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STABLY INVERSE SHADOWABLE TRANSITIVE SETS AND DOMINATED SPLITTING

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Abstract. Let f be a diffeomorphism of a closed n-dimensional smooth manifold. In this paper, we show that if f has the C^1-stably inverse shadowing property on a transitive set, then it admits a dominated splitting.

1. Introduction

It has been a main subject in differentiable dynamical systems during the last decades to understand the influence of a robust dynamic property on the behavior of the tangent map of the system. The best known result of this type might be the (now verified) stability conjecture of Palis and Smale, which states that structural stability implies Axiom A and the strong transversality. In the context of the stability conjecture, Mãné [10] introduced the notion of *dominated splitting* which is more general than that of uniform hyperbolicity.

In this paper we are concerned with the so-called *stably inverse shadowable property*, and we show that if a transitive set is C^1 stably inverse shadowable, then it admits a dominated splitting.

Let us be more precise. Let X be a compact metric space with metric d, and let $\mathcal{H}(X)$ denote the space of homeomorphisms on X with the C^0-metric d_0. Let $f \in \mathcal{H}(X)$, and let Λ be a closed f-invariant set. Denote by $f|_\Lambda$ the restriction of f to a subset Λ of X.

Let $X^{\mathbb{Z}}$ be the space of all two-sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in X$, endowed with the product topology. For a fixed $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f. A mapping $\varphi : X \to \Phi_f(\delta) \subset X^{\mathbb{Z}}$ is said to be a *δ-method* for f if $\varphi(x)_0 = x$, where $\varphi(x)_0$ denotes the 0-th component of $\varphi(x)$. Then each $\varphi(x)$ is a δ-pseudo orbit of f through x. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k\in\mathbb{Z}}$. The set of all δ-methods for f will be denoted by $\mathcal{T}_0(f,\delta)$. Say that φ is a *continuous δ-method* for f if φ is continuous. The set of all continuous δ-methods for f will be denoted by $\mathcal{T}_c(f,\delta)$. If $g : M \to M$ is a homeomorphism with $d_0(f,g) < \delta$, then g induces a continuous δ-method φ_g for f by defining

$$\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\}.$$

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Let $\mathcal{T}_h(f,\delta)$ denote the set of all continuous δ-methods φ_g for f which are induced by $g \in \mathcal{H}(M)$ with $d_0(f,g) < \delta$.

Let M be a C^∞ closed manifold, and let $\mathrm{Diff}^r(M)(r \geq 1)$ be the space of C^r diffeomorphisms of M endowed with C^r topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. If $g : M \to M$ is a diffeomorphism with $d_1(f,g) < \delta$, then g induces a continuous δ-method φ_g for f by defining

$$\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\},$$

where d_1 is the C^1 metric on $\mathrm{Diff}^r(M)$. Let $\mathcal{T}_d(f,\delta)$ denote the set of all continuous δ-methods φ_g for f which are induced by $g \in \mathrm{Diff}(M)$ with $d_1(f,g) < \delta$. We define the class Θ by

$$\Theta = \bigcup_{\delta>0} \mathcal{T}_\alpha(f,\delta),$$

where $\alpha = 0, c, h, d$. We know that

$$\mathcal{T}_d(f) \subset \mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f),$$

where $\mathcal{T}_\alpha(f) = \bigcup_{\delta>0} \mathcal{T}_\alpha(f,\delta), \alpha = 0, c, h, d$.

Let Λ be a closed f-invariant set. We say that f has the *inverse shadowing property* on Λ (or Λ is *inverse shadowable*) with respect to the class Θ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-method $\varphi \in \Theta$ and for a point $x \in \Lambda$ there exists a point $y \in M$ such that

$$d(f^k(x), \varphi(y)_k) < \epsilon,\ k \in \mathbb{Z}.$$

Note that f has the inverse shadowing property with respect to the class Θ if and only if f^n has the inverse shadowing property with respect to the class Θ for $n \in \mathbb{N} \setminus \{0\}$.

Given $f \in \mathrm{Diff}(M)$, a closed f-invariant set $\Lambda \subset M$ is said to be *transitive* if there exists a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. We say that Λ is *nontrivial* if Λ is not just a periodic orbit.

