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C^α-CLASSICAL SOLUTIONS FOR ABSTRACT NON-AUTONOMOUS INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the existence of strict and classical solutions for a class of abstract non-autonomous integro-differential equations in Banach spaces. Some applications of the abstract results are considered.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. In this paper we study the existence of solutions for a class of abstract non-autonomous integro-differential equations in the form

$$\frac{d}{dt}\left[u(t) + \int_0^t B(t,\tau)u(\tau)d\tau\right] = A(t)u(t) + f(t), \quad t \in [0,a], \tag{1.1}$$

$$u(0) = x \in X, \tag{1.2}$$

where $(A(t))_{t\in[0,a]}$ is a family of sectorial operators defined on a common domain $\mathcal{D}$ dense in X, $B(t,s)$, $t \geq s$, are bounded linear operators and $f(\cdot)$ is a suitable function.

The abstract system (1.1)-(1.2) appears, for example, in the theory of heat conduction in materials with fading memory. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature $u(\cdot)$ and on its gradient $\nabla u(\cdot)$. Under these conditions, the classic heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [9, 18], the internal energy and the heat flux are described as functionals of $u(\cdot)$ and $u_x(\cdot)$. The next system (see for instance [2, 3, 16, 19]) has been frequently used to describe this phenomena:

$$\begin{aligned} \frac{d}{dt}\left[u(t,x) + \int_{-\infty}^t k_1(t-s)u(s,x)ds\right] &= c\triangle u(t,x) + \int_{-\infty}^t k_2(t-s)\triangle u(s,x)ds, \\ u(t,y) &= 0. \end{aligned}$$

In this system, $(t,x) \in \mathbb{R}\times\Omega$, $\Omega \subset \mathbb{R}^n$ is an open bounded set with smooth boundary $\partial\Omega$, $y \in \partial\Omega$, $u(t,x)$ represents the temperature in x at the time t, c is a physical constant and $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are the internal energy and the heat flux

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relaxation respectively. By assuming that the solution $u(\cdot)$ is known on $(-\infty, 0]$, the function $t \to \int_{-\infty}^{0} B(t,\tau)u(\tau)d\tau$ is differentiable on $[0,a]$ and $k_2 \equiv 0$, we obtain an abstract system as (1.1)-(1.2).

For additional details on the theory of abstract integro-differential equations (general motivations, existence and regularity of solutions, asymptotic behavior of solutions, existence and qualitative properties of resolvent of operator, etc.) we refer the reader to [1]-[17], [19]-[21] and the book by Prüss [20]. To the best of our knowledge, the problem of the existence of strict and classical Hölder type solutions for the non-autonomous abstract system (1.1)-(1.2) is an untreated topic in the literature, and it is the main motivation of this paper.

The results in this paper are proved assuming different Lipschitz type conditions on the operator functions $B(\cdot)$ and $A(\cdot)$ and implementing some specific maximal regularity type results. This paper has three sections. In section 2 we study the existence of strict and classical solutions in spaces of α-Hölder continuous functions with weight. In the last section, a concrete application is considered.

We now introduce some notation. Let $(W, \|\cdot\|_W)$, $(Z, \|\cdot\|_Z)$ be Banach spaces. We denote by $\mathcal{L}(W,Z)$ the space of bounded linear operators from W into Z endowed with the uniform operator norm $\|\cdot\|_{\mathcal{L}(W,Z)}$ and we write simply $\mathcal{L}(W)$ and $\|\cdot\|_{\mathcal{L}(W)}$ when $Z = W$. As usual, $C(I;Z)$, $I \subset \mathbb{R}$, is the space of continuous functions from I into Z endowed with the uniform convergence norm $\|\cdot\|_{C([0,b];Z)}$.

Let $\gamma \in (0,1)$ and $\beta \in (0,1]$. The notation $C^{\gamma}([0,b];Z)$ represents the space of all the γ-Hölder Z-valued continuous functions from $[0,b]$ into Z provided with the norm $\| u \|_{C^{\gamma}([0,b];Z)} = \| u \|_{C([0,b];Z)} + [| u |]_{C^{\gamma}([0,b];Z)}$, where $[| u |]_{C^{\gamma}([0,b];Z)} = \sup_{t,s\in[0,b], t\neq s} \frac{\|u(t)-u(s)\|_Z}{(t-s)^{\gamma}}$. In addition,

$$C_{\beta}((0,b];Z) = \{u \in C((0,b];Z) : \| u \|_{C_{\beta}((0,b];Z)} = \sup_{t\in(0,b]} t^{\beta} \| u(t) \|_Z < \infty\},$$

endowed with the norm $\|\cdot\|_{C_{\beta}((0,b];Z)}$, and $C_{\beta}^{\gamma}((0,b];Z)$ is the space

$$\begin{aligned} C_{\beta}^{\gamma}((0,b];Z) \quad = \quad & \{u \in C_{\beta-\gamma}((0,b];Z) : \, u \in C^{\gamma}([\varepsilon,b];Z), \ \forall \varepsilon \in (0,b] \text{ and} \\ & [| u |]_{C_{\beta}^{\gamma}((0,b];Z)} = \sup_{\varepsilon\in(0,b]} \varepsilon^{\beta} \| u \|_{C^{\gamma}([\varepsilon,b];Z)} < \infty\}, \end{aligned}$$

provided with the norm $\|\cdot\|_{C_{\beta}^{\gamma}((0,b];Z)} = \|\cdot\|_{C_{\beta-\gamma}((0,b];Z)} + [|\cdot|]_{C_{\beta}^{\gamma}((0,b];Z)}$. The notation $C^{1+\gamma}([0,b];Z)$ is used for the space formed for the functions $u \in C^{\gamma}([0,b];Z)$ such that $u' \in C^{\gamma}([0,b];Z)$ endowed with the norm $\| u \|_{C^{\gamma}([0,b];Z)} = \| u \|_{C^{\gamma}([0,b];Z)} + \| u' \|_{C^{\gamma}([0,b];Z)}$. The space $C_{\beta}^{1+\gamma}((0,b];Z)$ is defined in a similar manner.

2. Existence of solutions

To establish our existence results, we assume that the next conditions are verified. Next, $R(\lambda, A(t))$ is the operator $R(\lambda, A(t)) = (\lambda - A(t))^{-1}$.

