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# LOCALLY NILPOTENT DERIVATIONS WITH A PID RING OF CONSTANTS

MOULAY A. BARKATOU AND M'HAMMED EL KAHOUI

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Abstract. Let $\mathcal{K}$ be a commutative field of characteristic zero, $\mathcal{A}$ be a domain containing $\mathcal{K}$ and $\partial$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$. We give in this paper a description of the differential $\mathcal{K}$-algebra $(\mathcal{A}, \partial)$ under the assumptions that the ring of constants $\mathcal{A}^{\partial}$ of $\partial$ is a PID, $\partial$ is fixed point free and its special fibers are reduced.

## 1. Introduction

Let $\mathcal{K}$ be a commutative field of characteristic zero, with $\overline{\mathcal{K}}$ as its algebraic closure, and let $\mathcal{A}$ be a commutative ring with unity containing $\mathcal{K}$. A $\mathcal{K}$-derivation $\partial$ of $\mathcal{A}$ is called *locally nilpotent* if for any $a \in \mathcal{A}$ there exists $m \geq 1$ such that $\partial^m(a) = 0$. When $\mathcal{A} = \mathcal{K}[\mathscr{V}]$ is the coordinate ring of an affine algebraic variety $\mathscr{V}$ defined over $\mathcal{K}$, a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$ corresponds to an action of the group $\mathbb{G}_a = (\overline{\mathcal{K}}, +)$ on the variety $\mathscr{V}$ defined by a regular map $\overline{\mathcal{K}} \times \mathscr{V} \longrightarrow \mathscr{V}$ with coefficients in the field $\mathcal{K}$.

Given a locally nilpotent $\mathcal{K}$-derivation $\partial$ of a $\mathcal{K}$*-domain* $\mathcal{A}$, i.e., a domain containing $\mathcal{K}$, we let $\mathcal{A}^{\partial}$ be its *ring of constants* and $\mathfrak{s}^{\partial} = \partial(\mathcal{A}) \cap \mathcal{A}^{\partial}$ be its *plinth ideal*; see [2] for more details on the plinth ideal. Given a prime ideal $\mathfrak{p}$ of $\mathcal{A}^{\partial}$, the derivation $\partial$ uniquely extends to $\mathcal{A} \otimes_{\mathcal{A}^{\partial}} \mathcal{A}^{\partial}_{\mathfrak{p}} = \mathcal{A}_{S_{\mathfrak{p}}}$, where $\mathcal{A}^{\partial}_{\mathfrak{p}}$ stands for the localization of $\mathcal{A}^{\partial}$ at $\mathfrak{p}$ and $S_{\mathfrak{p}} = \mathcal{A}^{\partial} \setminus \mathfrak{p}$. If $\mathfrak{s}^{\partial} \nsubseteq \mathfrak{p}$, then $\mathcal{A}_{S_{\mathfrak{p}}}$ is a univariate polynomial ring over $\mathcal{A}^{\partial}_{\mathfrak{p}}$ by a classical result of Wright [7] (see Lemma 2.1). In particular, if $\mathcal{F}_p$ is the residue field of $\mathcal{A}^{\partial}_{\mathfrak{p}}$, then $\mathcal{A} \otimes_{\mathcal{A}^{\partial}} \mathcal{F}_{\mathfrak{p}}$ is a univariate polynomial ring over $\mathcal{F}_{\mathfrak{p}}$. But when $\mathfrak{s}^{\partial} \subseteq \mathfrak{p}$ the structure of $\mathcal{A} \otimes_{\mathcal{A}^{\partial}} \mathcal{A}^{\partial}_{\mathfrak{p}}$ is not trivial and the fiber $\mathcal{A} \otimes_{\mathcal{A}^{\partial}} \mathcal{F}_{\mathfrak{p}}$ is degenerate. In the sequel, the fibers corresponding to the prime ideals containing $\mathfrak{s}^{\partial}$ will be called *the special fibers of* $\partial$.

Recently, M. Miyanishi established in [6] a structure theorem for $(\mathcal{A}, \partial)$ under the assumptions that $\mathcal{A}^{\partial}$ is a discrete valuation ring, with $\mathfrak{m}$ as its unique maximal ideal, $\mathcal{A}$ is finitely generated over $\mathcal{A}^{\partial}$ and the unique special fiber $\mathcal{A} \otimes_{\mathcal{A}^{\partial}} \mathcal{F}_{\mathfrak{m}}$ is irreducible; i.e., $\mathcal{A} \otimes_{\mathcal{A}^{\partial}} \mathcal{F}_{\mathfrak{m}}$ is a domain. More precisely, if $x$ is a uniformizer of $\mathcal{A}^{\partial}$, then Miyanishi's result may be stated as follows. The differential algebra $(\mathcal{A}, \partial)$ is $\mathcal{A}^{\partial}$-isomorphic to $(\mathcal{A}^{\partial}[z_1, \ldots, z_{r+1}]/\mathfrak{p}, a\zeta)$, where $\mathfrak{p}$ is an ideal of $\mathcal{A}^{\partial}[z_1, \ldots, z_{r+1}]$ generated by a system of the form $x^{m_1} z_2 - h_1(z_1), \ldots, x^{m_r} z_{r+1} - h_r(z_1, \ldots, z_r)$,

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$a$ is a constant in $\mathcal{A}^\partial$ and $\zeta$ is induced by the Jacobian derivation $\text{Jac}(x^{m_1}z_2 - h_1, \ldots, x^{m_r}z_{r+1} - h_r)$.

The assumptions in Miyanishi's result imply in particular that $\mathcal{A}$ is a UFD [6, Lemma 2.3] and $\partial = b\partial_1$, where $b \in \mathcal{A}^\partial$ and $\partial_1$ is fixed point free [6, Remark 2.2]. Thus, Miyanishi's result essentially concerns the case of a UFD endowed with a fixed point free locally nilpotent derivation. In this paper we show that the result holds true under the weaker assumptions that $\partial$ is fixed point free, $\mathcal{A}^\partial$ is a PID and the special fibers of $(\mathcal{A}, \partial)$ are reduced. This is a nontrivial generalization of Miyanishi's result since it goes beyond the factorial case. However, even in the case where $\mathcal{A}^\partial$ is a DVR, the techniques developed in this paper do not apply when $\partial$ is not fixed point free or when some of the special fibers of $\partial$ are not reduced.

## 2. Basics

In this section we recall the basic facts on locally nilpotent derivations to be used in this paper, and we refer to the books [4, 1, 2] for more details. We also recall the concept of affine modification [3]. Throughout this paper all the considered rings are commutative with unity.

2.1. **Locally nilpotent derivations.** Let $\mathcal{A}$ be a ring and $\partial$ be a locally nilpotent derivation of $\mathcal{A}$. We let $\mathcal{A}^\partial$ be the *ring of constants*, also called the *kernel*, of $\partial$. An element $s$ of $\mathcal{A}$ is called a *slice* of $\partial$ if $\partial(s) = 1$. The following fundamental result characterizes locally nilpotent derivations having a slice; see [7].