Let $\Lambda \subset M$ be a closed f-invariant set. We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that

$$\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda(U).$$

We say that Λ admits a *dominated splitting of dimension* $i \in \{1, 2, \ldots, n-1\}$ if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

Definition 1.1. Let Λ be a closed f-invariant set. We say that f has the C^1-*stably inverse shadowing property* on Λ (or Λ is C^1 *stably inverse shadowable*) with respect to the class Θ if there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that

(1) $\Lambda = \Lambda(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$ (locally maximal),
(2) for any $g \in \mathcal{U}(f)$, g has the inverse shadowing property on $\Lambda_g(U)$ with respect to the class Θ, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is the *continuation* of $\Lambda(U)$.

Remark 1.2. Let Λ be a closed f-invariant set. A splitting $T_\Lambda M = E \oplus F$ is called an l-dominated splitting for a positive integer l and dimension $i \in \{1, 2, \dots, n-1\}$ if E and F are Df-invariant and

$$\|Df^l|_{E(x)}\|/m(Df^l|_{F(x)}) \leq \frac{1}{2},$$

for all $x \in \Lambda$, where $m(A) = \inf\{\|Av\| : \|v\| = 1\}$ denotes the mininorm of a linear map A.

Let Λ be a closed f-invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^{-n}$$

for all $x \in \Lambda$ and $n \geq 0$. Note that if a set Λ is hyperbolic, then it admits a dominated splitting.

From now on, we consider only the class $\mathcal{T}_d(f)$ when we mention the inverse shadowing property; that is, the "inverse shadowing property" implies the "inverse shadowing property with respect to the class $\mathcal{T}_d(f)$".

Now we are in a position to state the main theorem of our paper.

Theorem A. *If $f \in \mathrm{Diff}(M)$ has the C^1-stably inverse shadowing property on a transitive set Λ, then Λ admits a dominated splitting.*

The result above motivates the following conjecture.

Conjecture. *If $f \in \mathrm{Diff}(M)$ has the C^1-stably inverse shadowing property on a transitive set Λ, then Λ is hyperbolic.*

2. Some known results

Let M be as before, and let $f \in \mathrm{Diff}(M)$. To prove our theorem, first of all we need the following lemma which can be found in [4].

Lemma 2.1. *Let $\mathcal{U}(f)$ be any given C^1-neighborhood of f. Then there exists $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \dots, x_N\}$, a neighborhood U of $\{x_1, x_2, \dots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \leq \epsilon$ for all $1 \leq i \leq N$, there exists $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \dots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \leq i \leq N$.*

We recall Mãné's result in [10] for the uniformly hyperbolic family of periodic sequences of linear maps of $\mathbb{R}^n$. Let $GL(n)$ be the group of linear isomorphisms of $\mathbb{R}^n$. We say that a sequence $\xi : \mathbb{Z} \to GL(n)$ is *periodic* if there is $k > 0$ such that $\xi_{j+k} = \xi_j$ for $j \in \mathbb{Z}$. In the case, the finite subset $\mathcal{A} = \{\xi_i : 0 \leq i \leq k-1\} \subset GL(n)$ is called a *periodic family with period* k. For a periodic family $\mathcal{A} = \{\xi_i : 0 \leq i \leq n-1\}$, we denote

$$\mathcal{C}_\mathcal{A} = \xi_{n-1} \circ \xi_{n-2} \circ \cdots \circ \xi_0.$$

Definition 2.2 ([8])**.** We say that the periodic family $\mathcal{A} = \{\xi_i : 0 \leq i \leq n-1\}$ admits an l-dominated splitting and dimension $i \in \{1, 2, \dots, n-1\}$ if there is a splitting $\mathbb{R}^n = E \oplus F$ satisfying:

(1) E and F are $\mathcal{C}_\mathcal{A}$ invariant; i.e., $\mathcal{C}_\mathcal{A}(E) = E$ and $\mathcal{C}_\mathcal{A}(F) = F$.

(2) For any $k = 0, 1, 2, \ldots,$

$$\frac{\|\mathcal{A}^l|_{E_k}\|}{m(\mathcal{A}^l|_{F_k})} \leq \frac{1}{2},$$

where

$$E_k = \xi_{k-1} \circ \xi_{k-2} \circ \cdots \circ \xi_0(E),$$
$$F_k = \xi_{k-1} \circ \xi_{k-2} \circ \cdots \circ \xi_0(F),$$

and $\mathcal{A}^l = \xi_{k+l-i}$ for $i = 0, \ldots k-1$.