- $(\mathbf{H_1})$ There are $M > 0$, $\vartheta \in (\pi/2, \pi)$ and a neighborhood of zero Σ, such that $\rho(A(t)) \supset \Lambda_{\vartheta} = \{\lambda \in \mathbb{C} :| \arg(\lambda) |< \vartheta\} \cup \Sigma$ and $\| R(\lambda, A(t)) \| \leq \frac{M}{|\lambda|}$ for all $(\lambda, t) \in \Lambda_{\vartheta} \times [0,a]$.

Under condition $\mathbf{H_1}$, each operator $A(t)$ is the generator of an analytic semigroup on X. In the rest of this work, $A = A(0)$, $\mathcal{D}$ is the space $D(A)$ provided with the graph norm $\| x \|_{\mathcal{D}} = \| Ax \|$ and $(T(t))_{t\geq 0}$ is the semigroup generated by A.

($\mathbf{H_2}$) The spaces $D(A(t))$ are independent of t and there is $\alpha \in (0,1)$ such that $A(\cdot) \in C^\alpha([0,a];\mathcal{L}(\mathcal{D},X))$.

($\mathbf{H_3}$) $\{B(t,s) : t \geq s,\ t,s \in [0,a]\} \subset \mathcal{L}(\mathcal{D})$ and the function $B(t,\cdot)x : [0,t] \to \mathcal{D}$ is strongly measurable for all $(t,x) \in [0,a] \times \mathcal{D}$.

In this paper, $(X,\mathcal{D})_{\eta,\infty}$ ($\eta \in (0,1)$) is the interpolation space

$$(X,\mathcal{D})_{\eta,\infty} = \{x \in X : [x]_{\eta,\infty} = \sup_{t\in(0,1)} \| t^{1-\eta}AT(t)x \| < \infty\},$$

endowed with the norm $\| x \|_{\eta,\infty} = [x]_{\eta,\infty} + \| x \|$. In addition, we assume that $C_k = \sup_{s\in(0,a]} \| s^k A^k T(s) \|$ and $C^k_{\eta,\infty} = \sup_{s\in(0,a]} s^{1-\eta} \| A^k T(s) \|_{\mathcal{L}((X,\mathcal{D})_{\eta,\infty},X)}$ are finite for all $k \in \mathbb{N} \cup \{0\}$.

Definition 1. A function $u \in C([0,b];X)$, $b \in (0,a]$, is called a strict solution of (1.1)-(1.2) in $C^\alpha([0,b];\mathcal{D})$ if $u \in C^\alpha([0,b];\mathcal{D})$ and (1.1)-(1.2) are verified.

Definition 2. A function $u \in C([0,b];X)$, $b \in (0,a]$, is called a classical solution of (1.1)-(1.2) on $[0,b]$ if $u(0) = x$, $u \in C((0,b];\mathcal{D}) \cap C^1((0,b];X)$ and (1.1) is verified on $(0,b]$.

Remark 1. In what follows, for $b \in (0,a]$ and $u : [0,b] \mapsto \mathcal{D}$, we denote by H_u and g_u the functions $H_u : [0,b] \to X$ and $g_u : [0,b] \to X$ defined by $H_u(t) = \int_0^t B(t,\tau)u(\tau)d\tau$ and $g_u(t) = (A(t) - A)u(t) + f(t)$.

2.1. **Strict solutions in $C^\alpha(\mathcal{D})$.** In this section we discuss the existence of strict solutions in $C^\alpha([0,b];\mathcal{D})$. We begin by introducing the next condition.

($\mathbf{H_4}$) The function $B(t,\cdot)$ belongs to $L^{\frac{1}{(1-\alpha)}}([0,t],\mathcal{L}(\mathcal{D}))$ for all $t \in [0,a]$,

$$\Theta_1(b) = \sup_{t\in[0,b]} \left(\int_0^t \| B(t,\sigma) \|^{\frac{1}{1-\alpha}}_{\mathcal{L}(\mathcal{D})} d\sigma \right)^{(1-\alpha)} < \infty,$$

for all $b < a$, and there is $L^1_B \in L^1([0,a];\mathbb{R}^+)$ such that

$$\| B(t,t-\tau) - B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \leq L_B(\tau) \mid t-s \mid^\alpha, \quad t \geq s,\ \tau \in [0,s].$$

The proof of the next lemma is similar to the proof of [15, Proposition 4.6], and we include it by convenience and completeness. Next, H_u is the function introduced in Remark 1.

Lemma 1. *If $u \in C^\alpha([0,b];\mathcal{D})$ and $\mathbf{H_4}$ is valid, then $H_u \in C^\alpha([0,b];\mathcal{D})$ and*

$$\begin{aligned} \| H_u \|_{C([0,b];\mathcal{D})} &\leq \| u \|_{C([0,b];\mathcal{D})}\ \Theta_1(b)b^\alpha, \\ [| H_u |]_{C^\alpha([0,b];\mathcal{D}))} &\leq (\| L_B \|_{L^1([0,b])} + \Theta_1(b)(1+b^\alpha)) \| u \|_{C^\alpha([0,b];\mathcal{D})}\ . \end{aligned}$$

If, in addition, $u(0) = 0$, then $\| H_u \|_{C([0,b];\mathcal{D})} \leq [| u |]_{C^\alpha([0,b];\mathcal{D})}\Theta_1(b)b^{2\alpha}$ and

$$[| H_u |]_{C^\alpha([0,b];\mathcal{D}))} \leq (\| L_B \|_{L^1([0,b])} + \Theta_1(b)b^\alpha(1 + (\frac{2}{3})^\alpha b^{1-\alpha}) \| u \|_{C^\alpha([0,b];\mathcal{D})}\ .$$

Proof. From the definition of $H_u(\cdot)$, it follows that

$$\| H_u(t) \|_{\mathcal{D}} \leq \int_0^t \| B(t,\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(\tau) \|_{\mathcal{D}}\ d\tau \leq \| u \|_{C([0,b];\mathcal{D})}\ \Theta_1(b)b^\alpha,$$

so that $\| H_u \|_{C([0,b];\mathcal{D}))} \leq \| u \|_{C([0,b];\mathcal{D})}\ \Theta_1(b)b^\alpha$. If $u(0) = 0$, then

$$\| H_u(t) \|_{\mathcal{D}} \leq \int_0^t \| B(t,\tau)(u(\tau) - u(0)) \|_{\mathcal{D}}\ d\tau \leq [| u |]_{C^\alpha(\mathcal{D})} t^\alpha \Theta_1(b)b^\alpha,$$

and hence, $\| H_u \|_{C([0,b];\mathcal{D})} \leq [| u |]_{C^\alpha([0,b];\mathcal{D})} \Theta_1(b) b^{2\alpha}$.