**Lemma 2.1.** *Let $\mathcal{A}$ be a ring containing $\mathbb{Q}$ and $\partial$ be a locally nilpotent derivation of $\mathcal{A}$ having a slice $s$. Then $\mathcal{A} = \mathcal{A}^\partial[s]$ and $\partial = \partial_s$. Moreover, if $\mathcal{A}$ is a domain, then all locally nilpotent $\mathcal{A}^\partial$-derivations of $\mathcal{A}$ are of the form $c\partial_s$, where $c \in \mathcal{A}^\partial$.*

In general, a nonzero locally nilpotent derivation need not have a slice. Nevertheless, it always has a *local slice*, i.e., an element $s$ such that $\partial(s) \in \mathcal{A}^\partial \setminus \{0\}$. If $s$ is a local slice of $\partial$, with $\partial(s) = c$, and $\mathcal{A}$ is a domain, then $\partial$ uniquely extends to a locally nilpotent derivation of the localization ring $\mathcal{A}_c$. Moreover, we have $(\mathcal{A}_c)^\partial = (\mathcal{A}^\partial)_c$ and according to Lemma 2.1 we have $\mathcal{A}_c = \mathcal{A}_c^\partial[s]$ if $\mathcal{A}$ contains $\mathbb{Q}$. In particular, $\mathcal{A}$ has transcendence degree 1 over $\mathcal{A}^\partial$, and in case $\mathcal{A}$ has a finite transcendence degree $r$ over $\mathcal{K}$ the $\mathcal{K}$-domain $\mathcal{A}^\partial$ has transcendence degree $r-1$ over $\mathcal{K}$.

A locally nilpotent derivation $\partial$ of a $\mathcal{K}$-domain $\mathcal{A}$ is called *irreducible* if the image $\partial(\mathcal{A})$ is not contained in any principal ideal of $\mathcal{A}$. Assume that every infinite ascending sequence $(a_i\mathcal{A}^\partial)_i$ of principal ideals of $\mathcal{A}^\partial$ is stationary. Then we have $\partial = a\delta$, where $a \in \mathcal{A}^\partial$ and $\delta$ is an irreducible locally nilpotent derivation. If in addition to the above assumption the intersection of any two principal ideals of $\mathcal{A}^\partial$ is a principal ideal, i.e., $\mathcal{A}^\partial$ is a UFD, then the decomposition is unique up to the units of $\mathcal{A}$; see e.g. [2, section 2.1].

Let $\mathcal{A}$ be a ring containing $\mathcal{K}$ and $\partial$ be a $\mathcal{K}$-derivation of $\mathcal{A}$. Let $\mathfrak{i}$ be a proper *invariant ideal* of $\partial$, i.e., $\partial(\mathfrak{i}) \subseteq \mathfrak{i}$. Then $\partial$ induces a $\mathcal{K}$-derivation, denoted by $\partial_{|\mathfrak{i}}$, of the quotient algebra $\mathcal{A}/\mathfrak{i}$. The derivation $\partial_{|\mathfrak{i}}$ is nonzero if and only if $\partial(\mathcal{A})$ is not contained in the ideal $\mathfrak{i}$. If $\partial$ is locally nilpotent, so is $\partial_{|\mathfrak{i}}$. Given an ideal $\mathfrak{i}$ of $\mathcal{A}$ invariant under $\partial$, its radical is also an invariant ideal of $\partial$. If moreover $\mathcal{A}$ is Noetherian, then any minimal prime of $\mathfrak{i}$ is an invariant ideal $\partial$.

Given two locally nilpotent $\mathcal{K}$-derivations $\partial$ and $\delta$ of $\mathcal{A}$, their sum $\partial+\delta$ need not be locally nilpotent. Nevertheless, if $\partial$ and $\delta$ commute, then $\partial+\delta$ is locally nilpotent.

A derivation $\partial$ of a ring $\mathcal{A}$ is called *fixed point free* if the ideal generated by the range $\partial(\mathcal{A})$ of $\partial$ is equal to $\mathcal{A}$. This equivalently means that $\partial(A)$ is not contained in any proper ideal of $\mathcal{A}$.

2.2. **Plinth ideal.** Let $\mathcal{A}$ be a $\mathcal{K}$-domain and let $\partial$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$. The subset $\mathfrak{s}^{\partial}=\mathcal{A}^{\partial}\cap\partial(\mathcal{A})$ is actually an ideal of $\mathcal{A}^{\partial}$, called *the plinth ideal* of $\partial$; see [2] for more details. It is easy to see that $\mathfrak{s}^{\partial}=\{\partial(s):\partial^2(s)=0\}$ and that $\mathfrak{s}^{\partial}=\mathcal{A}^{\partial}$ if and only if $\partial$ has a slice. A local slice $s$ of $\partial$ is called minimal if for any local slice $v$ such that $\partial(v)\mid\partial(s)$ we have $\partial(s)=\mu\partial(v)$, where $\mu$ is a unit of $\mathcal{A}^{\partial}$. In case the ring $\mathcal{A}$ satisfies the ascending chain condition on principal ideals, it is proved in [2, section 2.2] that a minimal local slice exists.

Now assume that $\mathfrak{s}^{\partial}$ is a principal ideal. Then for any minimal local slice $s$ of $\partial$ the element $c=\partial(s)$ generates the ideal $\mathfrak{s}^{\partial}$. Although a minimal local slice $s$ is not uniquely determined, any other minimal local slice $s_1$ of $\partial$ is of the form $s_1=\mu s+a$, where $\mu\in\mathcal{A}^{\star}$ and $a\in\mathcal{A}^{\partial}$; see [2, Proposition 2.7]. This shows that the subring $\mathcal{A}^{\partial}[s]$ is uniquely determined.

2.3. **Affine modifications.** We recall in this subsection the concept of affine modification and refer to [3] for more details.

**Definition 2.2.** Let $\mathcal{A}$ be a ring, $\mathfrak{i}$ be an ideal of $\mathcal{A}$ and $c\in\mathfrak{i}$ be a regular element. The subring $\mathcal{A}[c^{-1}\mathfrak{i}]$ of $\mathcal{A}_c$ is called the *affine modification* of $\mathcal{A}$ with locus $(\mathfrak{i},c)$.

Notice that the affine modification $\mathcal{A}[c^{-1}\mathfrak{i}]$ contains $\mathcal{A}$ since $c$ is assumed to be regular in $\mathcal{A}$. Moreover, if $\mathfrak{i}$ is finitely generated and $h_1,\dots,h_r$ is a generating system of $\mathfrak{i}$, then $\mathcal{A}[c^{-1}\mathfrak{i}]$ is generated as an $\mathcal{A}$-algebra by $c^{-1}h_1,\dots,c^{-1}h_r$. It follows that an affine modification of an affine ring over $\mathcal{K}$ is also an affine ring over $\mathcal{K}$. Let us also recall the following result; see [3, Corollary 2.3].

**Proposition 2.3.** *Let $\mathcal{A}$ be a a ring containing $\mathcal{K}$ and $\mathcal{A}[c^{-1}\mathfrak{i}]$ be an affine modification of $\mathcal{A}$ with locus $(\mathfrak{i},c)$. Let $\partial$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$ such that $\partial(c)=0$ and $\partial(\mathfrak{i})\subseteq\mathfrak{i}$. Then $\partial$ may be lifted in a unique way to a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}[c^{-1}\mathfrak{i}]$.*

When a locally nilpotent derivation $\delta$ of an affine modification $\mathcal{A}[c^{-1}\mathfrak{i}]$ is obtained from a locally nilpotent derivation $\partial$ of $\mathcal{A}$ by using Proposition 2.3, we will say that $\delta$ is *the affine modification* of $\partial$ with locus $(\mathfrak{i},c)$.

