A SHARP REGULARITY RESULT OF SOLUTIONS OF A TRANSMISSION PROBLEM

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Abstract. In this paper we study the regularity of solutions of a transmission problem arising in studying a fiber-reinforced composite media. It is described through a divergence type equation \( \text{div}(A \nabla u) = h \), in an open set \( D \), where \( h \) is a bounded function and \( A \) is a uniformly elliptic matrix, bounded and with piecewise Hölder continuous coefficients. The subdomains \( D_i \) where \( A \) is of class \( C^{\alpha} \) have disjoint closure and are of class \( C^{1,\alpha} \). Exploiting an idea contained in a paper by Li and Vogelius, we obtain the optimal regularity result for solutions, proving that they are of class \( C^{1,\alpha}(\overline{D_i}) \).

1. Introduction

In this paper we show that the solutions of a divergence form elliptic equation with piecewise Hölder continuous coefficients have optimal regularity. The problem arises in studying a fiber-reinforced composite media described as a domain \( D \subset \mathbb{R}^n \) with \( l-1 \) subdomains \( D_m \) representing the fibers. The domains \( D_m \) are compactly embedded in \( D \), with \( C^{1,\alpha} \) boundary and non-intersecting closure. We will also denote \( D_l = D - \bigcup_{m=1}^l D_m \). On this set we study the differential equation

\[
\text{div}(A \nabla u) = h + \text{div}g, \quad \text{in } D,
\]

where \( A \) is a uniformly elliptic matrix and \( A|_{D_m} \in C^{\mu}, g|_{D_m} \in C^{\mu}(D_m), \) and \( h \in L^\infty(D) \).

In [6] and [11] examples of solutions whose gradient becomes unbounded have been provided, and in [1] the speed of blow up of the gradient has been estimated. We would like to cite also [2] and [3], where very fine estimates have been provided for the degenerate conductivity cases.

On the other hand, if the matrix \( A \) is uniformly elliptic, it is well known from the De Giorgi Nash theory (see [4]) that all the solutions belong to \( C^{0,\beta} \), where \( 0 < \beta = \min\{\mu, \alpha\} \). Some higher regularity results have been established using the fact that \( u \) is a weak solution across the boundary. Indeed, under the assumption that \( A \) is diagonal, it has been shown in [5] that the gradient remain bounded even for
circular touching fibers $D_m$. Successively Li Vogelius [9] established a general result for solutions of equations (1.1) for general uniformly elliptic matrix $A$ with piecewise Hölder continuous coefficients. Their proof implies in particular that the solution is of class $C^1_{loc}$, with $\beta = \min\{\mu, \frac{\alpha}{(n+1)n}\}$. We also refer to the paper [8], where a similar result has been proved for systems. These results allow the sub-domain $D_m$ to be tangent, but the exponent of regularity does not seem to be optimal. Recently [10] studied the problem with a completely different technique, based on instruments of complex analysis, and, assuming that the domain $D_m$ do not become tangent, together with slightly more restrictive assumption on the structure of the elliptic equation, obtained $C^{1,\beta}_{loc}$ regularity for the solution of the problem, for all $\beta < \min\{\mu, \alpha\}$. It is then natural to ask if, at least in this non-tangential situation, the optimal regularity $C^{1,\beta}_{loc}$, with $\beta = \min\{\mu, \alpha\}$, can be achieved. This is exactly the scope of this paper. We prove that in the absence of tangency between the boundaries of the $D_i$, slightly modifying the proof of [9], it is possible to obtain the optimal regularity for solutions.

Precisely, we assume that $A$ is a symmetric, positive definite matrix-valued function such that there exist $\tilde{\lambda}, \tilde{\Lambda} > 0$ such that for every $x \in D$ and for every $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$
\tilde{\lambda} \| \xi \|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \tilde{\Lambda} \| \xi \|^2.
$$

Moreover, we set, following the notation in [9]:

(1.3) \quad A(x) = A^{(m)}(x), \quad x \in D_m, \ 1 \leq m \leq l. \text{ Then } A^{(m)}(x) \in C^\alpha(\tilde{D}_m).

