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ON THE FUNDAMENTAL UNITS OF A TOTALLY REAL CUBIC ORDER GENERATED BY A UNIT

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Abstract. We give a new and short proof of J. Beers, D. Henshaw, C. McCall, S. Mulay and M. Spindler following a recent result: if ϵ is a totally real cubic algebraic unit, then there exists a unit $\eta \in \mathbf{Z}[\epsilon]$ such that $\{\epsilon, \eta\}$ is a system of fundamental units of the group U_ϵ of the units of the cubic order $\mathbf{Z}[\epsilon]$, except for an infinite family for which ϵ is a square in $\mathbf{Z}[\epsilon]$ and one sporadic exception. Not only is our proof shorter, but it enables us to prove a new result: if the conjugates ϵ' and ϵ'' of ϵ are in $\mathbf{Z}[\epsilon]$, then the subgroup generated by ϵ and ϵ' is of bounded index in U_ϵ, and if $\epsilon > 1 > |\epsilon'| \geq |\epsilon''| > 0$ and if ϵ' and ϵ'' are of opposite sign, then $\{\epsilon, \epsilon'\}$ is a system of fundamental units of U_ϵ.

1. Introduction

Let ϵ be an algebraic unit for which the rank of the group of units U_ϵ of the order $\mathbf{Z}[\epsilon]$ is equal to 1. According to [Nag], [Lou06], [Lou08], [PL] and [Lou10], ϵ is a fundamental unit of $\mathbf{Z}[\epsilon]$, except for some explicit infinite families for which ϵ is a square in $\mathbf{Z}[\epsilon]$ and a short list of sporadic exceptions (as in Theorem 1 below). Now, by [BHMMS], if ϵ is a totally real cubic algebraic unit, then U_ϵ is of rank 2 and there exists a unit $\eta \in \mathbf{Z}[\epsilon]$ such that $\{\epsilon, \eta\}$ is a system of fundamental units of the group U_ϵ of the units of the cubic order $\mathbf{Z}[\epsilon]$, here again, except for an infinite family for which ϵ is a square in $\mathbf{Z}[\epsilon]$ and one sporadic exception. In the present paper, we give a much shorter proof of this result (see Theorem 1). Not only is our proof shorter and self-contained (not based on [Tho] as that of [BHMMS]), but it enables us to obtain new results (Theorem 2).

From now on, let ϵ be **a totally real algebraic cubic unit**. Let $\Pi_\epsilon(X) = X^3 - aX^2 + bX \pm 1 \in \mathbf{Z}[X]$ be its $\mathbf{Q}$-irreducible cubic minimal polynomial. Let $d_\epsilon = (\epsilon-\epsilon')^2(\epsilon-\epsilon'')^2(\epsilon'-\epsilon'')^2 > 0$ be the discriminant of ϵ, i.e. the discriminant of $\Pi_\epsilon(X)$. We may assume that $|\epsilon| \geq |\epsilon'| \geq |\epsilon''| > 0$, where ϵ, ϵ' and ϵ'' are the three conjugates of ϵ, i.e. the three real roots of $\Pi_\epsilon(X)$. By changing ϵ into $1/\epsilon''$, we may assume that $|\epsilon| > 1 > |\epsilon'| \geq |\epsilon''| > 0$ (use $\epsilon\epsilon'\epsilon'' = \pm 1$). Finally, by changing ϵ into

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$-\epsilon$, we may and **will assume** that

$$\epsilon > 1 > |\epsilon'| \geq |\epsilon''| > 0 \tag{1}$$

(which implies $\Pi_\epsilon(1) = 1 - a + b \pm 1 < 0$).

Hence, we can and will write

$$|\epsilon'| = t/\sqrt{\epsilon} \text{ and } |\epsilon''| = 1/(t\sqrt{\epsilon}), \text{ with } 1 \leq t \leq \sqrt{\epsilon}. \tag{2}$$

We say that ϵ is **of type** $(+)$ if ϵ' and ϵ'' are both positive, **of type** $(-)$ if ϵ' and ϵ'' are both negative and **of type** $(\pm)$ if ϵ' and ϵ'' are of opposite sign.