## 3. Statement of the main result

Given a ring $\mathcal{B}$ and $g=g_1,\dots,g_r$ a list of polynomials in $\mathcal{B}[z_1,\dots,z_{r+1}]$, we let $\mathrm{Jac}(g,.)$ be the *Jacobian derivation* associated to $g$; i.e., for any $f\in\mathcal{B}[z_1,\dots,z_{r+1}]$, the polynomial $\mathrm{Jac}(g,f)$ is the determinant of the Jacobian matrix of $(g,f)$ with respect to $z_1,\dots,z_{r+1}$.

Assume that $\mathcal{B}$ is a UFD and let $c\in\mathcal{B}$. Then we may write $c=c_1\cdots c_r$, where the $c_i$'s are square-free and $c_{i+1}\mid c_i$. Moreover, this factorization is essentially unique in the sense that if $c=d_1\cdots d_t$, with $d_i$ square-free and $d_{i+1}\mid d_i$, then $r=t$ and there exist $\mu_1,\dots,\mu_r\in\mathcal{B}$ such that $d_i=\mu_i c_i$ and $\mu_1\cdots\mu_r=1$. Such a factorization will be called the *square-free factorization* of $c$.

The following theorem is the main result of this paper.

**Theorem 3.1.** *Let $\mathcal{A}$ be a $\mathcal{K}$-domain and $\partial$ be a fixed point free locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$ such that $\mathcal{A}^\partial$ is a PID and all the special fibers of $\partial$ are reduced. Let $c$ be a generator of the plinth ideal $\mathfrak{s}^\partial$ and $c = c_1 \cdots c_r$ be its square-free factorization. Then there exists a triangular system $h_1(z_1), \ldots, h_r(z_1, \ldots, z_r)$ with coefficients in $\mathcal{A}^\partial$ such that*

$$(\mathcal{A}, \partial) \simeq_{\mathcal{A}^\partial} (\mathcal{A}^\partial[z_1, \ldots, z_{r+1}]/\mathfrak{p}, \zeta),$$

*where $\mathfrak{p}$ is the ideal generated by $c_1 z_2 - h_1, \ldots, c_r z_{r+1} - h_r$ and $\zeta$ is induced by the Jacobian derivation* $\mathrm{Jac}(c_1 z_2 - h_1, \ldots, c_r z_{r+1} - h_r)$.

If $\mathcal{A}^\partial$ is a DVR and the unique special fiber of $(\mathcal{A}, \partial)$ is irreducible, then $\mathcal{A}$ is a UFD by [6, Lemma 2.3] and $\partial = b\partial_1$, with $b \in \mathcal{A}^\partial$ and $\partial_1$ is fixed point free, by [6, Remark 2.2]. Thus, Theorem 3.1 applies to $(\mathcal{A}, \partial_1)$ and we retrieve Theorem 4.3 of [6]. It is also worth mentioning that we do not need to assume $\mathcal{A}$ to be finitely generated over $\mathcal{A}^\partial$ since this property is automatically satisfied.

## 4. Proof of the main result

The main idea behind the proof of Theorem 3.1 is the following construction. Let $\mathcal{A}$ be a $\mathcal{K}$-domain and $\partial$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$ such that $\mathcal{A}^\partial$ is a UFD and $\mathfrak{s}^\partial$ is principal. Let $c = \partial(s)$ be a generator of the plinth ideal $\mathfrak{s}^\partial$ and write $c = c_1 \cdots c_r$ for a square-free factorization of $c$; i.e., the $c_i$'s are square-free and $c_{i+1} \mid c_i$. We consider the following sequence of affine modifications and ideals:

$$\begin{array}{ll}
\mathcal{T}_1^\partial = \mathcal{A}^\partial[s] & \mathfrak{i}_1^\partial = c_1\mathcal{A} \cap \mathcal{T}_1^\partial \\
\mathcal{T}_2^\partial = \mathcal{T}_1^\partial[c_1^{-1}\mathfrak{i}_1^\partial] & \mathfrak{i}_2^\partial = c_2\mathcal{A} \cap \mathcal{T}_2^\partial \\
\vdots & \vdots \\
\mathcal{T}_{i+1}^\partial = \mathcal{T}_i^\partial[c_i^{-1}\mathfrak{i}_i^\partial] & \mathfrak{i}_{i+1}^\partial = c_{i+1}\mathcal{A} \cap \mathcal{T}_{i+1}^\partial \\
\vdots & \vdots \\
\mathcal{T}_{r+1}^\partial = \mathcal{T}_r^\partial[c_r^{-1}\mathfrak{i}_r^\partial] & \mathfrak{i}_{r+1}^\partial = \mathcal{A}.
\end{array}$$

If $c = d_1 \ldots d_t$ is another square-free factorization of $c$, then we have $t = r$ and $d_i = \mu_i c_i$, with $\mu_1 \cdots \mu_r = 1$. Moreover, the sequence of affine modifications and ideals corresponding to the factorization $c = d_1 \ldots d_r$ is the same as the one corresponding to the factorization $c = c_1 \cdots c_r$ since the $\mu_i$'s are units of $\mathcal{A}^\partial$. Thus, the sequence $(\mathcal{T}_i^\partial, \mathfrak{i}_i^\partial)_i$ is independent of the choice of the square-free factorization. It will be called *the tower of square-free affine modifications* corresponding to $\partial$. Notice that the above argument also shows that for any unit $\mu$ of $\mathcal{A}^\partial$ the towers of affine modifications corresponding respectively to $\partial$ and $\mu\partial$ are the same.

One readily checks that $\mathcal{T}_1^\partial$ is a subring of $\mathcal{A}$ stable under $\partial$ and that $\mathcal{T}_i^\partial \subseteq \mathcal{T}_{i+1}^\partial$. Moreover, an easy induction using Proposition 2.3 shows that $\mathcal{T}_i^\partial$ is stable under $\partial$ and $\mathfrak{i}_i^\partial$ is an invariant ideal of the restriction to $\mathcal{T}_i^\partial$ of $\partial$. In particular, the restriction of $\partial$ to $\mathcal{T}_{i+1}^\partial$ is the affine modification of the restriction of $\partial$ to $\mathcal{T}_i^\partial$ with locus $(\mathfrak{i}_i^\partial, c_i)$. So, in case $\mathcal{T}_{r+1}^\partial = \mathcal{A}$ the derivation $\partial$ is obtained from the derivation $c\partial_s$ of $\mathcal{A}^\partial[s]$ by a sequence of affine modifications.

To prove Theorem 3.1 we will study the structure of the sequence $(\mathcal{T}_i^\partial, \mathfrak{i}_i^\partial)_i$ when $\mathcal{A}^\partial$ is a PID, $\partial$ is fixed point free and its special fibers are reduced. For this, we need the following couple of lemmata.

