.1. *Then for $\lambda > 0$ large enough, we have for all $p \in \mathbb{N}$:*

$$\sigma_{p+1}(u,\lambda) \leq [(pm+1)...(pm+m)]^{-\mu}\sigma_p(Pu,\lambda) + \sigma_p(u,\lambda) + \sigma_{p-1}(u,\lambda) + \sigma_0(u,\lambda). \tag{4.4}$$

In Propositions 4.1, 4.2 and in the sequel we may assume $u \in \mathcal{S}(\mathbb{R}^n)$. Note that $\sigma_0(u,\lambda) = |u|_0 = \|u\|_{L^2(\mathbb{R}^n)}$.

Proof of Proposition 4.2. We recall that $P : Q^m(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Fredholm. Assuming $\operatorname{Ker} P = \{0\}$ for simplicity, we have for a constant $C > 0$:

$$\|u\|_{Q^s} = \sum_{|\alpha|+|\beta|\leq s} \|x^\beta D_x^\alpha u\|_{L^2(\mathbb{R}^n)} \leq C\|Pu\|_{L^2(\mathbb{R}^n)}, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{4.5}$$

Consider then for $|\alpha| + |\beta| = (p+1)m$ the term $\|x^\beta D_x^\alpha u\|_{L^2(\mathbb{R}^n)}$ and write

$$x^\beta D^\alpha u = x^{\beta-\delta} x^\delta D_x^{\alpha-\gamma} D_x^\gamma u,$$

where we fix $\gamma \leq \alpha, \delta \leq \beta$ so that $|\gamma|+|\delta| = pm$ and $|\alpha-\gamma|+|\beta-\delta| = m$. Therefore we may estimate, using (4.5):

$$\begin{aligned} \|x^\beta D_x^\alpha u\|_{L^2\mathbb{R}^n)} &\leq \|x^{\beta-\delta} D_x^{\alpha-\gamma}(x^\delta D_x^\gamma u)\|_{L^2(\mathbb{R}^n)} + \|x^{\beta-\delta}[x^\delta, D_x^{\alpha-\gamma}]D_x^\gamma u\|_{L^2(\mathbb{R}^n)} \\ &\leq C\|P(x^\delta D_x^\gamma u)\|_{L^2(\mathbb{R}^n)} + \|x^{\beta-\delta}[x^\delta, D_x^{\alpha-\gamma}]D_x^\gamma u\|_{L^2(\mathbb{R}^n)} \\ &\leq I_1 + I_2 + I_3 \end{aligned} \tag{4.6}$$

with

$$\begin{aligned} I_1 &= C\|x^\delta D_x^\gamma (Pu)\|_{L^2(\mathbb{R}^n)}, \quad I_2 = C\|[P, x^\delta D_x^\gamma]u\|_{L^2(\mathbb{R}^n)}, \\ I_3 &= \|x^{\beta-\delta}[x^\delta, D_x^{\alpha-\gamma}]D_x^\gamma u\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{4.7}$$

Summing up I_1, I_2, I_3 over all (α, β) with $|\alpha| + |\beta| = (p+1)m$, we can estimate correspondingly $|u|_{(p+1)m}$ and $\sigma_{p+1}(u,\lambda)$ from (4.1), (4.3); for short:

$$|u|_{(p+1)m} \leq J_1 + J_2 + J_3, \quad \sigma_{p+1}(u,\lambda) \leq Y_1 + Y_2 + Y_3. \tag{4.8}$$

Since $J_1 \leq C|Pu|_p$ for a new constant C, then for λ sufficiently large,

$$Y_1 = ((p+1)m)!^{-\mu}\lambda^{-p-1}J_1 \leq [(pm+1)...(pm+m)]^{-\mu}\sigma_p(Pu,\lambda). \tag{4.9}$$

To treat Y_2, by using the expression of P in (1.3), we compute

$$[P, x^\delta D_x^\gamma] = \sum_{|\tilde\alpha|+|\tilde\beta|\leq m} c_{\alpha\beta}[x^{\tilde\beta} D_x^{\tilde\alpha}, x^\delta D_x^\gamma]$$

with

$$\begin{aligned} [x^{\tilde\beta} D_x^{\tilde\alpha}, x^\delta D_x^\gamma] = &\sum_{0\neq\tau\leq\tilde\alpha, \tau\leq\delta} C_{1\tilde\alpha\delta\tau} x^{\delta+\tilde\beta-\tau} D_x^{\gamma+\tilde\alpha-\tau} \\ &- \sum_{0\neq\tau\leq\tilde\beta, \tau\leq\gamma} C_{2\tilde\beta\gamma\tau} x^{\delta+\tilde\beta-\tau} D_x^{\gamma+\tilde\alpha-\tau}, \end{aligned}$$

where $|C_{1\tilde\alpha\delta\tau}|$ and $|C_{2\tilde\beta\gamma\tau}|$ can be estimated by $C_3(pm)^{|\tau|}$. The constants here and in the sequel do not depend on p. Hence