We will use the following two perturbation theorems, which can be found in [2].

Theorem 2.3. *Given any $\epsilon > 0$ and $C > 0$, there is $n(\epsilon, C) > 0$ which satisfies the following property: Given any periodic family $\mathcal{A} = \{A_0, A_1, \ldots, A_{k-1}\}$ which satisfies the period $n_0 \geq n_1(\epsilon, C)$ and $\max\{\|A_i\|, \|A_i^{-1}\|\} \leq C$ for $i = 0, 1, \ldots, k-1$, one can find the periodic family $\mathcal{B} = \{B_0, B_1, \ldots, B_{k-1}\}$ such that $\max\{\|B_i - A_i\|, \|B_i^{-1} - A_i^{-1}\|\} < \epsilon$ for $i = 0, 1, \ldots, k-1$, $\det(\mathcal{C}_\mathcal{A}) = \det(\mathcal{C}_\mathcal{A})$ and the eigenvalues of $\mathcal{C}_\mathcal{B}$ are all real, with multiplicity 1 and different moduli.*

Theorem 2.4. *Given any $\epsilon > 0$ and $C > 0$, there are $n(\epsilon, C) > 0$ and $l(\epsilon, C) > 0$ which satisfy the following property: Given any periodic family $\mathcal{A} = \{A_0, A_1, \ldots, A_{k-1}\}$ which satisfies the period $n_0 \geq n_1(\epsilon, C)$ and $\max\{\|A_i\|, \|A_i^{-1}\|\} \leq C$ for $i = 0, 1, \ldots, k-1$, if $\mathcal{A}$ does not admit any $l(\epsilon, C)$ dominated splitting, then one can find a periodic family $\mathcal{B} = \{B_0, B_1, \ldots, B_{k-1}\}$ such that $\max\{\|B_i - A_i\|, \|B_i^{-1} - A_i^{-1}\|\} < \epsilon$ for $i = 0, 1, \ldots, k-1$, $\det(\mathcal{C}_\mathcal{A}) = \det(\mathcal{C}_\mathcal{A})$, and the eigenvalues of $\mathcal{C}_\mathcal{B}$ are all real and have the same modulus.*

To prove Theorem A, we need another result regarding a uniformly contracting family which can be found in [10]. Let $\mathcal{A} = \{\xi_i : 0 \leq i \leq k-1\} \subset GL(n)$ be a periodic family. We say the sequence $\mathcal{A}$ is a *uniformly contracting family* if there is a constant $\delta > 0$ such that every δ-perturbation of $\mathcal{A}$ is sink; i.e, for any $\mathcal{B} = \{\zeta_i : 0 \leq i \leq k-1\}$ with $\|\zeta_i - \xi_i\| < \delta$, all eigenvalues of $\mathcal{C}_\mathcal{B}$ have moduli less than 1. Similarly, we can define the notion of a *uniformly expanding periodic family.*

Theorem 2.5 ([10])**.** *For any $\delta > 0$ and $K > 0$, there are constants $C > 0, 0 < \lambda < 1$ and a positive integer m such that if $\mathcal{A} = \{A_0, A_1, \ldots, A_{n-1}\}$ is a uniformly contracting periodic family which satisfies*

$$\max_{i=0,1,\ldots,n-1} \left\{\|A_i\|, \|A_i^{-1}\|\right\} < K$$

for $n > m$, then

$$\prod_{j=0}^{k-1} \left\| \prod_{i=0}^{m-1} A_{i+mj} \right\| \leq C\lambda^k,$$

where $k = [n/m]$.

3. Proof of Theorem A

Let M be as before, and let $f \in \mathrm{Diff}(M)$. In this section, we will use the notation of pre-sink (resp. pre-source). A periodic point p is called a *pre-sink* (resp. *pre-source*) if $Df^{\pi(p)}(p)$ has a multiplicity 1 eigenvalue equal to $+1$ or -1 and the other eigenvalues have a norm less than 1 (resp. larger than 1).