On the other hand, for $t, s \in [0,b]$ with $t \geq s$ we see that

$$\begin{aligned}
&\| H_u(t) - H_u(s) \|_{\mathcal{D}} \\
&\leq \int_0^s \| B(t,t-\tau) - B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(t-\tau) \|_{\mathcal{D}} \, d\tau \\
&\quad + \int_0^s \| B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(t-\tau) - u(s-\tau) \|_{\mathcal{D}} \, d\tau \\
&\quad + \int_s^t \| B(t,t-\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(t-\tau) \|_{\mathcal{D}} \, d\tau \\
&\leq (t-s)^\alpha \| u \|_{C([0,b];\mathcal{D})} \int_0^b L_B(\tau) d\tau \\
&\quad + [| u |]_{C^\alpha([0,b];\mathcal{D})} (t-s)^\alpha \int_0^s \| B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \, d\tau \\
&\quad + \| u \|_{C([0,b];\mathcal{D})} \int_s^t \| B(t,t-\tau) \|_{\mathcal{L}(\mathcal{D})} \, d\tau \\
&\leq \| L_B \|_{L^1([0,b])} \| u \|_{C([0,b];\mathcal{D})} \, (t-s)^\alpha + [| u |]_{C^\alpha([0,b];\mathcal{D})} \Theta_1(b) b^\alpha (t-s)^\alpha \\
&\quad + \| u \|_{C([0,b];\mathcal{D})} \, \Theta_1(b)(t-s)^\alpha,
\end{aligned}$$

so that $H_u \in C^\alpha(\mathcal{D})$ and

$$[| H_u |]_{C^\alpha(\mathcal{D})} \leq (\| L_B \|_{L^1([0,b])} + \Theta_1(b)(1+b^\alpha) \| u \|_{C^\alpha([0,b];\mathcal{D})} \, .$$

The inequality for $[| H_u |]_{C^\alpha([0,b];\mathcal{D}))}$ in the case in which $u(0) = 0$ follows from the above estimate by observing that

$$\begin{aligned}
&\int_s^t \| B(t,t-\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(t-\tau) \|_{\mathcal{D}} \, d\tau \\
&\leq \int_s^t \| B(t,t-\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(t-\tau) - u(0) \|_{\mathcal{D}} \, d\tau \\
&\leq b^{\frac{\alpha}{2}} [| u |]_{C^\alpha([0,b];\mathcal{D})} \int_s^t \| B(t,t-\tau) \|_{\mathcal{L}(\mathcal{D})} \, (t-\tau)^{\frac{\alpha}{2}} d\tau \\
&\leq b^{\frac{\alpha}{2}} [| u |]_{C^\alpha([0,b];\mathcal{D})} \Theta_1(b) (\int_s^t (t-\tau)^{\frac{1}{2}} d\tau)^\alpha \\
&\leq (\frac{2}{3})^\alpha b [| u |]_{C^\alpha([0,b];\mathcal{D})} \Theta_1(b)(t-s)^\alpha. \qquad \square
\end{aligned}$$

We also need the following regularity type result.

Lemma 2. *Let $\xi_1 \in C^\alpha([0,b];\mathcal{D})$, $\xi_2 \in C^\alpha([0,b];X)$ and $u : [0,b] \to X$ be the function defined by*

$$u(t) = T(t)(x + \xi_1(0)) - \xi_1(t) - \int_0^t AT(t-s)\xi_1(s) ds + \int_0^t T(t-s)\xi_2(s) ds.$$

If $x \in \mathcal{D}$ and $Ax + \xi_2(0) \in (X,\mathcal{D})_{\alpha,\infty}$, then $u \in C^\alpha([0,b];\mathcal{D})$, $u'(\cdot)$ is bounded with values in $(X,\mathcal{D})_{\alpha,\infty}$, $u + \xi_1 \in C^\alpha([0,b];\mathcal{D}) \cap C^{1+\alpha}([0,b];X)$ and

$$\frac{d}{dt}[u(t) + \xi_1(t)] = Au(t) + \xi_2(t), \quad \textit{for all } t \in [0,b].$$

Moreover,

$$[\mid u \mid]_{C^\alpha([0,b];\mathcal{D})} \leq \Lambda([\mid \xi_1 \mid]_{C^\alpha([0,b];\mathcal{D})} + [\mid \xi_2 \mid]_{C^\alpha([0,b];X)}) + \frac{C^1_{\alpha,\infty}}{\alpha} \parallel Ax + \xi_2(0) \parallel_{\alpha,\infty},$$

$$\begin{aligned} \parallel u \parallel_{C([0,b];\mathcal{D})} &\leq C_0 \parallel Ax \parallel + ([\mid \xi_1 \mid]_{C^\alpha([0,b];\mathcal{D})} + [\mid \xi_2 \mid]_{C^\alpha([0,b];X)})(C_0 + \frac{C_1}{\alpha})b^\alpha \\ &\quad + 2C_0 \parallel \xi_2 \parallel_{C([0,b];X)}, \end{aligned}$$

where $\Lambda = \frac{2C_1}{\alpha} + 3C_0 + 2 + \frac{C_2}{\alpha(1-\alpha)}$.

Proof. Let $v : [0,b] \to X$ be the function defined by

$$v(t) = T(t)(x + \xi_1(0)) + \int_0^t T(t-s)(-A\xi_1(s) + \xi_2(s))ds.$$

From [14, Theorem 4.3.1] we have that $v \in C^\alpha([0,b];\mathcal{D}) \cap C^1([0,b];X)$ and $v(\cdot)$ is a solution of

$$\begin{aligned} x'(t) &= Ax(t) - A\xi_1(t) + \xi_2(t), \quad t \in [0,b], \\ x(0) &= x + \xi_1(0). \end{aligned}$$

Since $v = u + \xi_1$, we obtain $u \in C^\alpha([0,b];\mathcal{D})$ and

$$\frac{d}{dt}[u(t) + \xi_1(t)] = A(u(t) + \xi_1(t)) - A\xi_1(t) + \xi_2(t) = Au(t) + \xi_2(t), \forall\, t \in [0,b].$$