Theorem 1. *Let ϵ be a totally real algebraic cubic unit satisfying* (1). *If $\Pi_\epsilon(X) \neq X^3 - 6X^2 + 5X - 1$ and $\Pi_\epsilon(X) \neq X^3 - B^2X^2 + 2BX - 1$, $B \geq 3$ (in which cases $\epsilon = (\epsilon^2 - B^2\epsilon + B)^2$ is a square in $\mathbf{Z}[\epsilon]$), then there exists a unit $\eta \in \mathbf{Z}[\epsilon]$ such that $\{\epsilon, \eta\}$ is a system of fundamental units of the group U_ϵ of the units of the cubic order $\mathbf{Z}[\epsilon]$.*

Theorem 2. *Let $\epsilon > 1$ range over the totally real cubic algebraic units satisfying* (1) *for which the cubic number field $\mathbf{Q}(\epsilon)$ is normal (i.e. for which d_ϵ is a square) and $\epsilon' \in \mathbf{Z}[\epsilon]$ (see Proposition* 10 *below). Then, the group of units generated by ϵ and ϵ' is of bounded index in the group U_ϵ of the units of the cubic order $\mathbf{Z}[\epsilon]$. More precisely, if $\epsilon \geq 729$, then this index is equal to* 1 *or* 3. *Moreover, if ϵ is of type $(\pm)$, then, $\{\epsilon, \epsilon'\}$ is a system of fundamental units of the cubic order $\mathbf{Z}[\epsilon]$.*

For example, if $\Pi_\epsilon(X) = X^3 - (m+3)X^2 + mX + 1$, $m \geq -1$ (the so-called simplest cubic fields), then $d_\epsilon = (m^2 + 3m + 9)^2$ is a perfect square, ϵ is of type $(\pm)$ (see Lemma 3) and $\epsilon' = 1 - 1/\epsilon = \epsilon^2 - (m+3)\epsilon + m + 1 \in \mathbf{Z}[\epsilon]$ (see the proof of Proposition 10). Hence, $\{\epsilon, \epsilon'\}$ is a system of fundamental units of the cubic order $\mathbf{Z}[\epsilon]$ (see also [Tho, Theorem (3.10)] and [Lou02] for a different proof).

2. Proof of Theorem 1

2.1. Useful results.

Lemma 3. *$\Pi_\epsilon(X) = X^3 - aX^2 + bX - c \in \mathbf{Z}[X]$ with $c \in \{\pm 1\}$ is $\mathbf{Q}$-irreducible with three real roots that can be sorted to satisfy* (1) *if and only if (i) its discriminant $d_\epsilon = -4ca^3 - 4b^3 + a^2b^2 + 18cab - 27$ is positive, (ii) $\Pi_\epsilon(-1) < 0$, and (iii) $\Pi_\epsilon(1) < 0$ (and (ii) and (iii) are equivalent to $-a - c \leq b \leq a + c - 2$, and imply $a \geq 0$). In that case, ϵ is of type $(\pm)$ if and only if $c = -1$, of type $(+)$ if and only if $c = 1$ and $b \geq 1$, and of type $(-)$ if and only if $c = 1$ and $b \leq -2$.*

Proof. For the last assertion, use $a = \epsilon + \epsilon' + \epsilon''$ and $b = \epsilon(\epsilon' + \epsilon'') + \epsilon'\epsilon''$. □

Lemma 4. *Let $\epsilon > 1$ be a totally real cubic algebraic unit satisfying* (1). *If $\epsilon \leq 4$, then we are in one of the following six cases:*

$\Pi_\epsilon(X)$	d_ϵ	ϵ	*Type*
$X^3 - 2X^2 - X + 1$	49	$2.24697\cdots$	$(\pm)$
$X^3 - 3X^2 + 1$	81	$2.87938\cdots$	$(\pm)$
$X^3 - 3X^2 - X + 1$	148	$3.21431\cdots$	$(\pm)$
$X^3 - 3X^2 - 2X + 1$	257	$3.49086\cdots$	$(\pm)$
$X^3 - 4X^2 + X + 1$	169	$3.65109\cdots$	$(\pm)$
$X^3 - 4X^2 + 1$	229	$3.93543\cdots$	$(\pm)$

Proof. We have $-1 < \epsilon - 2 < a = \epsilon + \epsilon' + \epsilon'' \leq \epsilon + (t+1/t)/\sqrt{\epsilon}$ and $|b| = |\epsilon\epsilon' + \epsilon\epsilon'' + \epsilon'\epsilon''| \leq (t+1/t)\sqrt{\epsilon} + 1/\epsilon$, by (2). Hence, $0 \leq a, |b| \leq \epsilon + 1 + 1/\epsilon$. If $1 < \epsilon \leq 4$, then $0 \leq a, |b| \leq \epsilon + 1 + 1/\epsilon < 6$, and the assertion can be checked by listing all the possible polynomials $\Pi_\epsilon(X)$ in this range. □