4.1. **Some lemmata.**

**Lemma 4.1.** *Let $\mathcal{A}$ be a reduced $\mathcal{K}$-algebra of transcendence degree* 1 *and $\partial$ be a locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$. Assume that $\partial$ is fixed point free and let $\mathcal{B} = \mathcal{A}^{\partial}$. Then $\mathcal{A} = \mathcal{B}[s]$, where $s$ is a slice of $\partial$, and $\mathcal{B}$ is reduced and algebraic over $\mathcal{K}$.*

*Proof.* The case where $\mathcal{A}$ is finitely generated over $\mathcal{K}$ is proven in [5, Theorem 2.1]. Now assume that $\mathcal{A}$ is an arbitrary reduced $\mathcal{K}$-algebra. By assumption, $\partial$ is fixed point free and so there exist $u_1, \ldots, u_t$ and $s_1, \ldots, s_t$ in $\mathcal{A}$ such that $\sum u_i \partial(s_i) = 1$. On the other hand, there exists $n \geq 0$ such that $\partial^{n+1}(s_i) = \partial^{n+1}(u_i) = 0$ for any $i$. Now consider the subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ generated by the $\partial^j(u_i)$'s and the $\partial^j(s_i)$'s, where $j = 0, \ldots, n$. Clearly, $\mathcal{A}_0$ is finitely generated over $\mathcal{K}$ and is stable under $\partial$, and $\partial$ restricts to a fixed point free locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}_0$. Moreover, since $\mathcal{A}$ is reduced, so is $\mathcal{A}_0$. The finitely generated case then yields that $\partial$ has a slice $s$ in $\mathcal{A}_0$, and according to Lemma 2.1 we have $\mathcal{A} = \mathcal{B}[s]$. The fact that $\mathcal{B}$ is algebraic over $\mathcal{K}$ is obtained as a by-product. □

**Lemma 4.2.** *Let $\mathcal{A}$ be a $\mathcal{K}$-domain and $\partial$ be an irreducible locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$. Assume that $\mathcal{A}^{\partial}$ is a PID and let $c = \partial(s)$ be a generator of $\mathfrak{s}^{\partial}$. Then for any prime factor $p$ of $c$ such that $\mathcal{A}/p\mathcal{A}$ is reduced, the transcendence degree of $\mathcal{A}/p\mathcal{A}$ over $\mathcal{F}_p = \mathcal{A}^{\partial}/p\mathcal{A}^{\partial}$ is* 1.

*Proof.* Let $p$ be a prime factor of $c$ and let us first prove that $\mathcal{A}/p\mathcal{A}$ has transcendence degree at most 1 over $\mathcal{F}_p$. Let $a, b$ be two elements of $\mathcal{A} \setminus \mathcal{A}^{\partial}$. Since $\mathcal{A}$ has transcendence degree 1 over $\mathcal{A}^{\partial}$ there exists a polynomial $f(x, y) \in \mathcal{A}^{\partial}[x, y]$ such that $f(a, b) = 0$. Since, on the other hand, $\mathcal{A}^{\partial}$ is a UFD and $\mathcal{A}$ is a domain we may assume, even if it means dividing the coefficients of $f$ by their greatest common divisor, that $f$ is primitive. This means, in particular, that $f \neq 0$ when viewed in $\mathcal{F}_p[x, y]$.

Now assume towards a contradiction that $f(x, y)$ is constant in $\mathcal{F}_p[x, y]$, say $a_0$. Then we have $\gcd(a_0, p) = 1$ according to the fact that $f \neq 0$ in $\mathcal{F}_p[x, y]$. This yields $f(x, y) = pf_1(x, y) + a_0$ and so $pf_1(a, b) + a_0 = 0$. Thus, $p \mid a_0$ in $\mathcal{A}$ and so $p \mid a_0$ in $\mathcal{A}^{\partial}$ according to the fact that $\mathcal{A}^{\partial}$ is factorially closed in $\mathcal{A}$. But this contradicts the assumption that $\gcd(p, a_0) = 1$. Therefore, $f(x, y)$ is nonconstant in $\mathcal{F}_p[x, y]$ and hence $a$ and $b$, viewed in $\mathcal{A}/p\mathcal{A}$, are algebraically dependent over $\mathcal{F}_p$.

Since $\partial$ is irreducible it induces a nonzero $\mathcal{F}_p$-derivation of $\mathcal{A}/p\mathcal{A}$ and so $\mathcal{A}/p\mathcal{A}$ is transcendental over $\mathcal{F}_p$ according to the fact that $\mathcal{F}_p$ has characteristic zero and $\mathcal{A}/p\mathcal{A}$ is reduced. □

Recall that an ideal $\mathfrak{i}$ of a ring $\mathcal{A}$ is called *zero-dimensional* if the quotient ring $\mathcal{A}/\mathfrak{i}$ has Krull dimension 0. Recall as well that two ideals $\mathfrak{i}$ and $\mathfrak{j}$ of $\mathcal{A}$ are called *co-maximal* if $\mathfrak{i} + \mathfrak{j} = \mathcal{A}$. The following lemma concerns reduced zero-dimensional ideals of a polynomial ring over a PID.

**Lemma 4.3.** *Let $\mathcal{B}$ be a PID and let $\mathfrak{i}$ be a reduced zero-dimensional ideal of the polynomial ring $\mathcal{B}[z] = \mathcal{B}[z_1, \ldots, z_r]$. In case $\mathcal{B}$ is not a field we assume that $\mathfrak{i} \cap \mathcal{B} \neq (0)$. Then the ideal $\mathfrak{i}$ is generated by a triangular system $c, h_1(z_1), \ldots, h_r(z_1, \ldots, z_r)$ which satisfies the following properties:*

*i) The $h_i$'s are primitive and $c \in \mathcal{B}$ is square-free ($c = 0$ if $\mathcal{B}$ is a field).*

*ii) For any factor $d$ of $c$ and any $i = 0, \dots, r$ the ideal $(d\mathcal{B}[z] + \mathfrak{i}) \cap \mathcal{B}[z_1, \dots, z_i]$ is generated by $d, h_1, \dots, h_i$.*

*iii) The polynomials $\partial_{z_i} h_i$ are units in the quotient ring $\mathcal{B}[z]/\mathfrak{i}$.*

*Proof.* Assume first that $\mathcal{B}$ is a field and let $\mathfrak{j} = \mathfrak{i} \cap \mathcal{B}[z_1, \dots, z_{r-1}]$. Then $\mathfrak{j}$ is reduced and by the Hilbert Nullstellensatz it is also zero-dimensional and $\mathcal{B}[z]/\mathfrak{i}$ is integral over $\mathcal{B}[z_1, \dots, z_{r-1}]/\mathfrak{j}$. We may thus write $\mathfrak{j} = \bigcap \mathfrak{p}_i$ for the primary decomposition of $\mathfrak{j}$, where the $\mathfrak{p}_i$'s are maximal. Since the $\mathfrak{p}_i$'s are pairwise co-maximal, we have $\mathfrak{i} = \bigcap(\mathfrak{p}_i\mathcal{B}[z] + \mathfrak{i})$ and for any $i$ the ideal $\mathfrak{i}_i = \mathfrak{p}_i\mathcal{B}[z] + \mathfrak{i}$ is proper according to the fact that $\mathcal{B}[z]/\mathfrak{i}$ is integral over $\mathcal{B}[z_1, \dots, z_{r-1}]/\mathfrak{j}$. If we let $\mathcal{F}_i = \mathcal{B}[z_1, \dots, z_{r-1}]/\mathfrak{p}_i$, then $\mathcal{B}[z]/\mathfrak{i}_i$ is algebraic over the field $\mathcal{F}_i$ and so $z_r$ has a minimal polynomial $g_i(w)$ over $\mathcal{F}_i$. This shows that $\mathfrak{i}_i$ is generated by $g_i(z_r)$ and $\mathfrak{p}_i$. The fact that $\mathfrak{i}$ is reduced and zero-dimensional implies that any ideal containing $\mathfrak{i}$ is reduced and zero-dimensional. In particular, $\mathcal{B}[z]/\mathfrak{i}_i$ is reduced and so $g_i(w)$ is square-free, which means that $\partial_{z_r} g_i(z_r)$ is a unit in $\mathcal{B}[z]/\mathfrak{i}_i$.