On the other hand, a review of the proof of [14, Theorem 4.3.1] permits us to obtain

$$\begin{aligned} [\mid v \mid]_{C^\alpha([0,b];\mathcal{D})} &\leq d\left([\mid \xi_1 \mid]_{C^\alpha([0,b];\mathcal{D})} + [\mid \xi_2 \mid]_{C^\alpha([0,b];X)}\right) \\ &\quad + \frac{C^1_{\alpha,\infty}}{\alpha} \parallel Ax + \xi_2(0) \parallel_{\alpha,\infty}, \end{aligned}$$

where $d = \frac{2C_1}{\alpha} + 3C_0 + 1 + \frac{C_2}{\alpha(1-\alpha)}$. Now, from the definition of $v(\cdot)$ we obtain

$$\begin{aligned} [\mid u \mid]_{C^\alpha([0,b];\mathcal{D})} &\leq \Lambda\left([\mid \xi_1 \mid]_{C^\alpha([0,b];\mathcal{D})} + [\mid \xi_2 \mid]_{C^\alpha([0,b];X)}\right) \\ &\quad + \frac{C^1_{\alpha,\infty}}{\alpha} \parallel Ax + \xi_2(0) \parallel_{\alpha,\infty}. \end{aligned}$$

Moreover, by rewriting $u(\cdot)$ in the form

$$\begin{aligned} u(t) &= T(t)x + T(t)(\xi_1(0) - \xi_1(t)) - \int_0^t T(t-s)[A\xi_1(s) - A\xi_1(t)]ds \\ &\quad + \int_0^t T(t-s)(\xi_2(s) - \xi_2(t))ds + \int_0^t T(t-s)\xi_2(t)ds, \end{aligned} \tag{2.1}$$

we get

$$\begin{aligned} \parallel Au(t) \parallel &\leq C_0 \parallel Ax \parallel + C_0[\mid \xi_1 \mid]_{C^\alpha([0,b];\mathcal{D})}t^\alpha + \int_0^t \frac{[\mid \xi_1 \mid]_{C^\alpha([0,b];\mathcal{D})}C_1}{(t-s)^{1-\alpha}}ds \\ &\quad + \int_0^t \frac{[\mid \xi_2 \mid]_{C^\alpha([0,b];X)}C_1}{(t-s)^{1-\alpha}}ds + \parallel (T(t) - I)\xi_2(t) \parallel \\ &\leq C_0 \parallel Ax \parallel + ([\mid \xi_1 \mid]_{C^\alpha([0,b];\mathcal{D})} + [\mid \xi_2 \mid]_{C^\alpha([0,b];X)})(C_0 + \frac{C_1}{\alpha})b^\alpha \\ &\quad + 2C_0 \parallel \xi_2 \parallel_{C([0,b];X)}. \end{aligned}$$

The proof is now complete. □

The proof of the next result is a straightforward estimation. We omit the details.

Lemma 3. *Let $u, v \in C^\alpha([0,b];\mathcal{D})$ with $u(0) = v(0)$ and $f \in C^\alpha([0,b];X)$. Then $g_u \in C^\alpha([0,b];X)$ and*

$$\begin{aligned} \| g_u \|_{C^\alpha([0,b];X)} &\leq [| A |]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))} b^\alpha \| u \|_{C^\alpha([0,b];\mathcal{D})} + \| f \|_{C^\alpha([0,b];X)}, \\ \| g_u - g_v \|_{C^\alpha([0,b];X)} &\leq [| A |]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))} b^\alpha (2 + b^\alpha) \| u \|_{C^\alpha([0,b];\mathcal{D})} . \end{aligned}$$

We can now establish our first result on the existence of strict solutions.

Theorem 1. *Assume condition $\mathbf{H_4}$ holds, $x \in \mathcal{D}$, $Ax + f(0) \in (X,\mathcal{D})_{\alpha,\infty}$ and $f \in C^\alpha([0,a];X)$. Then there exists a strict solution of (1.1)-(1.2) in $C^\alpha([0,b];\mathcal{D})$ for some $0 < b \leq a$. Moreover, $u + H_u \in C^\alpha([0,b];\mathcal{D}) \cap C^{1+\alpha}([0,b];X)$.*

Proof. Let $0 < b \leq a$ be such that

$$(\Lambda + (C_0 + \frac{C_1}{\alpha})b^\alpha + 2C_0)[\Lambda_1(b) + \Lambda_2(b)] < 1,$$

where Λ is the constant introduced in Lemma 2 and

$$\begin{aligned} \Lambda_1(b) &= \| L_B \|_{L^1([0,b])} + \Theta_1(b) b^\alpha (b^\alpha + 1 + (\frac{2}{3})^\alpha b^{1-\alpha}), \\ \Lambda_2(b) &= [| A |]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))} b^\alpha (2 + b^\alpha). \end{aligned}$$

Let $Y = \{u \in C^\alpha([0,b];\mathcal{D}) : u(0) = x\}$ and $\Gamma : Y \to Y$ be the map defined by

$$\Gamma u(t) = T(t)x - H_u(t) - \int_0^t AT(t-s)H_u(s)ds + \int_0^t T(t-s)g_u(s)ds, \;\; t \in [0,b].$$

From Lemmas 1, 2 and 3, we obtain that $\Gamma u \in Y$ and

$$\begin{aligned} &\| \Gamma u - \Gamma v \|_{C^\alpha([0,b];\mathcal{D})} \\ &\leq (\Lambda + (C_0 + \frac{C_1}{\alpha})b^\alpha + 2C_0) \left[\| H_{(u-v)} \|_{C^\alpha([0,b];\mathcal{D})} + \| g_{(u-v)} \|_{C^\alpha([0,b];X)} \right], \end{aligned}$$

for all $u, v \in Y$. Moreover, from Lemmas 1 and 3 we infer that

$$\| \Gamma u - \Gamma v \|_{C^\alpha([0,b];\mathcal{D})} \leq (\Lambda + (C_0 + \frac{C_1}{\alpha})b^\alpha + 2C_0)[\Lambda_1(b) + \Lambda_2(b)] \| u - v \|_{C^\alpha([0,b];\mathcal{D})},$$

which proves that Γ is a contraction and there exist a unique fixed point $u(\cdot)$ of Γ. Finally, Lemma 2 implies that $u(\cdot)$ is a strict solution of (1.1)-(1.2), $u \in C^\alpha([0,b];\mathcal{D})$ and $u + H_u \in C^\alpha([0,b];\mathcal{D}) \cap C^{1+\alpha}([0,b];X)$. The proof is complete. □

2.2. **Classical solutions in $C_1^\alpha((0,b];\mathcal{D})$.** We now discuss the problem of the existence of classical solutions for (1.1)-(1.2). In what follows, for $x \in X$, $u : [0,b] \to \mathcal{D}$ and $f : [0,a] \to X$, we use the notation $\tilde{g}_u(\cdot)$ for the function $\tilde{g}_u : [0,b] \to X$ given by $\tilde{g}_u(t) = (A(t) - A)(u(t) + T(t)x) + f(t)$. As in section 2.1, we need to introduce a specific assumption for the operators $B(t,s)$.