Lemma 5. *Let ϵ be a totally real algebraic cubic unit satisfying* (1). *Then $\epsilon = \eta^2$ for some $\eta \in \mathbf{Z}[\epsilon]$ if and only if either $\Pi_\epsilon(X) = X^3 - 6X^2 + 5X - 1$, in which case ϵ is of type $(+)$, $d_\epsilon = 49$ and $\epsilon = (\epsilon^2 - 5\epsilon + 2)^2$, or $\Pi_\epsilon(X) = X^3 - B^2X^2 + 2BX - 1$ with $B \geq 3$, in which case ϵ is of type $(+)$, $d_\epsilon = 4B^3 - 27$ is a square if and only if $B = 3$ and $\epsilon = (\epsilon^2 - B^2\epsilon + B)^2$.*

Proof. Since $\epsilon = \eta^2$ and $\eta \in \mathbf{Z}[\epsilon]$, we have $\mathbf{Z}[\eta^2] \subseteq \mathbf{Z}[\eta] \subseteq \mathbf{Z}[\epsilon] = \mathbf{Z}[\eta^2]$. Hence, $\mathbf{Z}[\eta] = \mathbf{Z}[\epsilon]$ and $d_\eta = d_\epsilon$. By changing η into $-\eta$, we may assume that $\Pi_\eta(X) = X^3 - aX^2 + bX - 1 \in \mathbf{Z}[X]$. Then, $\Pi_\epsilon(X) = X^3 - (a^2 - 2b)X^2 + (b^2 - 2a)X - 1$ and $1 = (\mathbf{Z}[\eta] : \mathbf{Z}[\eta^2]) = |ab - 1|$. Hence, either $ab = 2$ or $ab = 0$.

If $ab = 2$, then $\Pi_\eta(X) = X^3 - X^2 + 2X - 1$ and $d_\eta = -23$, a contradiction; or $\Pi_\eta(X) = X^3 - 2X^2 + X - 1$ and $d_\eta = -23$, a contradiction; or $\Pi_\eta(X) = X^3 + X^2 - 2X - 1$, $d_\eta = 49$, $\Pi_\epsilon(X) = X^3 - 5X^2 + 6X - 1$ and (1) is not satisfied (for $\Pi_\epsilon(1) = 1 > 0$); or $\Pi_\eta(X) = X^3 + 2X^2 - X - 1$, $\Pi_\epsilon(X) = X^3 - 6X^2 + 5X - 1$ and $d_\epsilon = 49$.

If $ab = 0$, then $\Pi_\eta(X) = X^3 - aX^2 - 1$, $d_\eta = -4a^3 - 27 > 0$, i.e. $a = -B$ with $B \geq 3$, and $\Pi_\epsilon(X) = X^3 - B^2X^2 + 2BX - 1$; or $\Pi_\eta(X) = X^3 + bX - 1$, $d_\eta = -4b^3 - 27 > 0$, i.e. $b = -B$ with $B \geq 3$, and $\Pi_\epsilon(X) = X^3 - 2BX^2 + B^2X - 1$; in that case, (1) is not satisfied (for $\Pi_\epsilon(1) = B^2 - 2B > 0$).

For the last assertion, it is known that $(x, y) = (3, \pm 9)$ are the only integral points on the elliptic curve $y^2 = 4x^3 - 27$. □

Lemma 6. *Let ϵ be a totally real algebraic cubic unit satisfying* (1). *Assume that $\mathbf{Q}(\epsilon)$ is a normal cubic number field, i.e. that d_ϵ is a square, that $\epsilon' \in \mathbf{Z}[\epsilon]$, that $\Pi_\epsilon(X) \neq X^3 - 6X^2 + 5X - 1$ and that $\Pi_\epsilon(X) \neq X^3 - 9X^2 + 6X - 1$. Then, the subgroup $\langle -1, \epsilon, \epsilon' \rangle$ generated by -1, ϵ and ϵ' is of odd index in U_ϵ.*

Proof. If this index were even, there would exist $\eta \in U_\epsilon \setminus \langle -1, \epsilon, \epsilon' \rangle$ such that $\eta^2 = \epsilon^a |\epsilon|'^b \in \langle -1, \epsilon, \epsilon' \rangle$. Hence, a or b is odd and ϵ, ϵ' or $\epsilon'' = \pm 1/(\epsilon\epsilon')$ would be a square in U_ϵ; hence ϵ would be a square in U_ϵ. The desired result follows from Lemma 5. □

2.2. **First case: ϵ is of type $(\pm)$.** We assume that ϵ' and ϵ'' are of opposite sign. Then, $(\epsilon' - \epsilon'')^2 = (|\epsilon'| + |\epsilon''|)^2 \geq 4|\epsilon'\epsilon''| = 4/\epsilon$ and $d_\epsilon \geq 4(\epsilon - 1)^2\epsilon^2/\epsilon = 4(\epsilon - 1)^2\epsilon$.