By the Chinese Remainder Theorem we may find a polynomial $h_r(z_r)$ such that $h_r = g_i \mod \mathfrak{p}_i$ for any $i$. The ideal $\mathfrak{i}_i$ is thus generated by $h_r$ and $\mathfrak{p}_i$. Since $\mathfrak{i}$ is the intersection of the $\mathfrak{i}_i$'s we have $h_r \in \mathfrak{i}$. Since moreover the ideals $\mathfrak{i}_i$ are pairwise co-maximal, we have the equality $\mathfrak{i} = \prod \mathfrak{i}_i$, which shows that $\mathfrak{i}$ is generated by $h_r$ and $\mathfrak{j}$, and $\partial_{z_r} h_r$ is a unit in $\mathcal{B}[z]/\mathfrak{i}$. Continuing this way we construct the polynomials $h_{r-1}, \dots, h_1$.

Now we deal with the case where $\mathcal{B}$ is not a field. By assumption the ideal $\mathcal{B} \cap \mathfrak{i}$ is generated by a nonzero element $c \in \mathcal{B}$. Since $\mathfrak{i}$ is reduced, $c$ is square-free and we may write $c = p_1 \dots p_t$, where the $p_i$'s are prime elements of $\mathcal{B}$. Let $\mathfrak{q}_i = p_i\mathcal{B}[z] + \mathfrak{i}$ and notice that these ideals are reduced and pairwise co-maximal. Moreover, if for some $i$ we have $\mathfrak{q}_i = \mathcal{B}[z]$, then we have $1 = a + bp_i$, with $a \in \mathfrak{i}$. The multiplication of $p_i^{-1}c$ to both sides of the equation yields $p_i^{-1}c \in \mathfrak{i}$. Thus, the $\mathfrak{q}_i$'s are proper reduced and zero-dimensional, and so are the ideals $\mathfrak{q}_i' = \mathfrak{q}_i\mathcal{F}_{p_i}[z_1, \dots, z_r]$, where $\mathcal{F}_{p_i} = \mathcal{B}/p_i\mathcal{B}$. Since $\mathcal{F}_{p_i}$ is a field we may find $h_{i,1}(z_1), \dots, h_{i,r}(z_1, \dots, z_r) \in \mathcal{B}[z]$ which generate the ideal $\mathfrak{q}_i'$ and $\partial_{z_j} h_{i,j}$ is a unit in $\mathcal{B}[z]/\mathfrak{q}_i$. As a by-product, $p_i, h_{i,1}, \dots, h_{i,r}$ generate $\mathfrak{q}_i$. By the Chinese Remainder Theorem we may construct polynomials $h_1(z_1), \dots, h_r(z_1, \dots, z_r)$ such that $h_j = h_{i,j} \mod p_j$ for any $i, j$. Now the fact that $\mathfrak{i}$ is the intersection of the $\mathfrak{q}_i$'s implies that $h_j \in \mathfrak{i}$ for any $j$. Since moreover the $\mathfrak{q}_i$'s are pairwise co-maximal we have $\mathfrak{i} = \prod \mathfrak{q}_i$, which shows that $\mathfrak{i}$ is generated by $c, h_1, \dots, h_r$. Even if it means removing the content of $h_j$, which is necessarily co-prime with $c$, we may assume that $h_j$ is primitive. On the other hand, since each $\partial_{z_j} h_{i,j}$ is a unit in $\mathcal{B}[z]/\mathfrak{q}_i$, it is so for $\partial_{z_j} h_j$ in $\mathcal{B}[z]/\mathfrak{i}$. Finally, if $d$ is a divisor of $c$ we may assume without loss of generality that $d = p_1 \cdots p_u$. In this case we have $(d\mathcal{B}[z] + \mathfrak{i}) = \bigcap_1^u \mathfrak{q}_i$, and the way the $h_j$'s are constructed shows that $(d\mathcal{B}[z] + \mathfrak{i}) \cap \mathcal{B}[z_1, \dots, z_i]$ is generated by $d, h_1, \dots, h_i$. □

**Lemma 4.4.** *Let $\mathcal{A}$ be a $\mathcal{K}$-domain and $\partial$ be a fixed point free locally nilpotent $\mathcal{K}$-derivation of $\mathcal{A}$ such that $\mathcal{A}^\partial$ is a PID and all the special fibers of $\partial$ are reduced. Let $c = \partial(s)$ be a generator of the plinth ideal $\mathfrak{s}^\partial$, $c = c_1 \cdots c_r$ be its square-free factorization and let $(\mathcal{T}_i^\partial, \mathfrak{i}_i^\partial)$ be the tower of square-free affine modifications corresponding to $\partial$. Then there exist $s_1, \dots, s_{r+1} \in \mathcal{A}$ and a triangular system $h_1(z_1), \dots, h_r(z_1, \dots, z_r)$ with coefficients in $\mathcal{A}^\partial$ such that the following hold:*

*i) $s_1 = s$, $c_i \mid h_i(s_1, \dots, s_i)$, $s_{i+1} = c_i^{-1}h_i(s_1, \dots, s_i)$ and $c_i \cdots c_r \mid \partial(s_i)$.*

*ii)* $\mathcal{T}_i^\partial = \mathcal{A}^\partial[s_1, \dots, s_i]$, $\mathfrak{i}_i^\partial = (c_i, h_1(s_1), \dots, h_i(s_1, \dots, s_i))\mathcal{T}_i^\partial$ *and* $\partial_{z_i} h_i$ *is a unit in* $\mathcal{T}_i^\partial / \mathfrak{i}_i^\partial$.

*iii) If* $p$ *is a prime factor of* $c$*, with* $c = p^i q$ *and* $\gcd(p, q) = 1$*, then we have* $\mathcal{A}_q = \left(\mathcal{T}_{i+1}^\partial\right)_q$.