($\mathbf{H_5}$) There are $c_1 > 0$ and a mensurable function $L_B : [0,a] \to \mathbb{R}^+$ such that

$$\begin{aligned} \Theta_2(b) = \sup_{t \in [0,b]} \int_0^t \frac{L_B(\theta)}{(t-\theta)^{1-\alpha}} d\theta &< \infty, \\ \| B(s, s-\tau) \|_{\mathcal{L}(\mathcal{D})} &\leq c_1 (s-\tau)^\alpha, \\ \| B(t, t-\tau) - B(s, s-\tau) \|_{\mathcal{L}(\mathcal{D})} &\leq L_B(\tau) \mid t - s \mid^\alpha, \end{aligned}$$

for all $b \geq t \geq s \geq \tau \geq 0$ and every $b \in [0,a]$.

Lemma 4. *Let $u \in C_1^\alpha((0,b];\mathcal{D})$ and assume that the condition* $\mathbf{H_5}$ *is satisfied. Then $H_u \in C^\alpha([0,b];\mathcal{D})$ and*

$$\| H_u \|_{C_1^\alpha((0,b];\mathcal{D})} \leq (b^{1-\alpha}+b)(1+b^\alpha)[\Theta_2(b)+\frac{2c_1b^\alpha}{\alpha}] \| u \|_{C_1^\alpha((0,b];\mathcal{D})} .$$

In particular, if $x \in (X,\mathcal{D})_{\alpha,\infty}$, then $H_{T(\cdot)x} \in C_1^\alpha((0,b];\mathcal{D})$ and

$$\| H_{T(\cdot)x} \|_{C_1^\alpha((0,b];\mathcal{D})} \leq (b^{1-\alpha}+b)(1+b^\alpha)[\Theta_2(b)+\frac{2c_1b^\alpha}{\alpha}](C^1_{\alpha,\infty}+\frac{C^2_{\alpha,\infty}}{\alpha(1-\alpha)}) \| x \|_{\alpha,\infty} .$$

Proof. Proceeding as in the proof of Lemma 1, for $t,s \in (0,b]$ with $t \geq s$, we get

$$\begin{aligned}
&\| H_u(t) - H_u(s) \|_{\mathcal{D}} \\
&\leq \int_0^s \| B(t,t-\tau) - B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(t-\tau) \|_{\mathcal{D}} \, d\tau \\
&\quad + \int_0^s \| B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(t-\tau) - u(s-\tau) \|_{\mathcal{D}} \, d\tau \\
&\quad + \int_s^t \| B(t,t-\tau) \|_{\mathcal{L}(\mathcal{D})} \| u(t-\tau) \|_{\mathcal{D}} \, d\tau \\
&\leq (t-s)^\alpha \| u \|_{C_{1-\alpha}((0,b];\mathcal{D})} \int_0^s \frac{L_B(\tau)}{(s-\tau)^{1-\alpha}} d\tau \\
&\quad + c_1 [| u |]_{C_1^\alpha((0,b];\mathcal{D})} \int_0^s \frac{(s-\tau)^\alpha (t-s)^\alpha}{(s-\tau)} d\tau \\
&\quad + \| u \|_{C_{1-\alpha}((0,b];\mathcal{D})} \, c_1 \int_s^t \frac{(t-\tau)^\alpha}{(t-\tau)^{1-\alpha}} d\tau \\
&\leq \Theta_2(b) \| u \|_{C_{1-\alpha}((0,b];\mathcal{D})} \, (t-s)^\alpha + \frac{c_1 b^\alpha}{\alpha} [| u |]_{C_1^\alpha((0,b];\mathcal{D})} (t-s)^\alpha \\
&\quad + \frac{c_1 b^\alpha}{\alpha} \| u \|_{C_{1-\alpha}((0,b];\mathcal{D})} \, (t-s)^\alpha,
\end{aligned}$$

so that $[| H_u |]_{C^\alpha([0,b];\mathcal{D})} \leq (\Theta_2(b) + \frac{2c_1b^\alpha}{\alpha}) \| u \|_{C_1^\alpha((0,b];\mathcal{D})}$ and $H_u \in C^\alpha([0,b];\mathcal{D})$. Since $H_u(0) = 0$, we have that $\| H_u \|_{C([0,b];\mathcal{D})} \leq b^\alpha[\Theta_2(b) + \frac{2c_1b^\alpha}{\alpha}] \| u \|_{C_1^\alpha((0,b];\mathcal{D})}$, which implies that $\| H_u \|_{C^\alpha([0,b];\mathcal{D})} \leq (1+b^\alpha)[\Theta_2(b) + \frac{2c_1b^\alpha}{\alpha}] \| u \|_{C_1^\alpha((0,b];\mathcal{D})}$. As a consequence, $H_u \in C_1^\alpha((0,b];\mathcal{D})$ and

$$\| H_u \|_{C_1^\alpha((0,b];\mathcal{D})} \leq (b^{1-\alpha}+b)(1+b^\alpha)[\Theta_2(b)+\frac{2c_1b^\alpha}{\alpha}] \| u \|_{C_1^\alpha(\mathcal{D})} .$$

Concerning the assertion for $T(\cdot)x$, we note only that $T(\cdot)x \in C_1^\alpha((0,b];\mathcal{D})$ and $\| T(\cdot)x \|_{C_1^\alpha((0,b];\mathcal{D})} \leq (C^1_{\alpha,\infty} + \frac{C^2_{\alpha,\infty}}{\alpha(1-\alpha)}) \| x \|_{\alpha,\infty}$ when $x \in (X,\mathcal{D})_{\alpha,\infty}$. □

Arguing as in the proof of Lemma 2 and [14, Theorem 4.3.1], we can prove the following result.