(1.4) \quad g^{(m)} \in C^\mu(\tilde{D}_m, \mathbb{R}^n), \quad g(x) = g^{(m)}(x), \quad x \in D_m, \ 1 \leq m \leq l,

(1.5) \quad h \in L^\infty(D),

(1.6) \quad \phi \in C^{1,\mu}(\partial D).

We can now state our main result.

**Theorem 1.1.** Let $A$, $g$ and $h$ satisfy (1.3)-(1.6). Suppose $\varepsilon > 0$. There exists a constant $C = C(D, n, \alpha', \alpha, \varepsilon, \tilde{\lambda}, \tilde{\Lambda}, \|A^{(m)}\|_{C^{\alpha'}(\tilde{D}_m)})$ and the $C^{1,\alpha}$ modulus of $\bigcup_{m=1}^{l} \partial D_m$ such that if $u \in H^1(D)$ is a solution of (1.1) with boundary datum $u = \phi$, then

$$
\max_{1 \leq m \leq M} \|u\|_{C^{1,\beta}(\tilde{D}_m \cap D_m)} \leq C(\|u\|_{L^\infty(D)} + \|h\|_{L^\infty(D)} + \max_{1 \leq m \leq l} \|g^{(m)}(\tilde{D}_m)\|),
$$

where $\beta = \min\{\alpha, \mu\}$ and $D_\varepsilon = \{x \in D : \text{dist}(x, \partial D) > \varepsilon\}$.

2. **Proof of the main theorem**

We first note that since the result is local and the boundary of the set $D_m$ is disjoint, we can fix a point $x \in \partial D_m$ and we can locally perform a change of coordinates $\psi$ of class $C^{1,\alpha}$ in such a way that $\psi(\partial D_m)$ in a neighborhood of the fixed point is a plane. Precisely, we can find a neighborhood $U$ of $x$ such that $U \cap D_j \neq \emptyset$ if and only if $j = m$ or $j = l$, $\psi(U)$ is an $n-$dimensional cube $\Omega$ of size 2 parallel to the axis, and the images of the regions $D_m \cap U$ and $D_l \cap U$ become respectively

$$
\Omega^+ = \{x \in \Omega, \ 0 < x_n\}, \quad \Omega^- = \{x \in \Omega, \ x_n < 0\}.
$$
We represent the set \( \Omega \) as the union of slices
\[ \Omega_m = \{ x \in \Omega, \ c_{m-1} < x_n < c_m \}, \ 1 \leq m \leq l + 1, \]
choosing one of the points \( c_m = 0 \) so that one of the boundaries of \( \Omega_m \) is the image of the boundary of \( D_m \cap U \).

The proof is a modification of the one contained in [9]. Hence we have to locally approximate the problem with a new one, with piecewise constant coefficients.

With \( \bar{A}(\lambda, \Lambda) \) we denote the set of symmetric matrices constant on the regions \( \Omega_j, j = 1, \ldots, l + 1, \) and with smallest and largest eigenvalue, respectively \( \bar{\lambda}, \bar{\Lambda}. \) For every matrix \( \bar{A} \in \bar{A}(\bar{\lambda}, \bar{\Lambda}) \) we will call
\[ \bar{A}_{ij}(x) = \bar{A}_{ij}^m \]
the constant value attained on \( \Omega_m, \ 1 \leq m \leq l + 1. \) Analogously we will indicate \( \bar{G} \) vectors constant on each considered slice \( \Omega_m: \)
\[ \bar{G}(x) = G(m), x \in \Omega_m, \ 1 \leq m \leq l + 1. \]

**Definition 2.1.** For any \( s > 0 \) and any \( p \in (1, \infty) \) we define the norm
\[ \| h \|_{Y^{s,p}} := \sup_{0 < r \leq 1} r^{1-s} \left( \int_{r \Omega} | h |^p \right)^{1/p}. \]

With this notation we recall the following result proved in Proposition 3.2 in [9]. Below we give a version of such result useful for the subsequent proofs of our paper:

**Proposition 2.2.** Suppose \( g = (g_1, \ldots, g_n) \in L^q(\Omega), \ h \in L^{q/2}(\Omega) \) for some \( q > n. \) Let \( \bar{\alpha} \in (0, 1) \) and let \( u \in H^1(\Omega) \) be a solution to
\[ \partial_i (A_{ij}(x) \partial_j u) = h + \partial_i g_i, \text{ in } \Omega, \]
with
\[ \| u \|_{L^\infty(\Omega)} \leq 1. \]

There exist constants \( \sigma \in (0, 1/4), \ \epsilon_0 > 0, \) and \( C > 0, \ C = C(n, q, \bar{\alpha}, \bar{\lambda}, \bar{\Lambda}) \) such that if
\[ \| A - \bar{A} \|_{Y^{1+\alpha,q}} \leq \epsilon_0, \quad \| g - \bar{G} \|_{Y^{1+\alpha,q}} + \| h - \bar{H} \|_{Y^{1+\alpha,q/2}} \leq \epsilon_0, \]
\[ \| \bar{G} \|_{L^\infty(\Omega)} + \| \bar{H} \|_{L^\infty(\Omega)} \leq 1, \]
then we can find a sequence of continuous, piecewise linear functions \( p_k, \)
\[ p_k(x) = a_k^{(m)} + b_k^{(m)} x, \ x \in \Omega_m \cap \left[ -\frac{1}{4}, \frac{1}{4} \right]^n, \]
such that
\[ \| u - p_k \|_{L^\infty(\sigma + \Omega)} \leq \sigma^{k(1+\bar{\alpha})}. \]

The limit \( p(x) = \lim_{k \to \infty} p_k(x) \) exists for \( x \in 1/4 \Omega \) is a continuous piecewise linear function with coefficients that are uniformly bounded by \( C \) and furthermore satisfies
\[ \partial_i (\bar{A}_{ij}(x) \partial_j p) = \partial_i (\bar{G}_i), \text{ in } 1/4 \Omega, \]
and
\[ | u(x) - p(x) | \leq C | x |^{1+\bar{\alpha}}, \text{ in } 1/4 \Omega. \]

Let us now prove the main approximation properties of the matrix \( A \) on the set \( \Omega \) and on a rescaled version of it:
Lemma 2.3. Let $B$ be the ball of radius $r$. Assume that $A \in C^{0,\mu}(B^+)$ and $A \in C^{0,\lambda}(B^-)$. Let us define $\tilde{A}$ as $A^+(0)$ if $x_1 > 0$ and as $A^-(0)$ if $x_1 < 0$. There exists a positive constant $C$ such that

$$\left( \int_B |A - \tilde{A}|^q \, dx \right)^{1/q} \leq Cr^\mu.$$  

Proof. Then

$$\left( \int_B |A - \tilde{A}|^q \, dx \right)^{1/q} = \frac{1}{|B|} \left( \int_{B^+} |A - \tilde{A}|^q \, dx + \int_{B^-} |A - \tilde{A}|^q \, dx \right)^{1/q}.$$  

Thus

$$\left( \int_B |A - \tilde{A}|^q \, dx \right)^{1/q} \leq \frac{1}{|B|} \left( \int_{B^+} |x|^{\mu q} \, dx + \int_{B^-} |x|^{\mu q} \, dx \right)^{1/q} \leq \max\{L_1, L_2\} \frac{1}{|B|} \left( \int_B |x|^{\mu q} \, dx \right)^{1/q} = \max\{L_1, L_2\} \frac{\alpha q-n-1}{\mu q} \left( \int_B x^{\mu q+n-1} \, dx \right)^{1/q} = C(r^{\mu q+n-1})^{1/q} = Cr^\mu.$$  

We define $A_{r_0} = A(r_0 x)$, $\tilde{A}_{r_0}(x) = \tilde{A}(r_0 x)$.  