$$\|[P, x^\delta D_x^\gamma]u\|_{L^2(\mathbb{R}^n)} \leq C_4 \sum_{|\tilde\alpha|+|\tilde\beta|\leq m} \sum_\tau (pm)^{|\tau|} \|x^{\delta+\tilde\beta-\tau} D_x^{\gamma+\tilde\alpha-\tau} u\|_{L^2(\mathbb{R}^n)}, \tag{4.10}$$

with $0 \neq \tau \leq \tilde{\alpha}$, $\tau \leq \delta$ or $0 \neq \tau \leq \tilde{\beta}$, $\tau \leq \gamma$. Set $s = |\delta + \tilde{\beta} - \tau| + |\gamma + \tilde{\alpha} - \tau| = (pm + |\tilde{\alpha}| + |\tilde{\beta}| - 2|\tau|)$. Since $|\tilde{\alpha}| + |\tilde{\beta}| \leq m$ and $0 < |\tau| \leq m$, we have $(p-1)m \leq s < (p+1)m$. Note also that $s \leq (p+1)m - 2|\tau|$; hence in (4.10) we can estimate $|\tau| \leq ((p+1)m - s)/2$. We may then write

$$J_2 \leq C_5(J_2' + (pm)^{m/2}|u|_{pm} + J_2'') \tag{4.11}$$

with

$$\begin{aligned} J_2' &= \sum_{pm<s<(p+1)m} (pm)^{((p+1)m-s)/2}|u|_s, \\ J_2'' &= \sum_{(p-1)m\leq s<pm} (pm)^{((p+1)m-s)/2}|u|_s. \end{aligned} \tag{4.12}$$

We estimate $|u|_s$ in (4.12) by Proposition 4.1. Namely, in the expression of J_2' we apply (4.2) with

$$\epsilon = (pm)^{-((p+1)m-s)/2}(4mC_5)^{-1}$$

and we get

$$J_2' \leq (4C_5)^{-1}|u|_{(p+1)m} + C_6(pm)^{m/2}|u|_{pm} + C_7^{p+1}((p+1)m)!^{1/2}|u|_0. \tag{4.13}$$

Similarly

$$J_2'' \leq C_8(pm)^{m/2}|u|_{pm} + C_9(pm)^m|u|_{(p-1)m} + C_9^{p+1}((p+1)m)!^{1/2}|u|_0. \tag{4.14}$$

Applying (4.13), (4.14) in (4.11) and taking λ sufficiently large, we conclude

$$Y_2 = ((p+1)m)!^{-\mu}\lambda^{-p-1},\ J_2 \leq \frac{1}{4}\big(\sigma_{p+1}(u,\lambda) + \sigma_p(u,\lambda) + \sigma_{p-1}(u,\lambda) + \sigma_0(u,\lambda)\big).$$

The same arguments give identical estimates for Y_3, and summing up with (4.9) in (4.8) we obtain (4.4). Proposition 4.2 is therefore proved.

End of the proof of Lemma 3.1. Rewrite the inequality (4.4) with $\sigma_q(u,\lambda)$ in the left-hand side, $q = p,\ p-1, \ldots.$ Applying these estimates successively to the terms $\sigma_p(u,\lambda),\ \sigma_{p-1}(u,\lambda), \ldots$ in the right-hand side of (4.4), we deduce, for λ sufficiently large,

$$\sigma_{p+1}(u,\lambda) \leq \sum_{q=0}^{p}[(qm+1)\ldots(qm+m)]^{-\mu}\sigma_q(Pu,\lambda) + \sigma_0(u,\lambda). \tag{4.15}$$

We then obtain by induction

$$\sigma_p(u,\lambda) \leq \sum_{r=0}^{p}\binom{p}{r}(rm)!^{-\mu}\sigma_0(P^r u,\lambda). \tag{4.16}$$

Namely, given (4.16) as granted for $q \leq p$, we want to prove it for $\sigma_{p+1}(u,\lambda)$. As a first step we apply (4.15), and then we estimate $\sigma_q(Pu,\lambda)$, $q \leq p$, in the right-hand side of (4.15) by the inductive assumptions. Easy computations (see for example [9]) give (4.16) for $\sigma_{p+1}(u,\lambda)$.

Finally, let us apply the inequality (3.1) with $M = r = 0, 1, \ldots, p$ to the right-hand side of (4.16). Standard factorial and binomial estimates show that $\sigma_p(u,\lambda) \leq C^{p+1}$ for a new constant $C > 0$. Enlarging λ, we conclude that the sequence $\sigma_p(u,\lambda)$ in (4.3) is bounded. Lemma 3.1 is proved.

Remark 4.3. The results in [4] lead to a natural conjecture that one can extend the result of Theorem 1.2 for $S^{kt/(k+m)}_{mt/(m+k)}(\mathbb{R}^n)$, $t \geq 1$, by means of eigenfunction expansions for anisotropic elliptic Shubin type operators modelled by $(-\Delta)^m + x^{2k}$.

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