Lemma 5. *Let $\xi_1 : [0,b] \to X$ with $\xi_1 \in C_1^\alpha((0,b];\mathcal{D})$, $\xi_2 \in C_1^\alpha((0,b];X)$ and $v : (0,b] \to X$ be the function defined by*

$$v(t) = T(t)(x+\xi_1(0)) - \xi_1(t) - \int_0^t AT(t-s)\xi_1(s)ds + \int_0^t T(t-s)\xi_2(s)ds.$$

If $(x+\xi_1(0)) \in (X,\mathcal{D})_{\alpha,\infty}$, *then* $u \in C_1^\alpha((0,b];\mathcal{D})$,

$$\frac{d}{dt}\left[u(t)+\xi_1(t)\right] = Au(t)+\xi_2(t), \quad \textit{for all} \;\; t \in (0,b],$$

and $u+\xi_1 \in C_1^\alpha((0,b];\mathcal{D}) \cap C_1^{1+\alpha}((0,b];X)$. *Moreover, there exists* $C>0$, *independent of* $\xi_i(\cdot)$, $i=1,2$, $f(\cdot)$ *and* $b \in [0,a]$, *such that*

$$\| u \|_{C_1^\alpha((0,b];\mathcal{D})} \leq C(\| x+\xi_1(0) \|_{(X,\mathcal{D})_{\alpha,\infty}} + \| \xi_1 \|_{C_1^\alpha((0,b];\mathcal{D})} + \| \xi_2 \|_{C_1^\alpha((0,b];X)}).$$

The proof of Lemma 6 follows from the proof of [14, Theorem 6.1.4].

Lemma 6. *Let* $u \in C_1^\alpha((0,b];\mathcal{D})$, $x \in X$ *and* $f \in C_1^\alpha((0,b];X)$. *Then* $\tilde{g}_u \in C_1^\alpha((0,b];X)$ *and*

$$\begin{aligned} \| \tilde{g}_u \|_{C_1^\alpha((0,b];X)} \quad &\leq \quad [| A |]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))} 2b^\alpha \| u \|_{C_1^\alpha((0,b];\mathcal{D})} \\ &\quad + [| A |]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))}(2C_1 + \frac{C_2}{\alpha}) \| x \| + \| f \|_{C_1^\alpha((0,b];X)} . \end{aligned}$$

Theorem 2. *Let condition* $\mathbf{H_5}$ *hold,* $x \in (X,\mathcal{D})_{\alpha,\infty}$, $f \in C_1^\alpha((0,a];X)$ *and* $\limsup_{b\to 0} b^{1-\alpha}\Theta_2(b) < 1$. *Then there exists a classical solution* $u \in C_1^\alpha((0,b];\mathcal{D})$ *of the system* (1.1)-(1.2) *on* $[0,b]$, *for some* $0 < b \leq a$. *Moreover,* $u + H_u \in C_1^\alpha((0,b];\mathcal{D}) \cap C_1^{1+\alpha}((0,b];X)$.

Proof. Let $b \in (0,a]$ be such that

$$\Theta = C[(b^{1-\alpha}+b)(1+b^\alpha)[\Theta_2(b) + \frac{2c_1 b^\alpha}{\alpha}] + 2b^\alpha[| A |]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))}] < 1,$$

where C is the constant introduced in Lemma 5. Let $\Gamma : C_1^\alpha((0,b];\mathcal{D}) \to C_1^\alpha((0,b];\mathcal{D})$ be the map defined by

$$\begin{aligned} \Gamma v(t) \quad &= \quad -H_v(t) - H_{T(\cdot)x}(t) - \int_0^t AT(t-s)\left(H_v(s) + H_{T(\cdot)x}(s)\right) ds \\ &\quad + \int_0^t T(t-s)\left[(A(s)-A)(v(s)+T(s)x) + f(s)\right] ds. \end{aligned}$$

From Lemmas 4, 6 and 5, we obtain that $\Gamma u \in C_1^\alpha((0,b];\mathcal{D})$ if $u \in C_1^\alpha((0,b];\mathcal{D})$. Moreover, from Lemmas 4 and 6, for $v,w \in C_1^\alpha((0,b];\mathcal{D})$ we get

$$\| \Gamma v - \Gamma w \|_{C_1^\alpha((0,b];\mathcal{D})} \quad \leq \quad \Theta \| v - w \|_{C_1^\alpha((0,b];\mathcal{D})},$$

which proves that Γ is a contraction. Let $v \in C_1^\alpha((0,b];\mathcal{D})$ be the fixed point of Γ. Now, Lemma 5 permits us to conclude that $u = v(\cdot) + T(\cdot)x$ is a classical solution of (1.1)-(1.2) in $C_1^\alpha((0,b];\mathcal{D})$ and $u + H_u \in C_1^\alpha((0,b];\mathcal{D}) \cap C_1^{1+\alpha}((0,b];X)$. □

In order to consider the case in which $x \in X$, we introduce the next assumption.

($\mathbf{H_6}$) There are $c_1 > 0$, $\mu \in (\alpha,1)$ and a mensurable function $L_B : [0,a] \to \mathbb{R}^+$ such that

$$\begin{aligned} \Theta_3(b) = \sup_{t\in[0,b]} \max\{\int_0^t \frac{L_B(\theta)}{(t-\theta)^\beta} d\theta : \beta \in \{1+\alpha-\mu, 1-\alpha\}\} < \infty, \\ \| B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \leq c_1(s-\tau)^\mu, \\ \| B(t,t-\tau) - B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})} \leq L_B(\tau) \mid t-s \mid^\mu, \end{aligned}$$

for all $t \geq s \geq \tau \geq 0$ and every $b \in [0,a]$.