Lemma 7. *Assume that $\epsilon > 1 > |\epsilon'| \geq |\epsilon''| > 0$ and that ϵ' and ϵ'' are of opposite sign. Then,*

$$4(\epsilon - 1)^2\epsilon \leq d_\epsilon \leq 4(\epsilon + 1)^2\epsilon^2. \tag{3}$$

Moreover, there exists a unit $\eta \in \mathbf{Z}[\epsilon]$ such that $\{\epsilon, \eta\}$ is a system of fundamental units of the group U_ϵ of the units of the cubic order $\mathbf{Z}[\epsilon]$.

Proof. To begin with, we cannot extract proper roots of $\pm\epsilon$ in $\mathbf{Z}[\epsilon]$: if $\epsilon = \pm\eta^n$ for some unit $\eta \in \mathbf{Z}[\epsilon]$, then $n \in \{\pm 1\}$. Indeed, we may assume that $n \geq 1$ and that $\epsilon = \eta^n$. Hence, η is also of type $(\pm)$ and n is odd. Since $\epsilon = \eta^n$ and $\eta \in \mathbf{Z}[\epsilon]$, then

$\mathbf{Z}[\epsilon] = \mathbf{Z}[\eta^n] \subseteq \mathbf{Z}[\eta] \subseteq \mathbf{Z}[\epsilon]$; hence $\mathbf{Z}[\epsilon] = \mathbf{Z}[\eta]$ and $d_\eta = d_\epsilon = d_{\eta^n}$. Now, $n \geq 3$ and (3) yield

$$4(\eta^3-1)^2\eta^3 \leq 4(\eta^n-1)^2\eta^n = 4(\epsilon-1)^2\epsilon \leq d_\epsilon = d_{\eta^n} = d_\eta \leq 4(\eta+1)^2\eta^2,$$

which leads to the following contradiction (use $\eta > 2$): $(\eta^3-1)^2\eta \leq (\eta+1)^2$. Now that we know that we cannot extract proper roots of ϵ in $\mathbf{Z}[\epsilon]$, let us prove that there exists a unit $\eta \in \mathbf{Z}[\epsilon]$ such that $\{\epsilon, \eta\}$ is a system of fundamental units of U_ϵ. Indeed, let $\{\eta_1, \eta_2\}$ be a system of fundamental units of U_ϵ. We may assume that $\eta_1 > 0$ and $\eta_2 > 0$. Write $\epsilon = \eta_1^a\eta_2^b$, $a, b \in \mathbf{Z}$. Then, $\gcd(a, b) = 1$. Let $u, v \in \mathbf{Z}$ be such that $au - bv = 1$. Set $\eta = \eta_1^v\eta_2^u$. Then, $\eta_1 = \epsilon^u\eta^{-b}$ and $\eta_2 = \epsilon^{-v}\eta^a$. Hence, $\{\epsilon, \eta\}$ is a system of fundamental units of U_ϵ. □

2.3. **Second case: ϵ is of type $(-)$.** We assume that ϵ' and ϵ'' are both negative, i.e. that $\epsilon > 1 > 0 > \epsilon'' > \epsilon' > -1$. Then, $d_\epsilon = ((\epsilon+|\epsilon'|)(\epsilon+|\epsilon''|)(|\epsilon'|-|\epsilon''|))^2 \leq ((\epsilon+1)(\epsilon+|\epsilon''|)(1-|\epsilon''|))^2 \leq ((\epsilon+1)\epsilon)^2$.

Lemma 8. *Assume that $\epsilon > 1 > 0 > \epsilon'' > \epsilon' > -1$. Either $\Pi_\epsilon(X) = X^3 - B(B+1)X^2 - (2B+1)X - 1$, $B \geq 4$, and $d_\epsilon = (B^2-B-3)^2 - 32$ is asymptotic to ϵ^2, d_B is a square if and only if $B = 4$ (in which case $d_B = 49$) and*

$$\epsilon^2/9 \leq d_\epsilon \leq \epsilon^2 \tag{4}$$

or

$$\epsilon^2 \leq d_\epsilon \leq (\epsilon+1)^2\epsilon^2. \tag{5}$$

Moreover, there exists a unit $\eta \in \mathbf{Z}[\epsilon]$ such that $\{\epsilon, \eta\}$ is a system of fundamental units of the group U_ϵ of the units of the cubic order $\mathbf{Z}[\epsilon]$.