Lemma 3.1. *Let Λ be a closed f-invariant set. If f has the C^1-stably inverse shadowing property on Λ, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g has neither a pre-sink nor a pre-source whose orbit lies in U.*

Proof. Suppose that f has the C^1-stably inverse shadowing property on Λ. Then there is a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g has the inverse shadowing property on $\Lambda_g(U)$, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Assume that there is $g \in \mathcal{U}(f)$ such that g has a pre-sink p with $\mathcal{O}(p) \subset U$.

By making use of Lemma 2.1, we linearize g at p with respect to the exponential coordinates $\exp_p$; i.e., choose $\epsilon_1 > 0$ small enough with $B_{\epsilon_1}(p) \subset U$ and take g_1 C^1-nearby g such that if $x \in B_{\epsilon_1}(g^i(p))$, then

$$g_1(x) = \exp_{g^{i+1}(p)} \circ D_{g^i(p)}g \circ \exp^{-1}_{g^i(p)}(x),$$

for $0 \le i \le \pi(p) - 2$, and if $x \in B_{\epsilon_1}(g^{\pi(p)-1}(p))$, then

$$g_1(x) = \exp_p \circ D_{g^{\pi(p)-1}(p)}g \circ \exp^{-1}_{g^{\pi(p)-1}}(x).$$

Then we have $g_1^{\pi(p)}(p) = g^{\pi(p)}(p) = p$. Since p is a pre-sink of g, $D_pg^{\pi(p)}$ has an eigenvalue λ of multiplicity 1 such that $|\lambda| = 1$ and the other eigenvalues of $D_pg^{\pi(p)}$ have moduli less than 1. Denote by E^c_p the eigenspace corresponding to λ and by E^s_p the eigenspace corresponding to the eigenvalues with moduli less than 1. Then $T_pM = E^c_p \oplus E^s_p$, and $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$.

First we consider the case of $\lambda \in \mathbb{R}$. For simplicity, we may assume $\lambda = 1$. Then we have $\dim E^c_p = 1$, and there is a small arc $\mathcal{I}_p \subset B_{\epsilon_1}(p) \cap \exp_p(E^c_p(\epsilon_1))$ with center at p such that $g_1^{\pi(p)}(\mathcal{I}_p) = \mathcal{I}_p$. Here $E^c_p(\epsilon_1)$ denotes the ϵ_1-ball in E^c_p center at the origin O_p. By the hypothesis, g_1 has the inverse shadowing property on $\Lambda_{g_1}(U)$. For any $\delta > 0$, we can get a class of continuous δ-method $\varphi \in \mathcal{T}_d(g_1)$ which is induced by g_1. To simplify, we may assume that $g_1^{\pi(p)}(p) = g_1(p) = p$. Thus we can choose $0 < \epsilon = \epsilon_1/8$ as in the above. Let $0 < \delta < \epsilon$ be a number which can be obtained by the definition of the inverse shadowing property. To get a contradiction, we can introduce the local coordinates $x = (x_1, x_2, \dots, x_n) \in B_{\epsilon_1}(p)$, where $B_{\epsilon_1}(p)$ is a neighborhood of p. Then E^c_p is the subspace of T_pM corresponding to the eigenvalue λ of D_pg_1, and the space E^c_p coincides with the subspaces

$$E^c_p = \{v \in T_pM : v_2 = \cdots = v_n = 0\}$$

in the coordinates of the corresponding neighborhoods. We can idenfity T_pM with $\mathbb{R}^n$. Then

$$E^c_p = \{x \in \mathbb{R}^n : x_2 = \cdots = x_n = 0\}.$$

For ϵ_1, the mapping

$$g_1|_{B_{\epsilon_1}(p)} : B_{\epsilon_1}(p) \to B_{\epsilon_1}(p)$$

is given by $g_1(x) = (x_1, Ax')$, where A is the contracting part of D_pg and $x' = (x_2, \dots, x_n)$. Then we can construct a diffeomorphism $h \in \mathrm{Diff}(M)$ with the following properties: for $x \in B_{\epsilon_1}(p)$,

$$h(x) = \Big(x_1 + \frac{\delta}{2}, Ax'\Big) \text{ and } h^{-1}(x) = \Big(x_1 + \frac{\delta}{2}, A^{-1}x'\Big).$$

Then $d_1(h, g_1) < \delta$. Let p be identified with 0. Take $x_0 = (\epsilon, 0, \dots, 0) \in E^c_p(2\epsilon_1)$ such that $d(x_0, 0) = 2\epsilon$. Since g_1 has the inverse shadowing property on $B_\epsilon(0)$, we

can see that $d(g_1^n(x_0), \varphi_h(y)_n) < \epsilon$ for $n \in \mathbb{Z}$. For $y \in B_\epsilon(0)$, if $y_1 = 0$, then it is clear that g_1 does not have the inverse shadowing property on $B_\epsilon(0)$.