Lemma 2.4. Suppose that $q > n$ and $0 < \alpha' \leq \min\{\mu, \alpha\}$. Given any $\epsilon_0 > 0$, depending on $n$, $q$, $l$, $\epsilon_0$, $\alpha$, $\alpha'$, $\lambda$, $\Lambda$, there exist constants $C$ and $r_0$ such that if $u \in H^1(\Omega)$ is a solution to

$$\partial_i(A_{ij}(x)\partial_j u) = h + \partial_i g_i, \text{ in } \Omega,$$

with

$$\|u\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)} + \max_{1 \leq m \leq l+1} \|g^{(m)}\|_{C^{\alpha'}}, \quad \frac{\lambda}{\alpha} \bar{\delta}_m \leq 1,$$

then one may find a continuous, piecewise linear function, $p$, whose coefficients are bounded in absolute value by $C$, and which satisfies

$$\partial_i(A_{ij}(x)\partial_j p) = \partial_i g_i, \text{ in } r_0 \Omega,$$

and

$$|u(x) - p(x)| \leq C |x|^{1+\alpha'}, \quad x \in r_0 \Omega.$$  

The constant $C$ and $r_0$ depend on $n$, $\alpha'$, $\alpha$, $q$, $\lambda$, $\Lambda$, the number $l$, $\max_{1 \leq m \leq l+1} \|A^{(m)}\|_{C^{\alpha'}, \bar{\delta}_m}$, and $\max_{1 \leq m \leq l+1} \|f^{(m)}\|_{C^{1, \alpha}[-1,1]^{n-1}}$.  

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1Here $\alpha$ appears. We get the best exponent because the boundary is flat. Indeed, in the paper by Li and Vogelius (see [2]) there is $\frac{\alpha}{(\alpha+1)q}$.  

2Notice that in the following inequality the best exponent is $1 + \min\{\mu, \alpha\}$, while in [9] it is $1 + \min\{\mu, \frac{\alpha}{(\alpha+1)q}\}$.  

Proof. The proof is similar to Proposition 5.3 in [9]. For the reader’s convenience we present the proof of this assertion. We consider $w(x) = u(r_0 x)$, which solves the equation

$$\partial_i((A_{r_0})_{ij} \partial_j w) = r_0^2 h + r_0 \partial_i g_i,$$

in $\Omega$.

Then we choose $r_0$ as prescribed in Lemma 2.4 and obtain the estimate

$$\|A_{r_0} - \bar{A}_{r_0}\|_{Y^{1+\alpha',q}} \leq \epsilon_0,$$

which allows us to apply Proposition 2.2. Applying Lemma 2.4 to the functions $g$ and $\bar{G}$ we may select $r_0$ sufficiently small that

$$\|g(r_0\cdot) - G(r_0\cdot)\|_{Y^{1+\alpha',q}} \leq \frac{1}{\epsilon_0}.$$

By selecting $r_0$ sufficiently small we thus get

$$r_0^2\|h(r_0\cdot)\|_{Y^{1+\alpha',q}} + r_0\|g(r_0\cdot) - \bar{G}(r_0\cdot)\|_{Y^{1+\alpha',q}} \leq r_0^2\|h\|_{L^\infty} + r_0\frac{\epsilon_0}{2} \leq \epsilon_0.$$

We also have

$$r_0\|G(r_0\cdot)\|_{L^\infty(\Omega)} \leq r_0 \max_{1 \leq m \leq l} \|g^{(m)}\|_{C^{\alpha'}(\bar{D}_m)} \leq 1.$$

We can now apply Proposition 2.2 to $u$. This leads to the existence of a continuous, piecewise linear polynomial $q$ whose coefficients are bounded in absolute value by $C$ and which satisfies

$$\partial_i((\bar{A}_{r_0})_{ij} \partial_j q) = r_0^2 h + r_0 \partial_i \bar{G}_i,$$

in $\Omega$.

The function $p(x) = q(x/r_0)$ satisfies all the requirements from the statement of this proposition.

The statement of Proposition 2.5 is analogous to Proposition 5.3 in Li and Vogelius [9], but, under the non-tangency assumption we made here on the boundary, we obtain a Hölder estimate of exponent $\alpha'$ instead of the exponent $\alpha' = \frac{\alpha}{(\alpha+1)q}$ obtained in [9]. Hence at this point the proof of the main theorem, Theorem 1.1, is the same as for Theorem 1.1 in [9] on p. 122.

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