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A SHARP REGULARITY RESULT OF SOLUTIONS OF A TRANSMISSION PROBLEM

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ABSTRACT. In this paper we study the regularity of solutions of a transmission problem arising in studying a fiber-reinforced composite media. It is described through a divergence type equation $\operatorname{div}(A\nabla u) = h$, in an open set D, where h is a bounded function and A is a uniformly elliptic matrix, bounded and with piecewise Hölder continuous coefficients. The subdomains D_i where A is of class C^α have disjoint closure and are of class $C^{1,\alpha}$. Exploiting an idea contained in a paper by Li and Vogelius, we obtain the optimal regularity result for solutions, proving that they are of class $C^{1,\alpha}(\bar{D}_i)$.

1. INTRODUCTION

In this paper we show that the solutions of a divergence form elliptic equation with piecewise Hölder continuous coefficients have optimal regularity. The problem arises in studying a fiber-reinforced composite media described as a domain $D \subset R^n$ with $l-1$ subdomains D_m representing the fibers. The domains D_m are compactly embedded in D, with $C^{1,\alpha}$ boundary and non-intersecting closure. We will also denote $D_l = D - \bigcup_{m=1}^{l} D_m$. On this set we study the differential equation

$$\operatorname{div}(A\nabla u) = h + \operatorname{div} g, \text{ in } D, \tag{1.1}$$

where A is a uniformly elliptic matrix and $A_{|_{\bar{D}_m}} \in C^\mu$, $g_{|_{\bar{D}_m}} \in C^\mu(D_m)$, and $h \in L^\infty(D)$.

In [6] and [11] examples of solutions whose gradient becomes unbounded have been provided, and in [1] the speed of blow up of the gradient has been estimated. We would like to cite also [2] and [3], where very fine estimates have been provided for the degenerate conductivity cases.

On the other hand, if the matrix A is uniformly elliptic, it is well known from the De Giorgi Nash theory (see [7]) that all the solutions belong to $C^{0,\beta}$, where $0 < \beta = \min\{\mu, \alpha\}$. Some higher regularity results have been established using the fact that u is a weak solution across the boundary. Indeed, under the assumption that A is diagonal, it has been shown in [5] that the gradient remain bounded even for

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circular touching fibers D_m. Successively Li Vogelius [9] established a general result for solutions of equations (1.1) for general uniformly elliptic matrix A with piecewise Hölder continuous coefficients. Their proof implies in particular that the solution is of class $C^{1,\beta}_{loc}$, with $\beta = \min\{\mu, \frac{\alpha}{(\alpha+1)n}\}$. We also refer to the paper [8], where a similar result has been proved for systems. These results allow the sub-domain D_m to be tangent, but the exponent of regularity does not seem to be optimal. Recently [10] studied the problem with a completely different technique, based on instruments of complex analysis, and, assuming that the domain D_m do not become tangent, together with slightly more restrictive assumption on the structure of the elliptic equation, obtained $C^{1,\beta}_{loc}$ regularity for the solution of the problem, for all $\beta < \min\{\mu, \alpha\}$. It is then natural to ask if, at least in this non-tangential situation, the optimal regularity $C^{1,\beta}_{loc}$, with $\beta = \min\{\mu, \alpha\}$, can be achieved. This is exactly the scope of this paper. We prove that in the absence of tangency between the boundaries of the D_i, slightly modifying the proof of [9], it is possible to obtain the optimal regularity for solutions.

Precisely, we assume that A is a symmetric, positive definite matrix-valued function such that there exist $\bar\lambda, \bar\Lambda > 0$ such that for every $x \in D$ and for every $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\bar\lambda \mid \xi \mid^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \bar\Lambda \mid \xi \mid^2 . \tag{1.2}$$

Moreover, we set, following the notation in [9]:

$$A(x) = A^{(m)}(x),\ x \in D_m,\ 1 \leq m \leq l. \text{ Then } A^{(m)}(x) \in C^{\mu}(\bar D_m). \tag{1.3}$$

$$g^{(m)} \in C^{\mu}(\bar D_m, \mathbb{R}^n),\ g(x) = g^{(m)}(x),\ x \in D_m,\ 1 \leq m \leq l, \tag{1.4}$$

$$h \in L^{\infty}(D), \tag{1.5}$$

$$\phi \in C^{1,\mu}(\partial D). \tag{1.6}$$

We can now state our main result.