Now, we consider $y = (y_1, \dots, y_n) \in B_\epsilon(0)$ with $y_1 \neq 0$ and $d(x_0, y) < \epsilon$. If $y_2 = \cdots = y_n = 0$, then for any $k \in \mathbb{Z}$, we have

$$h^k(y) = \Big(y_1 + \frac{\delta}{2}k, 0\Big).$$

Then we can take $i \in \mathbb{Z}$ such that

$$d(0, h^i(y)) = d\Big(0, y_1 + \frac{\delta}{2}i\Big) > \epsilon.$$

Thus

$$d(g_1^k(x_0), h^k(y)) = d\Big(x_0, y_1 + \frac{\delta}{2}k\Big) > \epsilon,$$

for all $k > i$. This means that g_1 does not have the inverse shadowing property.

From now on we consider a point $y = (y_1, \dots, y_n) \in B_\epsilon(0)$ with $y \neq 0$ and $d(x_0, y) < \epsilon$. Since A is a contraction matrix, we can choose $k < 0$ such that

$$d(g_1^k(x_0), h^k(y)) = d\Big(x_0, \Big(y_1 + \frac{\delta}{2}k, A^k y'\Big)\Big) > \epsilon,$$

where $y' = (y_2, \dots, y_n)$. Therefore we get

$$d(g_1^{k+i}(x_0), h^{k+i}(y)) = d\Big(x_0, \Big(y_1 + \frac{\delta}{2}(k+i), A^{k+i} y'\Big)\Big) > \epsilon$$

for $i < 0$. This means g_1 does not have the inverse shadowing property on $B_\epsilon(0)$.

Now we consider a point $y = (y_1, \dots, y_n) \in M \setminus B_\epsilon(0)$ with $d(x_0, y) < \epsilon$. Since $h(y) = (y_1, Ay')$, as in the proof of the above, there exists $j \in \mathbb{Z}$ such that $h^j(y) \notin B_\epsilon(0)$. Thus

$$d(g_1^{j+i}(x_0), \varphi_h(y)_{j+i}) = d(x_0, h^{j+i}(y)) \geq \epsilon,$$

for all $i \in \mathbb{Z}_-$. Consequently, we know that g_1 does not have the inverse shadowing property on $\Lambda_{g_1}(U)$.

Next we consider the case of $\lambda \in \mathbb{C}$. Then $\dim E_p^c = 2$. To simplify, we may assume that $g^{\pi(p)}(p) = g(p) = p$. Then by Lemma 2.1, there are $\epsilon_1 > 0$ and g_1 C^1-nearby g such that

$$g_1(x) = \begin{cases} \exp_p \circ D_p g \circ \exp_p^{-1}(x), & \text{if } x \in B_{\epsilon_1}(p), \\ p, & \text{if } x = p. \end{cases}$$

Then $g_1(p) = g(p) = p$. By a modification of the map $D_p g_1$, we may suppose that there is $m > 0$ (the minimal number) such that $D_p g_1^m(v) = v$ for any $v \in E_p^c(\epsilon_0) \cap \exp_p^{-1}(B_{\epsilon_1}(p))$. Hence we can get a small disk $\mathcal{D}_p \subset \exp_p(E_p^c(\epsilon_1)) \cap B_{\epsilon_1}(p)$ such that $g_1^m(\mathcal{D}_p) = \mathcal{D}_p$.