Proof. Write $\epsilon' = -t/\sqrt{\epsilon}$ and $\epsilon'' = -1/(t\sqrt{\epsilon})$ for some $t > 1$. Then,

$$d_\epsilon \geq (t-1/t)^2\epsilon^3,$$

$a = \epsilon + \epsilon' + \epsilon'' = \epsilon - (t+1/t)/\sqrt{\epsilon}$ and $b = \epsilon\epsilon' + \epsilon\epsilon'' + \epsilon'\epsilon'' = -(t+1/t)\sqrt{\epsilon} + 1/\epsilon$. Hence,

$$4a - b^2 = -(t-1/t)^2\epsilon - 2(t+1/t)/\sqrt{\epsilon} - 1/\epsilon^2 < 0.$$

Assume that $t - 1/t \geq 1/\sqrt{\epsilon}$. Then,

$$d_\epsilon \geq \epsilon^2.$$

Now, assume that $t - 1/t < 1/\sqrt{\epsilon}$. Then, $t + 1/t = \sqrt{(t-1/t)^2+4} < \sqrt{4+1/\epsilon}$ and

$$-1 - \frac{2}{\sqrt{\epsilon}}\sqrt{4+1/\epsilon} - \frac{1}{\epsilon^2} < 4a - b^2 = -(t-1/t)^2\epsilon - 2(t+1/t)/\sqrt{\epsilon} - 1/\epsilon^2 < 0.$$

Using $\epsilon > 3$, we obtain $-4 < 4a - b^2 < 0$; hence $4a - b^2 = -1$. Therefore, $b = -(2B+1)$, $B \geq 0$, $a = B(B+1)$ and $\Pi_\epsilon(X) = X^3 - B(B+1)X^2 - (2B+1)X - 1$, for which $d_\epsilon = B^4 - 2B^3 - 5B^2 + 6B - 23 > 0$ if and only if $B \geq 4$. Now, $\Pi_\epsilon(B^2+B) = -2B^3 - 3B^2 - B - 1 < 0$ and $\Pi_\epsilon(B^2+B+1) = B^4 - B - 1 > 0$ yield $B^2 + B < \epsilon < B^2 + B + 1$ and $1 > d_\epsilon/\epsilon^2 \geq d_\epsilon/(B^2+B+1)^2 \geq 1/9$.

As in the first case, if $1 < \epsilon = \eta^n$ for some unit $1 < \eta \in \mathbf{Z}[\epsilon]$, and some $n \geq 1$, then n is odd and η is of type $(-)$. It follows from (4) and (5) that $n \geq 3$ yields

$$\eta^6/9 \leq \eta^{2n}/9 = \epsilon^2/9 \leq d_\epsilon = d_{\eta^n} = d_\eta \leq (\eta+1)^2\eta^2$$

and leads to the following contradiction (use $\eta \geq 4$): $\eta^2 \leq 3(\eta+1)$. □

2.4. **Third case: ϵ is of type $(+)$.** We assume that ϵ' and ϵ'' are both positive, i.e. that ϵ is totally positive and that $\epsilon > 1 > \epsilon' > \epsilon'' > 0$. Then, $d_\epsilon = (\epsilon-\epsilon')^2(\epsilon-\epsilon'')^2(\epsilon'-\epsilon'')^2 \leq (\epsilon-\epsilon')^2\epsilon^2\epsilon'^2 \leq (\epsilon-1)^2\epsilon^2$.

Lemma 9. *Assume that $\epsilon > 1 > \epsilon' > \epsilon'' > 0$. Either $\Pi_\epsilon(X) = X^3 - B^2X^2 + 2BX - 1$, $B \geq 3$, and $d_\epsilon = 4B^3 - 27 \leq (\epsilon-1)^2\epsilon^2$ is asymptotic to $4\epsilon^{3/2}$ and d_B is a square if and only if $B = 3$ (in which case $d_B = 81$) or*

$$(\epsilon-1)^4/\epsilon^2 \leq d_\epsilon \leq (\epsilon-1)^2\epsilon^2; \tag{6}$$

and if $\Pi_\epsilon(X) \neq X^3 - 6X^2 + 5X - 1$, there exists a unit $\eta \in \mathbf{Z}[\epsilon]$ such that $\{\epsilon, \eta\}$ is a system of fundamental units of the group U_ϵ of the units of the cubic order $\mathbf{Z}[\epsilon]$.