Lemma 7. *Let condition* $\mathbf{H_6}$ *hold. Assume* $x \in X$ *and* $u \in C_1^\alpha((0,b];\mathcal{D})$. *Then* H_u *and* $H_{T(\cdot)x}$ *are functions in* $C_1^\alpha((0,b];\mathcal{D})$ *and*

$$\| H_u \|_{C_1^\alpha(\mathcal{D})} \leq (b+b^{1-\alpha})(1+b^\alpha)(b^{\mu-\alpha}\Theta_3(b)+c_1b^\mu(\frac{1}{\mu}+\frac{1}{\alpha}))\| u \|_{C_{1-\alpha}(\mathcal{D})},$$

$$\| H_{T(\cdot)x} \|_{C_1^\alpha(\mathcal{D})} \leq \frac{2b^{1+\mu-\alpha}}{\alpha(\mu-\alpha)}(L_BC_1b^\mu+c_1(C_1+C_2))\| x \| .$$

Proof. Proceeding as in the proof of Lemma 1, for $t \geq s$ we see that

$$\begin{aligned}
\| H_u(t)-H_u(s) \|_{\mathcal{D}} &\leq \| u \|_{C_{1-\alpha}((0,b];\mathcal{D})}\int_0^s \frac{L_B(\tau)(t-s)^\mu}{(t-\tau)^{1-\alpha}}d\tau \\
&\quad + c_1[\| u \|]_{C_1^\alpha((0,b];\mathcal{D})}\int_0^s \frac{(s-\tau)^\mu(t-s)^\alpha}{(s-\tau)}d\tau \\
&\quad + \| u \|_{C_{1-\alpha}((0,b];\mathcal{D})}\, c_1\int_s^t \frac{(t-\tau)^\mu}{(t-\tau)^{1-\alpha}}d\tau \\
&\leq [b^{\mu-\alpha}\Theta_3(b)+c_1b^\mu(\frac{1}{\mu}+\frac{1}{\alpha})]\| u \|_{C_1^\alpha((0,b];\mathcal{D})}\,(t-s)^\alpha,
\end{aligned}$$

which implies that $H_u \in C^\alpha([0,b];\mathcal{D})$,

$$[\| H_u \|]_{C^\alpha([0,b];\mathcal{D})} \leq (b^{\mu-\alpha}\Theta_3(b)+c_1b^\mu(\frac{1}{\mu}+\frac{1}{\alpha}))\| u \|_{C_1^\alpha((0,b];\mathcal{D})}$$

and $\| H_u \|_{C(\mathcal{D})} \leq b^\alpha(b^{\mu-\alpha}\Theta_3(b)+c_1b^\mu(\frac{1}{\mu}+\frac{1}{\alpha}))\| u \|_{C_1^\alpha((0,b];\mathcal{D})}$ since $H_u(0)=0$. As a consequence, $H_u \in C_1^\alpha((0,b];\mathcal{D})$ and

$$\| H_u \|_{C_1^\alpha(\mathcal{D})} \leq (b+b^{1-\alpha})(1+b^\alpha)(b^{\mu-\alpha}\Theta_3(b)+c_1b^\mu(\frac{1}{\mu}+\frac{1}{\alpha}))\| u \|_{C_1^\alpha((0,b];\mathcal{D})} .$$

Concerning the assertion for the function $H_{T(\cdot)x}$, we see that

$$\begin{aligned}
&\| H_{T(\cdot)x}(t)-H_{T(\cdot)x}(s) \|_{\mathcal{D}} \\
&\leq \int_0^s \| B(t,t-\tau)-B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})}\| T(t-\tau)x \|_{\mathcal{D}}\, d\tau \\
&\quad + \int_0^s \| B(s,s-\tau) \|_{\mathcal{L}(\mathcal{D})}\| T(t-\tau)x - T(s-\tau)x \|_{\mathcal{D}}\, d\tau \\
&\quad + \int_s^t \| B(t,t-\tau) \|_{\mathcal{L}(\mathcal{D})}\| T(t-\tau)x \|_{\mathcal{D}}\, d\tau \\
&\leq C_1(t-s)^\alpha \| x \| \int_0^s \frac{L_B(\tau)(t-s)^{\mu-\alpha}}{(t-\tau)}d\tau \\
&\quad + c_1\int_0^s (s-\tau)^\mu \int_{s-\tau}^{t-\tau} \| A^2T(\xi)x \|\, d\xi d\tau + c_1C_1 \| x \| \int_s^t \frac{(t-\tau)^\mu}{(t-\tau)}d\tau \\
&\leq C_1\Theta_3(b) \| x \| (t-s)^\alpha + c_1C_2 \| x \| \int_0^s \frac{(s-\tau)^\mu}{(s-\tau)^{1+\alpha}}\int_{s-\tau}^{t-\tau}\frac{d\xi d\tau}{\xi^{1-\alpha}} \\
&\quad + \frac{c_1C_1}{\mu} \| x \| (t-s)^\mu \\
&\leq C_1\Theta_3(b) \| x \| (t-s)^\alpha + \frac{c_1C_2 \| x \| b^{\mu-\alpha}}{\alpha(\mu-\alpha)}(t-s)^\alpha + \frac{c_1C_1 \| x \|}{\mu}(t-s)^\mu,
\end{aligned}$$

so that $[\mid H_{T(\cdot)x} \mid]_{C^\alpha([0,b];\mathcal{D})} \leq (\Theta_3(b)C_1 + \frac{c_1 b^{\mu-\alpha}}{\alpha(\mu-\alpha)}(C_2+C_1)) \parallel x \parallel$. Moreover, since $H_{T(\cdot)x}(0) = 0$, it follows that

$$\parallel H_{T(\cdot)x} \parallel_{C([0,b];\mathcal{D})} \leq b^\alpha(\Theta_3(b)C_1 + \frac{c_1 b^{\mu-\alpha}}{\alpha(\mu-\alpha)}(C_2+C_1)) \parallel x \parallel,$$

$H_{T(\cdot)x} \in C^\alpha([0,b];\mathcal{D})$ and

$$\parallel H_{T(\cdot)x} \parallel_{C^\alpha([0,b];\mathcal{D})} \leq (1+b^\alpha)(\Theta_3(b)C_1 + \frac{c_1 b^{\mu-\alpha}}{\alpha(\mu-\alpha)}(C_1+C_2)) \parallel x \parallel .$$

This proves that $H_{T(\cdot)x} \in C_1^\alpha((0,b];\mathcal{D})$ and

$$\parallel H_{T(\cdot)x} \parallel_{C_1^\alpha((0,b];\mathcal{D})} \leq (b+b^{1-\alpha})(1+b^\alpha)(\Theta_3(b)C_1 + \frac{c_1 b^{\mu-\alpha}}{\alpha(\mu-\alpha)}(C_1+C_2)) \parallel x \parallel .$$

□

Theorem 3. *Let condition* $\mathbf{H_6}$ *be valid. Assume* $x \in X$, $f \in C_1^\alpha((0,a];X)$ *and* $\limsup_{b\to 0} b^{1+\mu-2\alpha}\Theta_3(b) < 1$. *Then there exists a classical solution of* (1.1)-(1.2) *on* $[0,b]$, *for some* $0 < b \leq a$. *Moreover,* $u \in C_1^\alpha((0,b];\mathcal{D})$ *and* $u + H_u \in C_1^\alpha((0,b];\mathcal{D}) \cap C_1^{1+\alpha}((0,b];X)$.