Hence we treat only the case when the pseudo points are in $M \setminus \mathcal{D}_p$. The other case can be proved as in the above case. To prove this, we assume $g_1^m(\mathcal{D}_p) = \widetilde{g}(\mathcal{D}_p) = \mathcal{D}_p$. By applying Proposition 3.1 in [5], we can choose $h \in \mathrm{Diff}(M)$ and $\epsilon_1 > 0$ such that $d_1(h, \widetilde{g}) < \delta$, $h(p) = \widetilde{g}(p) = p$, $h(x) \neq \widetilde{g}(x)$ for $x \in B_{\epsilon_1}(p)$ and $h(x) = \widetilde{g}(x)$ for $x \notin B_{\epsilon_1}(p)$. Since p is pre-sink, for any point $y \in M \setminus \mathcal{D}_p$ we have

$$h^k(y) \to z \in \mathcal{D}_p \text{ as } k \to \infty \text{ and } h^k(y) \to \infty \text{ as } k \to -\infty.$$

Thus we get a contradiction by the inverse shadowing property for g_1. The contradiction shows that if $f \in \mathrm{Diff}(M)$ has the C^1-stably inverse shadowing property on Λ, then it does not have the pre-sinks.

The case for pre-source can be proved in a similar way. $\square$

Lemma 3.2. *Let Λ be a nontrivial transitive set. Then there are a sequence $\{g_n\}_n$ of diffeomorphisms and a sequence $\{P_n\}$ of periodic orbits of g_n with period $\pi(P_n) \to \infty$ such that $g_n \to f$ in the C^1-topology and $P_n \to_H \Lambda$ as $n \to \infty$, where $\to_H$ is the Hausdorff limit and $\pi(P_n)$ is the period of P_n.*

Proof. See Corollary 2.7.1 in [11]. □

From Lemma 3.2, we can choose $p_n \in P_n$ such that

$$\mathcal{A}_n = \{D_{p_n} g_n, D_{g_n(p_n)} g_n, \dots, D_{g_n^{\pi(p_n)-1}(p_n)} g_n\}$$

is a family of periodic linear maps.

Lemma 3.3. *Let Λ and P_n be as in Lemma 3.2, and let $\mathcal{A}_n$ be given as in the above. Then for any $\epsilon > 0$ there exists $n_0(\epsilon) > 0$ such that for any $n > n_0(\epsilon), \mathcal{A}_n$ is neither ϵ-uniformly contracting nor ϵ-uniformly expanding.*

Proof. We prove the lemma by a contradiction. Suppose that there is an infinite subsequence $\mathcal{A}_{n_k}$ and $\epsilon > 0$ such that $\mathcal{A}_{n_k}$ are ϵ-uniformly contracting. Then there are constants $C > 0, 0 < \lambda < 1$, and a positive integer m such that

$$\prod_{i=0}^{[\pi(p_{n_k})/m]-1} \|Df^m(f^{im}(p_{n_k}))\| \le C\lambda^k,$$

where $p_{n_k} \in P_{n_k}$ and $\pi(p_{n_k})$ is the minimum period of p_{n_k}. To simplify the notion, we assume that $m = 1$. Let $0 < \lambda < \gamma < 1$. Then there is a positive integer L such that for any $k > L$,

$$\prod_{j=0}^{\pi(p_{n_k})-1} \|Df(f^i(p_{n_k}))\| \le \gamma^{\pi(p_{n_k})}.$$

For every $k > L$, one can choose

$$q_{n_k} \in \{f^i(p_{n_k}) : j = 0, 1, \dots, \pi(p_{n_k}) - 1\}$$

such that for $l > 0$,

$$\prod_{i=0}^{l-1} \|Df(f^i(q_{n_k}))\| \le \gamma^l.$$

Otherwise, one can choose an infinite sequence $0 = l_0 < l_1 < \cdots$ such that

$$\prod_{i=l_i}^{l_{i+1}-1} \|Df(f^i(q_{n_k}))\| > \gamma^{l_{i+1}-l_i},$$

for $i = 0, 1, \dots$. However, we can choose two constants i, j with $i < j$ such that $l_j - l_i$ is a multiple of $\pi(p_{n_k})$. This contradicts

$$\prod_{i=0}^{\pi(p_{n_k})-1} \|Df(f^i(p_{n_k}))\| \le \gamma^{\pi(p_{n_k})}.$$

For given ϵ, choose $\gamma < \gamma_1 < 1$ such that for any $x, y \in \Lambda$, if $d(x, y) < \epsilon$, then

$$\frac{\gamma}{\gamma_1} < \frac{\|Df(x)\|}{\|Df(y)\|} < \frac{\gamma_1}{\gamma}.$$