Theorem 1.1. *Let A, g and h satisfy* (1.3)-(1.6). *Suppose $\varepsilon > 0$. There exists a constant $C = C(D, n, \alpha', \alpha, \varepsilon, \bar\lambda, \bar\Lambda, \|A^{(m)}\|_{C^{\alpha'}(\bar D_m)})$ and the $C^{1,\alpha}$ modulus of $\bigcup_{m=1}^{l} \partial D_m$ such that if $u \in H^1(D)$ is a solution of* (1.1) *with boundary datum $u = \phi$, then*

$$\max_{1\leq m\leq M} \|u\|_{C^{1,\beta}(\bar D_m \cap D_\varepsilon)} \leq C(\|u\|_{L^\infty(D)} + \|h\|_{L^\infty(D)} + \max_{1\leq m\leq l} \|g^{(m)}(\bar D_m)\|),$$

where $\beta = min(\alpha, \mu)$ and $D_\varepsilon = \{x \in D :\ dist(x, \partial D) > \varepsilon\}$.

2. Proof of the main theorem

We first note that since the result is local and the boundary of the set D_m is disjoint, we can fix a point $x \in \partial D_m$ and we can locally perform a change of coordinates ψ of class $C^{1,\alpha}$ in such a way that $\psi(\partial D_m)$ in a neighborhood of the fixed point is a plane. Precisely, we can find a neighborhood U of x such that $U \cap D_j \neq \emptyset$ if and only if $j = m$ or $j = l$, $\psi(U)$ is an $n-$dimensional cube Ω of size 2 parallel to the axis, and the images of the regions $D_m \cap U$ and $D_l \cap U$ become respectively

$$\Omega^+ = \{x \in \Omega,\ 0 < x_n\}, \quad \Omega^- = \{x \in \Omega,\ x_n < 0\}.$$

We represent the set Ω as the union of slices

$$\Omega_m = \{x \in \Omega,\ \ c_{m-1} < x_n < c_m\},\ 1 \leq m \leq l+1,$$

choosing one of the points $c_m = 0$ so that one of the boundaries of Ω_m is the image of the boundary of $D_m \cap U$.

The proof is a modification of the one contained in [9]. Hence we have to locally approximate the problem with a new one, with piecewise constant coefficients.

With $\bar{\mathcal{A}}(\bar\lambda, \bar\Lambda)$ we denote the set of symmetric matrices constant on the regions Ω_j, $j = 1, \ldots, l+1$, and with smallest and largest eigenvalue, respectively $\bar\lambda, \bar\Lambda$. For every matrix $\bar A \in \bar{\mathcal{A}}(\bar\lambda, \bar\Lambda)$ we will call

$$\bar A_{i,j}(x) = \bar A^m_{ij}$$

the constant value attained on Ω_m, $1 \leq m \leq l+1$. Analogously we will indicate $\bar G$ vectors constant on each considered slice Ω_m:

$$\bar G(x) = G^{(m)},\ x \in \Omega_m,\ 1 \leq m \leq l+1.$$

Definition 2.1. For any $s > 0$ and any $p \in (1, \infty)$ we define the norm

$$\|h\|_{Y^{s,p}} := \sup_{0<r\leq 1} r^{1-s} \Big(\fint_{r\Omega} \mid h \mid^p \Big)^{1/p}.$$

With this notation we recall the following result proved in Proposition 3.2 in [9]. Below we give a version of such result useful for the subsequent proofs of our paper:

Proposition 2.2. *Suppose $g = (g_1, \ldots, g_n) \in L^q(\Omega)$, $h \in L^{q/2}(\Omega)$ for some $q > n$. Let $\bar\alpha \in (0,1)$ and let $u \in H^1(\Omega)$ be a solution to*

$$\partial_i (A_{ij}(x) \partial_j u) = h + \partial_i g_i,\ \ in\ \Omega,$$

with

$$\|u\|_{L^\infty(\Omega)} \leq 1.$$

There exist constants $\sigma \in (0, \frac{1}{4})$, $\epsilon_0 > 0$, and $C > 0$, $C = C(n, q, \bar\alpha, \bar\lambda, \bar\Lambda)$ such that if