Proof. The proof is similar to the proof of Theorem 2. We note only that in this case it is sufficient to select $b \in (0,a]$ such that

$$\Theta = C(b+b^{1-\alpha})(1+b^\alpha)(b^{\mu-\alpha}\Theta_3(b) + \frac{2c_1 b^\mu}{\alpha}) + 2C[\mid A \mid]_{C^\alpha([0,b];\mathcal{L}(\mathcal{D},X))} b^\alpha < 1,$$

where C is the constant introduced in Lemma 5. □

3. Application

We now discuss the existence of solutions for a class of delayed integro-differential equations which arises in the theory of heat conduction in fading memory material. Consider the partial integro-differential system

$$\frac{\partial}{\partial t}\left[u(t,\xi) + \int_{-\infty}^{t} \gamma(t,t-s)\sigma(s)u(s,\xi)ds\right] = \zeta(t)\frac{\partial^2}{\partial \xi^2}u(t,\xi) + g(t,\xi), \tag{3.1}$$

$$u(s,0) = u(s,\pi) = 0, \quad s \in [0,a], \tag{3.2}$$

$$u(s,\xi) = \phi(s,\xi), \quad s \leq 0, \tag{3.3}$$

for $(t,\xi) \in [0,a] \times [0,\pi]$, where $\zeta \in C^\alpha([0,a];(0,\infty))$ and $\gamma : [0,a] \times [0,\infty) \to \mathbb{R}$, $\sigma : (-\infty,a] \to \mathbb{R}$, $g : [0,a] \times [0,\pi] \to \mathbb{R}$ and $\phi : (-\infty,0] \times [0,\pi] \to \mathbb{R}$ are appropriate functions.

In what follows, we assume that the expression

$$\Phi(t,\xi) = \frac{d}{dt}\int_{-\infty}^{0} \gamma(t,t-\tau)\sigma(\tau)\phi(\tau,\xi)d\tau$$

defines an $\mathbb{R}$-valued function Φ on $[0,a] \times [0,\pi]$ and $\Phi \in C^\alpha([0,a];L^2([0,\pi]))$.

Proposition 1. *Assume* $\gamma \in C^\beta([0,a]\times[0,a];\mathbb{R})$, $\sigma \in C^\beta([0,a];\mathbb{R})$ *for some* $\alpha,\beta \in (0,1)$, $\sigma(0) = 0$ *and there are* $\eta \in L^2(0,\pi)$ *such that*

$$\mid g(t,\xi) - g(s,\xi) \mid \leq \eta(\xi) \mid t-s \mid^\alpha, \quad t,s \in (0,a],\ \xi \in [0,\pi]. \tag{3.4}$$

(a) *If* $\phi(0,\cdot) \in L^2([0,\pi])$ *and* $\alpha < \beta$, *then there exist* $b \in (0,a]$ *and a function* $u : (-\infty, b] \to X$ *such that* $u \in C_1^\alpha((0,b]; W^2([0,\pi])) \cap W_0^1([0,\pi])$, $u(\tau,\xi) = \phi(\tau,\xi)$ *a.e. for* $(\tau,\xi) \in (-\infty,0] \times [0,\pi]$ *and* $u(\cdot)$ *satisfies* (3.1)-(3.2) *a.e. on* $[0,b] \times [0,\pi]$.

(b) *If* $\phi(0,\cdot) \in W^2([0,\pi]) \cap W_0^1([0,\pi])$ *and* $\alpha \le \beta$, *then there exist* $b \in (0,a]$ *and* $u : (-\infty, b] \to X$ *such that* $u \in C^\alpha([0,b]; W^2([0,\pi]) \cap W_0^1([0,\pi])$, $u(\tau,\xi) = \phi(\tau,\xi)$ *a.e. for* $(\tau,\xi) \in (-\infty,0] \times [0,\pi]$ *and* $u(\cdot)$ *satisfies* (3.1)-(3.2) *a.e. on* $[0,b] \times [0,\pi]$.

Proof. Let $X = L^2([0,\pi])$, $\mathcal{D} := \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$ and $A(t) : \mathcal{D} \subset X \to X$, $B(t,s) : \mathcal{D} \to \mathcal{D}$ $(a \ge t \ge s \ge 0)$ be the operators defined by $A(t)x = \zeta(t)x''$ and $B(t,s)x = \gamma(t,t-s)\sigma(s)x$. Let $f : [0,a] \to X$ be defined by

$$f(t)(\xi) = g(t,\xi) - \frac{d}{dt}\int_{-\infty}^{0} \gamma(t,t-\tau)\sigma(\tau)\phi(\tau,\xi)d\tau.$$

Under the above conditions and notation, we can represent the system (3.1)-(3.3) in the abstract form (1.1)-(1.2), the conditions $\mathbf{H_1}$ and $\mathbf{H_2}$ are verified and $f \in C^\alpha([0,a]; X)$.

If the assumptions in (**a**) are valid, then $\mathbf{H_6}$ is verified and the conclusions in (**a**) follow from Theorem 3. Item (**b**) is proved using Theorem 1. □

To finish, we note that the results in section 2 are easily applicable in the convolution case.

Proposition 2. *Assume there is* $\chi \in C([0,a]; X)$ *such that* $\gamma(t,t-s)\sigma(s) = \chi(t-s)$ *for* $t \ge s$, $t,s \in [0,a]$. *Suppose* $x \in W^2([0,\pi]) \cap W_0^1([0,\pi])$ *and there is a function* $\eta \in L^2(0,\pi)$ *such that* (3.4) *is verified. Then there exists* $b \in (0,a]$ *and* $u \in C^\alpha([0,b]; W^2([0,\pi]) \cap W_0^1([0,\pi]))$ *such that* $u(\tau,\xi) = \phi(\tau,\xi)$ *a.e. for* $(\tau,\xi) \in (-\infty,0] \times [0,\pi]$ *and* $u(\cdot)$ *satisfies* (3.1)-(3.2) *a.e. on* $[0,b] \times [0,\pi]$.

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