Let P_{n_k} be given with $k > L$. Then for any $y \in B_\epsilon(q_{n_k})$, we can get $\|Df(y)\| < \gamma_1$. Then we have

$$f(B_\epsilon(q_{n_k})) \subset B_{\gamma_1\epsilon}(f(q_{n_k})) \subset B_\epsilon(f(q_{n_k})).$$

This implies that for any $y \in B_\epsilon(q_{n_k})$,

$$\|Df^2(y)\| \leq \|Df(f(y))\| \cdot \|Df(y)\| < \gamma_1^2.$$

Thus we have

$$f^2(B_\epsilon(q_{n_k})) \subset B_{\gamma_1^2\epsilon}(f^2(q_{n_k})) \subset B_\epsilon(f^2(q_{n_k})).$$

By induction, we can get that

$$f^n(B_\epsilon(q_{n_k})) \subset B_{\gamma_1^n\epsilon}(f^n(q_{n_k})) \subset B_\epsilon(f^n(q_{n_k})),$$

for $n > 0$. This implies that $B_\epsilon(q_{n_k}) \subset W^s(q_{n_k})$. Passing to a subsequence, we may assume that $q_{n_k} \to x \in \Lambda$ as $k \to \infty$. Then we can choose $m > 0$ sufficiently large such that $d(q_{n_k}, q_{n_m}) < \epsilon$ for any $k > m$. Thus we get that $d(f^n(q_{n_k}), f^n(q_{n_m})) \to 0$ as $n \to \infty$. Therefore, we have $q_{n_k} = q_{n_m}$ for $k > m$. This contradicts $\pi(q_n) \to \infty$.

Similarly, we can show that there are only finite ϵ-uniformly expanding in $\mathcal{A}_n$. This completes the proof of the lemma. □

The proof of the following lemma can be found in [8]. For convenience we cite the proof here.

Lemma 3.4. *Let Λ and P_n be as in Lemma 3.2. Then for any $\epsilon > 0$ there are $n(\epsilon)$ and $l(\epsilon) > 0$ such that for any $n > n(\epsilon)$, if P_n does not admit an $l(\epsilon)$-dominated splitting, then there exists g C^1-nearby f which preserves the orbit of P_n, and P_n is pre-sink or pre-source for g.*

Proof. We will prove the lemma by a contraction. Let $\epsilon > 0$ be arbitrary. To simplify the notion, we denote $g_n = f$. Let $K = \sup\{\|Df(x)\| : x \in M\}$, and let $\delta(\epsilon) > 0$ be as in Theorem 2.5. Then we can find constants $n_1(\delta/2, C)$, $n_2(\delta/2, C)$ and $l(\delta/2, C)$ by Theorems 2.3 and 2.4. Let $n_0(\delta/2)$ be as in Lemma 3.3. We choose $m > n_0$ such that $\pi(P_n) > \max\{n_1, n_2\}$ for any $n > m$. For $n > m$, we may assume that P_n does not admit an l-dominated splitting. This means that the periodic family $\mathcal{A}_n = \{D_{p_n}f, D_{f(p_n)}f, \ldots, D_{f^{\pi(p_n)-1}(p_n)}f\}$ does not admit an l-dominated splitting, where $p_n \in P_n$. By Theorem 2.4, we can choose

$$\mathcal{B}_n = \{D_{p_n}g, D_{g(p_n)}g, \ldots, D_{g^{\pi(p_n)-1}(p_n)}g\}$$

such that

$$\max_{0\leq i\leq \pi(p_n)-1} \left\{\|A_i - B_i\|, \|A_i^{-1} - B_i^{-1}\|\right\} < \epsilon, \ \ \det(\mathcal{C}_{\mathcal{A}_n}) = \det(\mathcal{C}_{\mathcal{B}_n}),$$

and the eigenvalues of $\mathcal{C}_{\mathcal{B}_n}$ are all real and have the same moduli, where $A_i = D_{f^i(p_n)}f$ and $B_i = D_{g^i(p_n)}g$.