*Proof.* We will prove the assertions *i)* and *ii)* by induction on $i$. Let us write $c_1 = p_1, \dots, p_t$, where the $p_j$'s are prime and pairwise distinct, and recall that each $\mathcal{A}_j = \mathcal{A}/p_j$ is reduced by assumption. By Lemma 4.2, $\mathcal{A}_j$ is of transcendence degree 1 over $\mathcal{F}_j = \mathcal{A}^\partial / p_j$. Moreover, $\partial$ induces a fixed point free locally nilpotent $\mathcal{F}_j$-derivation $\delta_j$ on $\mathcal{A}_j$. Let $\mathcal{B}_j$ be the ring of constants of $\delta_j$ and notice that $\mathcal{B}_j$ is algebraic over $\mathcal{F}_j$ by Lemma 4.1.

Let $\mathfrak{j}_{1,j} = \mathcal{T}_1^\partial \cap p_j \mathcal{A}$. Then $\mathcal{T}_1^\partial / \mathfrak{j}_{1,j} \subset \mathcal{A}/p_j$ and moreover $\delta_j = 0$ on $\mathcal{T}_1^\partial / \mathfrak{j}_{1,j}$ according to the fact that $p_j \mid \partial(s)$. This yields $\mathcal{T}_1^\partial / \mathfrak{j}_{1,j} \subset \mathcal{B}_j$, and so $\mathcal{T}_1^\partial / \mathfrak{j}_{1,j}$ is reduced and algebraic over $\mathcal{F}_j$ since it is the case for $\mathcal{B}_j$. This shows that $\mathcal{T}_1^\partial / \mathfrak{j}_{1,j}$ is zero-dimensional. On the other hand, since $\mathcal{A}^\partial$ is a PID the $\mathfrak{j}_{1,j}$'s are pairwise co-maximal and so $\mathfrak{i}_1^\partial = \bigcap \mathfrak{j}_{1,j}$ and $\mathcal{T}_1^\partial / \mathfrak{i}_1^\partial \simeq \prod \mathcal{T}_1^\partial / \mathfrak{j}_{1,j}$ by the Chinese Remainder Theorem. Therefore, $\mathcal{T}_1^\partial / \mathfrak{i}_1^\partial$ is reduced zero-dimensional. By Lemma 4.3 the ideal $\mathfrak{i}_1^\partial$ is generated by $c_1, h_1(s_1)$, where $h_1(z_1) \in \mathcal{A}^\partial[z_1]$ is primitive. If we let $s_2 = c_1^{-1} h_1(s_1)$, then clearly $\mathcal{T}_2^\partial = \mathcal{A}^\partial[s_1, s_2]$ and $\partial(s_2) = c_2 \cdots c_r \partial_{z_1} h_1(s_1)$. We have thus proven the properties *i)* and *ii)* for $i = 1$.

Assume that *i)* and *ii)* hold for $i \leq r$. Let us write $\mathcal{T}_i^\partial = \mathcal{A}^\partial[s_1, \dots, s_i]$ and let $\mathfrak{i}_i^\partial$ be generated by $c_i, h_1(s_1), \dots, h_i(s_1, \dots, s_i)$. It follows immediately that $\mathcal{T}_{i+1}^\partial = \mathcal{A}^\partial[s_1, \dots, s_{i+1}]$, with $s_{i+1} = c_i^{-1} h_i$. Notice that if $i = r$, then we are done. Thus, we assume in the sequel that $i < r$. By the induction hypothesis we have $c_j \cdots c_r \mid \partial(s_j)$ for any $j \leq i$. This fact together with the relation $\partial(s_{i+1}) = c_i^{-1} \sum_1^i \partial_{z_j} h_i(s_1, \dots, s_i) \partial(s_j)$ implies that $c_{i+1} \cdots c_r \mid \partial(s_{i+1})$.

Since $c_{i+1} \mid c_1$ it is the product of some of the $p_j$'s. Without loss of generality we may assume that $c_{i+1} = p_1 \cdots p_u$. Let $\mathfrak{j}_{i+1,j} = \mathcal{T}_{i+1}^\partial \cap p_j \mathcal{A}$ and notice that $\mathcal{T}_{i+1}^\partial / \mathfrak{j}_{i+1,j} \subset \mathcal{A}/p_j$. Moreover, we have $\delta_j(s_{i+1}) = 0$ since $c_{i+1} \mid \partial(s_{i+1})$ and so $\mathcal{T}_{i+1}^\partial / \mathfrak{j}_{i+1,j} \subset \mathcal{B}_j$. This shows that $\mathcal{T}_{i+1}^\partial / \mathfrak{j}_{i+1,j}$ is reduced and zero-dimensional. The ideals $\mathfrak{j}_{i+1,j}$ are clearly pairwise co-maximal and $\mathfrak{i}_{i+1}^\partial = \bigcap \mathfrak{j}_{i+1,j}$, and by the Chinese Remainder Theorem we have $\mathcal{T}_{i+1}^\partial / \mathfrak{i}_{i+1}^\partial \simeq \prod \mathcal{T}_{i+1}^\partial / \mathfrak{j}_{i+1,j}$. This shows that $\mathcal{T}_{i+1}^\partial / \mathfrak{i}_{i+1}^\partial$ is reduced and zero-dimensional.

Let $\phi : \mathcal{A}^\partial[z_1, \dots, z_{i+1}] \longrightarrow \mathcal{T}_{i+1}^\partial$ be the $\mathcal{A}^\partial$-algebra homomorphism defined by $\phi(z_i) = s_i$. Clearly, $\phi$ is onto. Since moreover $\mathfrak{i}_r$ is reduced and zero-dimensional, so is the ideal $\mathfrak{i} = \phi^{-1}(\mathfrak{i}_r)$. We may thus find a generating system $c_{i+1}, h_1(z_1), \dots, h_{i+1}(z_1, \dots, z_{i+1})$ of $\mathfrak{i}$ which satisfies the properties of Lemma 4.3, and therefore $c_{i+1}, h_1(s_1), \dots, h_{i+1}(s_1, \dots, s_{i+1})$ generates $\mathfrak{i}_{i+1}^\partial$. Now we need to show that we may always choose $h_1, \dots, h_i$ in such a way that the system $c_i, h_1(s_1), \dots, h_i(s_1, \dots, s_i)$ generates the ideal $\mathfrak{i}_i^\partial$. Taking into account the property *ii)* of Lemma 4.3 this reduces to showing that $c_{i+1}\mathcal{A} \cap \mathcal{T}_i^\partial = c_{i+1}\mathcal{T}_i^\partial + \mathfrak{i}_i^\partial$. Let us write $c_i = c_{i+1} d$, and notice that $\gcd(c_{i+1}, d) = 1$ since $c_i$ is square-free. Since moreover $\mathcal{A}^\partial$ is a PID the ideals $c_{i+1}\mathcal{A} \cap \mathcal{T}_i^\partial$ and $d\mathcal{A} \cap \mathcal{T}_i^\partial$ are co-maximal and so $\mathfrak{i}_i^\partial$ is their product. The fact that these two ideals are co-maximal also yields $c_{i+1}\mathcal{T}_i^\partial + \mathfrak{i}_i^\partial = (c_{i+1}\mathcal{T}_i^\partial + c_{i+1}\mathcal{A} \cap \mathcal{T}_i^\partial) \cap (c_{i+1}\mathcal{T}_i^\partial + d\mathcal{A} \cap \mathcal{T}_i^\partial)$ and finally $c_{i+1}\mathcal{T}_i^\partial + \mathfrak{i}_i^\partial = c_{i+1}\mathcal{A} \cap \mathcal{T}_i^\partial$ since $c_{i+1}\mathcal{T}_i^\partial + d\mathcal{A} \cap \mathcal{T}_i^\partial = \mathcal{T}_i^\partial$.