$$\|A - \bar A\|_{Y^{1+\bar\alpha,q}} \leq \epsilon_0, \quad \|g - \bar G\|_{Y^{1+\bar\alpha,q}} + \|h - \bar H\|_{Y^{1+\bar\alpha,q/2}} \leq \epsilon_0,$$
$$\|\bar G\|_{L^\infty(\Omega)} + \|\bar H\|_{L^\infty(\Omega)} \leq 1,$$

then we can find a sequence of continuous, piecewise linear functions p_k,

$$p_k(x) = a^{(m)}_k + b^{(m)}_k x,\ \ x \in \Omega_m \cap [-\frac{1}{4}, \frac{1}{4}]^n,$$

such that

$$\|u - p_k\|_{L^\infty(\sigma^k\Omega)} \leq \sigma^{k(1+\bar\alpha)}.$$

The limit $p(x) = \lim_{k\to\infty} p_k(x)$ exists for $x \in \frac{1}{4}\Omega$ is a continuous piecewise linear function with coefficients that are uniformly bounded by C and furthermore satisfies

$$\partial_i (\bar A_{ij}(x) \partial_j p) = \partial_i (\bar G_i),\ \ in\ \frac{1}{4}\Omega,$$

and

$$\mid u(x) - p(x) \mid \leq C \mid x \mid^{1+\bar\alpha},\ \ in\ \frac{1}{4}\Omega. \tag{2.1}$$

Let us now prove the main approximation properties of the matrix A on the set Ω and on a rescaled version of it:

Lemma 2.3. *Let B be the ball of radius r. Assume that $A \in C^{0,\mu}(B^+)$ and $A \in C^{0,\mu}(B^-)$. Let us define $\bar{A}$ as $A^+(0)$ if $x_n > 0$ and as $A^-(0)$ if $x_n < 0$. There exists a positive constant C such that*

$$(\fint_B \mid A - \bar{A} \mid^q dx)^{1/q} \leq Cr^\mu.$$

Proof. Then

$$(\fint_B \mid A - \bar{A} \mid^q dx)^{1/q} = \frac{1}{\mid B \mid}(\int_{B^+} \mid A - \bar{A} \mid^q dx + \int_{B^-} \mid A - \bar{A} \mid^q dx)^{1/q}.$$

Thus

$$\begin{aligned}
(\fint_B \mid A - \bar{A} \mid^q dx)^{1/q} &\leq \frac{1}{\mid B \mid}(L_1 \int_{B^+} \mid x \mid^{\mu q} dx + L_2 \int_{B^-} \mid x \mid^{\mu q} dx)^{1/q} \\
&\leq \max\{L_1, L_2\} \frac{1}{\mid B \mid}(\int_B \mid x \mid^{\mu q} dx)^{1/q} \\
&= \max\{L_1, L_2\} \frac{\omega_{n-1}}{\mid B \mid}(\int_0^r \rho^{\mu q + n - 1} dx)^{1/q} \\
&= C(r^{\mu q + n - n})^{1/q} = Cr^\mu.
\end{aligned}$$

□

We define $A_{r_0} = A(r_0 x)$, $\bar{A}_{r_0}(x) = \bar{A}(r_0 x)$.

Lemma 2.4. *Suppose that $q > n$ and $0 < \alpha' \leq \min\{\mu, \alpha\}$. Given any $\epsilon_0 > 0$, depending on n, q, l, ϵ_0, α, α', $\bar{\lambda}$, $\bar{\Lambda}$, $\max_{1 \leq m \leq l} \|f_m\|_{C^{1,\alpha}[-1,1]^{n-1}}$, and $\max_{1 \leq m \leq l} \|A^{(m)}\|_{C^{\alpha'}(\bar{\tilde{D}}_m)}$, we have that*

$$(\fint_{r\Omega} \mid A_{r_0}(x) - \bar{A}_{r_0}(x) \mid^q)^{1/q} \leq \epsilon_0 r^\mu \leq \epsilon_0 r^{\alpha'},$$

for every r, $0 < r \leq 1$.

Now combining Proposition 2.2 with Lemma 2.4 we can prove the following.