If $\det(\mathcal{C}_{\mathcal{B}_n}) = 1$, then we can find a periodic family $\mathcal{E}_n$ arbitrarily close to $\mathcal{B}_n$ such that $\mathcal{C}_{\mathcal{E}_n}$ has a simple eigenvalue 1 or -1, and the other eigenvalues of $\mathcal{C}_{\mathcal{E}_n}$ have moduli less than 1. By Lemma 2.1, we can find a pre-sink.

Thus we can consider the case $\det(\mathcal{C}_{\mathcal{B}_n}) < 1$. Since $n > n_0$, $\mathcal{A}_n$ is not $\delta/2$-uniformly contracting. Thus we can choose a periodic family

$$\mathcal{F}_n = \{F_0, F_1, \ldots, F_{\pi(p_n)-1}\}$$

such that $\mathcal{C}_{\mathcal{F}_n}$ has an eigenvalue with modulus greater than or equal to 1. We denote

$$\mathcal{D}_n(t) = \{tB_0 + (1-t)F_0, tB_1 + (1-t)F_1, \dots, tB_{\pi(P_n)-1} + (1-t)F_{\pi(p_n)-1}\}.$$

Then we can easily show that for any $0 \le i < \pi(p_n)$,

$$\|A_i - (tB_i + (1-t)F_i)\| < \delta/2.$$

Take $t_0 = \sup\{t \in [0,1] : \mathcal{C}_{\mathcal{D}_n(t)} \text{ is a sink}\}$, and let

$$\mathcal{D}_n(t_0) = \{D_0, D_1, \dots, D_{\pi(P_n)-1}\}.$$

It is clear that $\mathcal{C}_{\mathcal{D}_n(t_0)}$ has an eigenvalue with modulus equal to 1 and has no eigenvalue with modulus larger than 1. Let λ be an eigenvalue of $\mathcal{C}_{\mathcal{D}_n(t_0)}$ satisfying $|\lambda| = 1$. If $\lambda \in \mathbb{R}$, then by taking a little change of $\mathcal{D}_n(t_0)$, we get a pre-sink. If $\lambda \in \mathbb{C}$, by applying Theorem 2.3, we can get a periodic family $\mathcal{E}_n = \{E_0, E_1, \dots, E_{\pi(p_n)}\}$ such that

- $\|E_i - D_i\| < \delta/2$ for any $0 \le i < \pi(p_n)$,
- $\mathcal{C}_{\mathcal{D}_n}$ has an eigenvalue of multiplicity 1, and the other eigenvalues are the same with the eigenvalue of $\mathcal{C}_{\mathcal{D}_n(t_0)}$ except λ, and $\bar{\lambda}$, where $\bar{\lambda}$ is conjugate of λ.

By making a little change of $\mathcal{E}_n$, we can see that P_n is a pre-sink with respect to a ϵ-perturbation of f if we apply Lemma 2.1.

Similarly, if $\det(\mathcal{C}_{\mathcal{B}_n}) > 1$, then we can take a small perturbation of f having P_n as a pre-source. This completes the proof of the lemma. □

From Lemmas 2.1 and 3.1-3.4, we get the following proposition.

Proposition 3.5. *Let Λ and P_n be as in Lemma 3.2. If f has the C^1-stably inverse shadowing property on Λ, then there are constants N and $l > 0$ such that P_n admits an l-dominated splitting for any $n > N$.*

Proposition 3.6. *Let $\{g_n\}$ be a sequence in* Diff(M) *which converges to f, and let Λ_{g_n} be a closed g_n-invariant set for each $n \in \mathbb{N}$. Suppose $\lim \Lambda_{g_n} = \Lambda$. If each Λ_{g_n} admits an l-dominated splitting for g_n, then Λ admits an l-dominated splitting for f.*

Proof. See [1, Lemma 1.4]. □

End of the proof of Theorem A. Let Λ be a transitive set of $f \in \mathrm{Diff}(M)$. Then by Lemma 3.2, there exist a sequence $\{g_n\}_{n\in\mathbb{Z}}$ in Diff(M) and a sequence $\{P_n\}$ of periodic orbits of g_n such that $g_n \to f$ in the C^1-topology and $P_n \to \Lambda$ in the Hausdorff limit. By Proposition 3.5, each P_n admits an l-dominated splitting. Hence if we apply Proposition 3.6, then we can see that Λ admits an l-dominated splitting. This completes the proof of Theorem A. □

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