Proof. Write $\epsilon' = t/\sqrt{\epsilon}$ and $\epsilon'' = 1/(t\sqrt{\epsilon})$ for some $t > 1$. We have

$$d_\epsilon \geq (\epsilon-1)^4(t-1/t)^2/\epsilon,$$

and as in the second case,

$$-(t-1/t)^2\epsilon < 4a - b^2 = -(t-1/t)^2\epsilon + 2(t+1/t)/\sqrt{\epsilon} - 1/\epsilon^2 < 2(t+1/t)/\sqrt{\epsilon}.$$

Assume that $t - 1/t > 1/\sqrt{\epsilon}$. Then,

$$d_\epsilon \geq (\epsilon-1)^4/\epsilon^2.$$

Now, assume that $t - 1/t \leq 1/\sqrt{\epsilon}$. Then, $-1 < 4a - b^2 < 2\sqrt{4+1/\epsilon}/\sqrt{\epsilon} < 3$, since $\epsilon > 2$ (by Lemma 4). Hence, $4a = b^2$, i.e. $b = 2B$ and $a = B^2$. Therefore, $\Pi_\epsilon(X) = X^3 - B^2X^2 + 2BX - 1$, $d_\epsilon = 4B^3 - 27 > 0$, i.e. $B \geq 3$, $\Pi_\epsilon(B^2) = 2B^3 - 1 > 0$ and $\Pi_\epsilon(B^2-1) = -(B^2-1)(B^2-2B-1) - 1 < 0$. Hence, $B^2 - 1 < \epsilon < B^2$ and d_ϵ is asymptotic to $4\epsilon^{3/2}$.

As in the first case, if $1 < \epsilon = \eta^n$ for some unit $1 < \eta \in \mathbf{Z}[\epsilon]$ of type $(+)$ and some $n \geq 2$, then (6) yields

$$(\eta-\eta^{-1})^4 \leq (\eta^{n/2}-\eta^{-n/2})^4 = (\epsilon-1)^4/\epsilon^2 \leq d_\epsilon = d_{\eta^n} = d_\eta \leq (\eta-1)^2\eta^2,$$

which leads to the following contradiction (use $\eta \geq 2$): $(\eta-\eta^{-1})^2 \leq (\eta-1)\eta$.

In the same way, if $1 < \epsilon = \eta^n$ for some unit $1 < \eta \in \mathbf{Z}[\epsilon]$ of type $(-)$ and some $n \geq 2$, then n is even, and (4), (5) and $n \geq 4$ yield

$$\eta^8/9 \leq \epsilon^2/9 \leq d_\epsilon = d_{\eta^n} = d_\eta \leq (\eta+1)^2\eta^2,$$

which leads to the following contradiction (use $\eta > 3$): $\eta^3 \leq 3(\eta+1)$. Hence, we must have $\epsilon = \eta^2$ and $\Pi_\epsilon(X) = X^3 - 6X^2 + 5X - 1$, by Lemma 5.

Finally, if $1 < \epsilon = \eta^n$ for some unit $1 < \eta \in \mathbf{Z}[\epsilon]$ of type $(\pm)$ and some $n \geq 1$, then $n \geq 2$ is even. Moreover, (3) and $n \geq 4$ yield

$$(\eta^2-\eta^{-2})^4 \leq (\epsilon-1)^4/\epsilon^2 \leq d_\epsilon = d_{\eta^n} = d_\eta \leq 4(\eta+1)^2\eta^2,$$

and the following contradiction (use $\eta \geq 2$): $(\eta^2-\eta^{-2})^2 \leq 2(\eta+1)\eta$. Hence, we must have $\epsilon = \eta^2$ and $\Pi_\epsilon(X) = X^3 - 6X^2 + 5X - 1$, by Lemma 5. □

3. Proof of Theorem 2

From now on, **we assume** that the cubic extension $\mathbf{Q}(\epsilon)/\mathbf{Q}$ is normal, i.e. that d_ϵ is a square. Let σ be a generator of its Galois group. We may assume that $\epsilon' = \sigma(\epsilon)$ and $\sigma^2(\epsilon) = \epsilon''$. Then ϵ and ϵ' are multiplicatively independent (if $\epsilon^a\epsilon'^b = 1$ with $\gcd(a,b) = 1$, then $\epsilon^{-b}\epsilon'^{a-b} = \epsilon'^a(\epsilon\epsilon')^{-b} = \epsilon'^a\epsilon''^b = \sigma(\epsilon^a\epsilon'^b) = \sigma(1) = 1$, hence $\epsilon^{a(a-b)+b^2} = 1$, thus $a(b-a) = b^2$, hence $a = \pm 1$ and $b^2 = \pm b - 1$, which is impossible). We may ask whether $\{\epsilon, \epsilon'\}$ is a system of fundamental units of the cubic order $\mathbf{Z}[\epsilon]$, provided that $\epsilon' \in \mathbf{Z}[\epsilon]$. Let $\{\epsilon_1, \epsilon_2\}$ be a system of independent units of $\mathbf{Z}[\epsilon]$. Let $\{\eta_1, \eta_2\}$ be a system of fundamental units of this order. Set

$$\mathrm{Reg}(\epsilon_1, \epsilon_2) := \left|\det \begin{pmatrix} \log|\epsilon_1| & \log|\epsilon_2| \\ \log|\epsilon_1'| & \log|\epsilon_2'| \end{pmatrix}\right|.$$