*iii)* Let $p$ be a prime factor of $c$ and write $c = p^i q$, with $\gcd(p, q) = 1$. Notice first that $p \mid c_i$ and so $p^{i-j+1} \mid \partial(s_j)$ for any $j \leq i$. On the other hand, even if it means replacing $\mathcal{A}$ by the localization $\mathcal{A}_q$ we may assume that $c = p^i$. Now let $a \in \mathcal{A}$ and assume that $pa \in \mathcal{T}_{i+1}^{\partial}$. We may then write $pa = a_0(s_1, \ldots, s_i) + \cdots + a_m(s_1, \ldots, s_i)s_{i+1}^m$. We claim that $a_i \in \mathfrak{i}_i^{\partial}$ for any $i = 0, \ldots, m$. Indeed, the result is obviously true for $m = 0$. So, assume it holds true for $m-1$ and let us prove it for $m$. By applying $\partial$ to the relation $pa = \sum_j a_j s_{i+1}^j$ we get

$$p\partial(a) = \sum_j \partial(a_j)s_{i+1}^j + \partial(s_{i+1})\sum_{j\geq 1} j a_j s_{i+1}^{j-1}.$$

Since $\partial(a_j) = \sum_0^i \partial_{z_k} a_j \partial(s_k)$ and $p^{i-k+1} \mid \partial(s_k)$ we have $p \mid \sum_j \partial(a_j)s_{i+1}^j$. On the other hand, we have $\partial(s_{i+1}) = p^{-1}\sum_k \partial_{z_k} h_i \partial(s_k)$, and according to the fact that $p^{i-k+1} \mid \partial(s_k)$ we have $\partial(s_{i+1}) = pb_1 + \partial_{z_i} h_i \left(p^{-1}\partial(s_i)\right)$, where $b_1 \in \mathcal{A}$. By inductively repeating the same process we ultimately get

$$\begin{aligned}\partial(s_{i+1}) &= pb_i + \left(\prod_1^i \partial_{z_k} h_k\right) p^{-i}\partial(s_1)\\ &= pb_i + \prod_1^i \partial_{z_k} h_k.\end{aligned}$$

We therefore have $p \mid (\prod_k \partial_{z_k} h_k)\sum_{j\geq 1} j a_j s_{i+1}^{j-1}$, and since, by Lemma 4.3, $\prod_k \partial_{z_k} h_k$ is a unit modulo $p$ we have $p \mid \sum_{j\geq 1} j a_j s_{i+1}^{j-1}$. By the induction hypothesis we have $a_j \in \mathfrak{i}_i^{\partial}$ for any $j = 1, \ldots, m$. The fact that $p \mid \sum_0^m a_j s_{i+1}^j$ and $p \mid a_j$ for $j \geq 1$ implies that $p \mid a_0$ and so $a_0 \in \mathfrak{i}_i^{\partial}$.

Since $\mathfrak{i}_i^{\partial} = (p, h_1, \ldots, h_i)\mathcal{T}_i^{\partial}$ we may write $a_j = a_{0,j}p + a_{1,j}h_1 + \cdots + a_{i,j}h_i$ for $j = 1, \ldots, m$. This gives $a = p^{-1}\sum_j a_j s_{i+1}^j = \sum_j a_{0,j} + s_2 s_{i+1}\sum_j a_{1,j} + \cdots + s_{i+1}^{m+1}\sum_j a_{i,j}$. We have thus shown that $a \in \mathcal{T}_{i+1}^{\partial}$ whenever $pa \in \mathcal{T}_{i+1}^{\partial}$. A straightforward induction shows that for any $n \geq 1$ such that $p^n a \in \mathcal{T}_{i+1}^{\partial}$ we actually have $a \in \mathcal{T}_{i+1}^{\partial}$. Now let $a \in \mathcal{A}$ and notice that by Lemma 2.1 we have $\mathcal{A}_p = \mathcal{A}_p^{\partial}[s_1]$. In particular, there exists $n \geq 0$ such that $p^n a = \ell(s_1) \in \mathcal{T}_{i+1}^{\partial}$ and so $a \in \mathcal{T}_{i+1}^{\partial}$. □

Given two ideals $\mathfrak{i}$ and $\mathfrak{j}$ of a ring $\mathcal{A}$ recall that $\mathfrak{i} : \mathfrak{j}$ stands for the quotient ideal of $\mathfrak{i}$ and $\mathfrak{j}$. In case $\mathfrak{j}$ is generated by a single element $c$ we use the notation $\mathfrak{i} : c$ instead of $\mathfrak{i} : c\mathcal{A}$. The sequence $(\mathfrak{i} : \mathfrak{j}^n)_n$ is ascending, and so $\bigcup_n (\mathfrak{i} : \mathfrak{j}^n)$ is an ideal of $\mathcal{A}$ denoted by $\mathfrak{i} : \mathfrak{j}^{\infty}$. In case $\mathfrak{j}$ is generated by a single element $c$ we use the notation $\mathfrak{i} : c^{\infty}$ instead of $\mathfrak{i} : (c\mathcal{A})^{\infty}$.

**Lemma 4.5.** *Let $\mathcal{A}$ be a domain, $\mathfrak{i}$ be an ideal of $\mathcal{A}$ and let $c \in \mathcal{A}$. Let $\mathfrak{j} = \mathfrak{i} : c^{\infty}$ and assume that $\mathfrak{j} \subseteq c\mathcal{A} + \mathfrak{i}$. Then for any $n \geq 1$ we have $\mathfrak{j} \subseteq c^n\mathfrak{j} + \mathfrak{i}$. As a consequence, if $\mathfrak{j}$ is finitely generated, then we have $\mathfrak{i} = \mathfrak{j}$.*

*Proof.* Let $a \in \mathfrak{j}$ and let $v \in \mathbb{N}$ be such that $c^v a \in \mathfrak{i}$. Since $\mathfrak{j} \subseteq c\mathcal{A} + \mathfrak{i}$ we can write $a = ca_1 + b_1$, where $a_1 \in \mathcal{A}$ and $b_1 \in \mathfrak{i}$. This gives $c^v a = c^{v+1}a_1 + c^v b_1$, and so we have $c^{v+1}a_1 \in \mathfrak{i}$ since both $c^v a$ and $c^v b_1$ belong to $\mathfrak{i}$. Therefore $a_1 \in \mathfrak{j}$, and so $a \in c\mathfrak{j} + \mathfrak{i}$. We have thus proven that $\mathfrak{j} \subseteq c\mathfrak{j} + \mathfrak{i}$.