Proposition 2.5. *Let $A \in \mathcal{A}(\bar{\lambda}, \bar{\Lambda})$ and $\bar{A} \in \bar{\mathcal{A}}(\bar{\lambda}, \bar{\Lambda})$. Assume that $h \in L^\infty(\Omega)$. For any $q > n$, and any $0 < \alpha' \leq \min\{\mu, \alpha\}$ there[1] exist constants C and r_0 such that if $u \in H^1(\Omega)$ is a solution to*

$$\partial_i(A_{ij}(x)\partial_j u) = h + \partial_i g_i, \ \text{in } \Omega,$$

with

$$\|u\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)} + \max_{1 \leq m \leq l+1} \|g^{(m)}\|_{C^{\alpha'} \bar{\tilde{D}}_m} \leq 1,$$

then one may find a continuous, piecewise linear function, p, whose coefficients are bounded in absolute value by C, and which satisfies

$$\partial_i(A_{ij}(x)\partial_j p) = \partial_i G_i, \ \text{in } r_0\Omega,$$

and[2]

$$\mid u(x) - p(x) \mid \leq C \mid x \mid^{1+\alpha'}, \ x \in r_0\Omega.$$

The constant C and r_0 depend on n, α', α, q, $\bar{\lambda}$, $\bar{\Lambda}$, the number l, $\max_{1 \leq m \leq l+1} \|A^{(m)}\|_{C^{\alpha'} \bar{\tilde{D}}_m}$, and $\max_{1 \leq m \leq l+1} \|f^{(m)}\|_{C^{1,\alpha}[-1,1]^{n-1}}$.

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[1] Here α appears. We get the best exponent because the boundary is flat. Indeed, in the paper by Li and Vogelius (see [9]) there is $\frac{\alpha}{(\alpha+1)q}$.

[2] Notice that in the following inequality the best exponent is $1 + \min\{\mu, \alpha\}$, while in [9] it is $1 + \min\{\mu, \frac{\alpha}{(\alpha+1)q}\}$.

Proof. The proof is similar to Proposition 5.3 in [9]. For the reader's convenience we present the proof of this assertion. We consider $w(x) = u(r_0 x)$, which solves the equation

$$\partial_i((A_{r_0})_{ij}\partial_j w) = r_0^2 h + r_0 \partial_i g_i, \text{ in } \Omega.$$

Then we choose r_0 as prescribed in Lemma 2.4 and obtain the estimate

$$||A_{r_0} - \bar{A}_{r_0}||_{Y^{1+\alpha',q}} \leq \epsilon_0,$$

which allows us to apply Proposition 2.2. Applying Lemma 2.4 to the functions g and $\bar{G}$ we may select r_0 sufficiently small that

$$||g(r_0\cdot) - G(r_0\cdot)||_{Y^{1+\alpha',q}} \leq \frac{1}{\epsilon_0}.$$

By selecting r_0 sufficiently small we thus get

$$r_0^2||h(r_0\cdot)||_{Y^{1+\alpha',q/2}} + r_0||g(r_0\cdot) - \bar{G}(r_0\cdot)||_{Y^{1+\alpha',q}} \leq r_0^2||h||_{L^\infty} + r_0\frac{\epsilon_0}{2} \leq \epsilon_0.$$

We also have

$$r_0||G(r_0\cdot)||_{L^\infty(\Omega)} \leq r_0 \max_{1\leq m\leq l} ||g^{(m)}||_{C^{\alpha'}(\bar{D}_m)} \leq 1.$$

We can now apply Proposition 2.2 to u. This leads to the existence of a continuous, piecewise linear polynomial q whose coefficients are bounded in absolute value by C and which satisfies

$$\partial_i((\bar{A}_{r_0})_{ij}\partial_j q) = r_0^2 h + r_0 \partial_i \bar{G}_i, \text{ in } \Omega.$$

The function $p(x) = q(x/r_0)$ satisfies all the requirements from the statement of this proposition. □

The statement of Proposition 2.5 is analogous to Proposition 5.3 in Li and Vogelius [9], but, under the non-tangency assumption we made here on the boundary, we obtain a Hölder estimate of exponent α' instead of the exponent $\alpha' = \frac{\alpha}{(\alpha+1)q}$ obtained in [9]. Hence at this point the proof of the main theorem, Theorem 1.1, is the same as for Theorem 1.1 in [9] on p. 122.

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