The index of the subgroup $\langle -1, \epsilon_1, \epsilon_2\rangle$ generated by -1, ϵ_1 and ϵ_2 in the group U_ϵ of the units of $\mathbf{Z}[\epsilon]$ generated by -1, η_1 and η_2 is equal to $\mathrm{Reg}(\epsilon_1, \epsilon_2)/\mathrm{Reg}(\eta_1, \eta_2)$. In particular, $\mathrm{Reg}(\mathbf{Z}[\epsilon]) = \mathrm{Reg}(\eta_1, \eta_2)$ does not depend on the set $\{\eta_1, \eta_2\}$ of fundamental units of $\mathbf{Z}[\epsilon]$. By [Cus, Theorem 1], it follows that

$$\mathrm{Reg}(\mathbf{Z}[\epsilon]) \geq \frac{1}{16}\log^2(d_\epsilon/4). \tag{7}$$

Now, we have (recall that $|\epsilon'| = t/\sqrt{\epsilon}$ and $|\epsilon''| = 1/(t\sqrt{\epsilon})$ with $1 < t \leq \sqrt{\epsilon}$):

$$\mathrm{Reg}(\epsilon, \epsilon') = \left|\det \begin{pmatrix} \log|\epsilon| & \log|\epsilon'| \\ \log|\epsilon'| & \log|\epsilon''| \end{pmatrix}\right| = \frac{3}{4}(\log\epsilon)^2 + \log^2 t \leq (\log\epsilon)^2. \tag{8}$$

By (3), (4), (5), (6) and Lemma 5, we obtain

$$\mathrm{Reg}(\epsilon, \epsilon') \leq (1/4 + o(1))\log^2 d_\epsilon$$

and

$$\mathrm{Reg}(\epsilon, \epsilon')/\mathrm{Reg}(\mathbf{Z}[\epsilon]) \leq 4 + o(1),$$

which together with Lemma 6 proves Theorem 2. In fact, if $\Pi_\epsilon(X) \neq X^3 - 6X^2 + 5X - 1$ (for which $\epsilon = 5.04891\cdots$), $\Pi_\epsilon(X) \neq X^3 - 9X^2 + 6X - 1$ (for which $\epsilon = 8.29085\cdots$), $\Pi_\epsilon(X) \neq X^3 - 20X^2 - 9X - 1$ (for which $\epsilon = 20.44264\cdots$) and $\mathbf{Q}(\epsilon)$ is normal, then $d_\epsilon \geq \min(4(\epsilon-1)^2\epsilon, \epsilon^2, (\epsilon-1)^4/\epsilon^2) = (\epsilon-1)^4/\epsilon^2$, by Lemmas 7, 8 and 9. Hence, by (7) and (8), for $\epsilon \geq 729$ we have

$$\mathrm{Reg}(\epsilon, \epsilon')/\mathrm{Reg}(\mathbf{Z}[\epsilon]) \leq \left(2\frac{\log\epsilon}{\log((\epsilon-1)^2/(2\epsilon))}\right)^2 < 5.$$

If $\epsilon > 1$ is of type $(\pm)$, then $d_\epsilon/4 \geq (\epsilon-1)^2\epsilon$, by (3). Hence, for $\epsilon > 3.1$ we have

$$\mathrm{Reg}(\epsilon, \epsilon')/\mathrm{Reg}(\mathbf{Z}[\epsilon]) \leq \left(\frac{4\log\epsilon}{\log((\epsilon-1)^2\epsilon)}\right)^2 < 3.$$

Finally, for the two cases of type $(\pm)$ with $1 < \epsilon \leq 3.1$ (Lemma 4), we also have

$$\mathrm{Reg}(\epsilon, \epsilon')/\mathrm{Reg}(\mathbf{Z}[\epsilon]) \leq 4\frac{3\log^2\epsilon + 4\log^2 t}{\log^2(d_\epsilon/4)} = 4\frac{3\log^2\epsilon + \log^2(\epsilon|\epsilon'|^2)}{\log^2(d_\epsilon/4)} < 3,$$

which completes the proof of Theorem 2.