The fact that $\mathfrak{j} \subseteq c^n\mathfrak{j} + \mathfrak{i}$ for any $n \geq 1$ follows immediately from the inclusion $\mathfrak{j} \subseteq c\mathfrak{j} + \mathfrak{i}$. In case $\mathfrak{j}$ is finitely generated we have $c^n\mathfrak{j} \subseteq \mathfrak{i}$ for $n$ large enough, and so $\mathfrak{j} \subseteq \mathfrak{i}$. Since $\mathfrak{i} \subseteq \mathfrak{j}$ we have the equality $\mathfrak{i} = \mathfrak{j}$. □

4.2. **Proof of Theorem 3.1.** Let us write $c = p_1^{n_1} \cdots p_t^{n_t}$, where the $p_i$'s are prime and pairwise distinct, and let $q_i = \prod_{j \neq i} p_j^{n_j}$. If $a \in \mathcal{A}$, then, according to Lemma 4.4 *iii*), for any $i = 1, \ldots, t$ there exists $m_i \geq 0$ such that $q_i^{m_i} a = \ell_i(s_1, \ldots, s_{n_i+1})$. Since $(q_1^{m_1}, \ldots, q_t^{m_t})\mathcal{A}^\partial = \mathcal{A}^\partial$ there exist $u_1, \ldots, u_t \in \mathcal{A}^\partial$ such that $\sum u_i q_i^{m_i} = 1$. This yields $a = \sum u_i q_i^{m_i} a = \sum u_i \ell_i(s_1, \ldots, s_{n_i+1})$, and so $\mathcal{A} = \mathcal{T}_{r+1}^\partial$.

Let $\phi : \mathcal{A}^\partial[z_1, \cdots, z_{r+1}] \longrightarrow \mathcal{A}$ be the $\mathcal{A}^\partial$-algebra homomorphism defined by $\phi(z_i) = s_i$. Since $\mathcal{A} = \mathcal{T}_{r+1}^\partial = \mathcal{A}^\partial[s_1, \ldots, s_{r+1}]$ the map $\phi$ in onto. Now consider the ideal $\mathfrak{p} = (c_1 z_2 - h_1(z_1), \ldots, c_r z_{r+1} - h_r(z_1, \ldots, z_r))\mathcal{A}^\partial[z]$ and let $\mathfrak{q} = \mathfrak{p} : c_1^\infty$. We prove in the sequel that $\mathfrak{q}$ is the kernel of $\phi$.

Given $a \in \mathfrak{q}$ there exists a nonnegative integer $v$ such that $c_1^v a \in \mathfrak{p}$. According to Lemma 4.4 *i*) we have $\phi(c_i z_{i+1} - h_i) = 0$, and so $c_1^v \phi(a) = 0$. This gives $\phi(a) = 0$ since $\mathcal{A}$ is a domain. Conversely, let $a \in \mathcal{A}^\partial[z_1, \ldots, z_{r+1}]$ be such that $\phi(a) = 0$. Since $c_r \mid c_1$ we can multiply $a$ by a suitable power $c_1^{v_r}$ and then perform Euclidean division of $c_1^{v_r} a$ by $c_r z_{r+1} - h_r(z_1, \ldots, z_r)$, with respect to $z_{r+1}$ to obtain

$$c_1^{v_r} a = u_r(z)(c_r z_{r+1} - h_r) + a_r(z),$$

where $u_r, a_r \in \mathcal{A}^\partial[z_1, \ldots, z_{r+1}]$ and $a_r$ depends only on $z_1, \ldots, z_r$. Using inductively this process and taking into account the fact that $c_i \mid c_1$ we ultimately get an identity of the form

$$c_1^v a = u_r(z)(c_r z_{r+1} - h_r) + \cdots + u_1(z)(c_1 z_2 - h_1) + a_1(z_1). \tag{1}$$

From Lemma 4.4 *i*) we have $\phi(c_i z_{i+1} - h_i) = 0$, and so by applying $\phi$ to the identity (1) we get $a_1(s_1) = 0$. Since $s_1$ is transcendental over $\mathcal{A}^\partial$ we have $a_1(z_1) = 0$, and so $a \in \mathfrak{q}$.

Let us now prove that $\mathfrak{p} = \mathfrak{q}$. Even if it means localizing and then using Lemma 4.4 *iii*) we may assume without loss of generality that $c = p^r$, where $p$ is prime. On the other hand, since $\mathcal{A}^\partial[z]$ is Noetherian it suffices, according to Lemma 4.5, to show that $\mathfrak{q} \subseteq p\mathcal{A}^\partial[z] + \mathfrak{p} = (p, h_1, \ldots, h_r)\mathcal{A}^\partial[z]$.

First, let us recall the following fact established in the proof of Lemma 4.4 *iii*). Let $b \in \mathcal{A}$ and write $b = b_0 + b_1 s_{r+1} + \cdots + b_m s_{r+1}^m$, with $b_i \in \mathcal{T}_r = \mathcal{A}^\partial[s_1, \ldots, s_r]$, and assume that $p \mid b$ in $\mathcal{A}$. Then $p \mid b_i$ in $\mathcal{T}_r$; i.e., $b_i \in \mathfrak{i}_r$, for any $i = 0, \ldots, m$. Now let $a(z) \in \mathfrak{q}$ and write $a(z) = a_0 + a_1 z_{r+1} + \cdots + a_m z_{r+1}^m$, with $a_i \in \mathcal{A}^\partial[z_1, \ldots, z_r]$. Then $\phi(a) = 0$ and so $p \mid \phi(a)$ in $\mathcal{A}$. It follows that $\phi(a_i) \in \mathfrak{i}_r$ for any $i = 0, \ldots, m$, and so $a_i \in \phi^{-1}(\mathfrak{i}_r) \cap \mathcal{A}^\partial[z_1, \ldots, z_r]$. As established in the proof of Lemma 4.4 *i*), *ii*), $\phi^{-1}(\mathfrak{i}_r) \cap \mathcal{A}^\partial[z_1, \ldots, z_r]$ is nothing but the ideal of $\mathcal{A}^\partial[z_1, \ldots, z_r]$ generated by $p, h_1(z_1), \ldots, h_r(z_1, \ldots, z_r)$. Therefore, $a(z)$ belongs to $p\mathcal{A}^\partial[z] + \mathfrak{p}$.

The canonical decomposition of the homomorphism $\phi$ yields an $\mathcal{A}^\partial$-algebra isomorphism $\psi : \mathcal{A}^\partial[z_1, \ldots, z_{r+1}]/\mathfrak{p} \longrightarrow \mathcal{A}$. On the other hand, consider the Jacobian derivation $\delta = \mathrm{Jac}(c_1 z_2 - h_1, \ldots, c_r z_{r+1} - h_r)$ of $\mathcal{A}^\partial[z_1, \ldots, z_{r+1}]$. An easy induction on $r$ shows that $\delta$ is triangular and so locally nilpotent. Moreover, we have $\delta(c_i z_{i+1} - h_i) = 0$ for $i = 1, \ldots, r$, and so $\mathfrak{p}$ is invariant under $\delta$. If we let $\zeta = \delta_{|\mathfrak{p}}$, then $\partial(\psi(z_i)) = \psi(\zeta(z_i))$ for any $i$. This shows that $\psi$ is a differential algebra isomorphism.

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Laboratoire XLIM, UMR 6172, CNRS-Université de Limoges, Avenue Albert-Thomas 123, 87060, Limoges Cedex, France

*E-mail address*: moulay.barkatou@unilim.fr

Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, P.O. Box 2390, Marrakesh, Morocco

*E-mail address*: elkahoui@ucam.ac.ma