3.1. **Complements.**

Proposition 10. *Assume that $\Pi(X) = X^3 - aX^2 + bX - c \in \mathbf{Z}[X]$ is $\mathbf{Q}$-irreducible and of discriminant $\Delta = -4a^3c - 4b^3 + a^2b^2 + 18cab - 27c^2 = D^2$, a square. Let α, α' and α'' be the three real roots of $\Pi(X)$. Then α' is in $\mathbf{Z}[\alpha]$ if and only if $2D$ divides $2a^2 - 6b$, $2a^3 - 7ab + 9c + D$ and $a^2b + 3ac - 4b^2 + Da$.*

Proof. Since the characterization does not depend on the sign of D, we may assume that the sign of D is such that $D = (\alpha-\alpha')(\alpha-\alpha'')(\alpha'-\alpha'')$. We have $\alpha'+\alpha'' = a-\alpha$ and $\alpha' - \alpha'' = D/\Pi'(\alpha) = D/(3\alpha^2 - 2a\alpha + b)$. Since

$$\Delta/\Pi'(\alpha) = (2a^2 - 6b)\alpha^2 - (2a^3 - 7ab + 9c)\alpha + a^2b + 3ac - 4b^2$$

(check that $\Pi(X)$ divides $\Pi'(X)((2a^2-6b)X^2-(2a^3-7ab+9c)X+a^2b+3ac-4b^2)-\Delta$, we obtain

$$\alpha' = \frac{(2a^2-6b)\alpha^2-(2a^3-7ab+9c+D)\alpha+(a^2b+3ac-4b^2+Da)}{2D}$$

and

$$\alpha'' = \frac{-(2a^2-6b)\alpha^2+(2a^3-7ab+9c-D)\alpha-(a^2b+3ac-4b^2-Da)}{2D}.$$

The desired result follows. □

If we assume that $\Pi(X)$ is as in Lemma 3, then $\alpha > 1 > |\alpha'| \geq |\alpha''| > 0$ and the sign of D is the same as the sign of α', hence the same as the sign of $a-\alpha = \alpha'+\alpha''$. Hence $\alpha' > 0$ if and only if $a > \alpha$, i.e. if and only if $\Pi(a) = ab-c > 0$. In conclusion, the sign of D is the same as that of $ab-c$.

4. Conclusion

It remains to decide whether it is possible for the index given in Theorem 2 to be equal to 3.

Lemma 11. *Assume that ϵ is a totally real cubic algebraic unit, that d_ϵ is a perfect square and that $\epsilon' \in \mathbf{Z}[\epsilon]$. Then, ϵ is never a cube in $\mathbf{Z}[\epsilon]$. Moreover,* 3 *divides the index $(U_\epsilon : \langle -1, \epsilon, \epsilon' \rangle)$ if and only if $\eta := \epsilon/\epsilon'$ is a cube in $\mathbf{Z}[\epsilon]$.*

Proof. Assume that $\epsilon = \eta^3$ with $\eta \in \mathbf{Z}[\epsilon]$. Then $\mathbf{Z}[\epsilon] = \mathbf{Z}[\eta]$. We may assume that $\Pi_\eta(X) = X^3 - AX^2 + BX - 1 \in \mathbf{Z}[X]$. Then, $(\mathbf{Z}[\eta] : \mathbf{Z}[\eta^3]) = |A^3+B^3-A^2B^2|$. Hence, $A^3+B^3-A^2B^2 = \pm 1$ and $0 < d_\eta = -4A^3-4B^3+A^2B^2+18AB-27 = -3A^2B^2+18AB-27 \mp 4$. Hence, we must have $\pm = -$ and $AB = 2$, 3 or 4, which together with $A^3+B^3-A^2B^2 = -1$ leads to a contradiction. Now, as in the proof of Lemma 6, 3 divides the index $(U_\epsilon : \langle -1, \epsilon, \epsilon' \rangle)$ if and only if ϵ or ϵ/ϵ' is a cube in $\mathbf{Z}[\epsilon]$. The desired result follows. Notice that $\Pi_\epsilon(X) = \Pi_{\eta^3}(X) = X^3 - (A^3-3AB+3)X^2 + (B^3-3AB+3) - 1$. □

Now, at least for a given $\Pi_\epsilon(X)$, we can check whether $\epsilon/\epsilon' = \eta^3$ is a cube or not in $\mathbf{Z}[\epsilon]$ by computing numerical approximations to ϵ, ϵ' and ϵ'' and then a numerical approximation to $(\epsilon/\epsilon')^{1/3} + (\epsilon'/\epsilon'')^{1/3} + (\epsilon''/\epsilon)^{1/3}$ which, being equal to the trace of the algebraic integer $\eta = (\epsilon/\epsilon')^{1/3}$, must be a rational integer.

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