PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 2, February 2012, Pages 635–644
S 0002-9939(2011)10930-4
Article electronically published on June 17, 2011

THE KDV CURVE AND SCHRÖDINGER-AIRY CURVE

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(Communicated by Jianguo Cao)

ABSTRACT. Among other things, we introduce the notion of KdV curves and Schrödinger-Airy curves. These curves are stable solutions to the geometric KdV-Airy flow equation and Schrödinger-Airy flow equation respectively, which were recently proposed by Sun and Wang. We demonstrate that the KdV curves can be regarded as a 3rd-order analogue of geodesics. Other interesting properties of these curves will be addressed. Explicit examples of these curves will be provided. In addition, we will consider a perturbed KdV curve system and show the existence of multiple solutions to this system on the torus.

1. THE KDV CURVE

Suppose that (N, ω, J) is a Kähler manifold with symplectic form ω and complex structure J and that $u(x,t)$ is a smooth map from $S^1 \times \mathbb{R}^1$ into N. Let ∇_x denote the covariant derivative $\nabla_{\frac{\partial}{\partial x}}$ on the pull-back bundle $u^{-1}TN$ induced from the Levi-Civita connection ∇ on N. In [5], Sun and Wang introduced the so-called geometric KdV-Airy flow as follows:

$$u_t = \nabla_x^2 u_x + \frac{1}{2} R(u_x, J u_x) J u_x, \tag{1.1}$$

where R is the curvature tensor on N. Here we denote $u_t = \nabla_t u$, $u_x = \nabla_x u$ and write $J = J(u)$ for simplicity. Equation (1.1) is a geometric flow which stems from the vortex filament dynamics and belongs to the same family as the Schrödinger flow (cf. [4]):

$$u_t = J \nabla_x u_x. \tag{1.2}$$

In certain circumstances, equation (1.1) transforms into the well-known modified KdV equation. We refer to [5] for more details and background knowledge.

The *KdV curve* is defined as the stable solution to equation (1.1). Namely, a map $u \in C^\infty(S^1, N)$ is called a KdV curve if it satisfies

$$\nabla_x^2 u_x + \frac{1}{2} R(u_x, J u_x) J u_x = 0. \tag{1.3}$$

Suppose N is embedded in a Euclidean space $\mathbb{R}^K$. Define the KdV energy functional on Sobolev space,

$$W = W^{2,2}(S^1, N) := \{W^{2,2}(S^1, \mathbb{R}^K) | u(x) \in N \text{ for a.e. } x \in S^1\},$$

Received by the editors October 22, 2010 and, in revised form, November 28, 2010 and December 2, 2010.

2010 *Mathematics Subject Classification.* Primary 37K25, 58E50, 53C99.

as follows:

$$F(u) = \frac{1}{2}\int_{S^1} \langle \nabla_x u_x, J u_x\rangle dx. \tag{1.4}$$

Let $u_s : [0,\delta] \to W$ be a variation of u and

$$\frac{\partial u_s}{\partial s}|_{s=0} = \xi \in T_u W = W^{2,2}(S^1, T_u N).$$

Then a direct computation yields

$$(dF(u), \xi) = \frac{d}{ds}F(u_s)|_{s=0} = \int_{S^1} \langle J\nabla_x^2 u_x + \frac{1}{2}R(Ju_x, u_x)u_x, \xi\rangle.$$

So the KdV curves are actually critical points of the functional F.

There is a symplectic form Ω on W naturally induced by the symplectic form ω on N. Indeed, for any vector fields $X, Y \in T_u W$, we can define

$$\Omega(X,Y) = \int_{S^1} \omega(X(u), Y(u))dx.$$

Similarly, there is an induced complex structure on W, which we still denote by J. The Hamiltonian vector field X_F associated with F is defined by

$$\Omega(X_F, \cdot) = dF.$$

Then the KdV-Airy flow (1.1) can be written in the following form:

$$u_t = X_F = J\nabla F(u),$$

which is the Hamiltonian flow of F on the infinite dimensional symplectic manifold (W, Ω). On the other hand, the Schrödinger flow (1.2) is known to be the Hamiltonian flow of the normal energy functional

$$E(u) = \frac{1}{2}\int_{S^1} |u_x|^2 dx,$$

and the stable solutions for Schrödinger flow are just geodesics. It turns out that the KdV-Airy flow and the Schrödinger flow belong to the same integrable system [5]. Therefore, the KdV curves are not only important for understanding the KdV-Airy flow, but are also of special interest as a higher order analogue of geodesics.

It is a basic fact that geodesics have constant speed. In other words, the energy density $e(u) = |u_x|^2$ remains the same along geodesics. There is an analogous result for KdV curves.

Theorem 1.1. *For any given KdV curve u, the quantity*

$$f(u) = \langle \nabla_x u_x, J u_x\rangle$$

is a constant.

Proof. A simple calculation yields

$$\begin{aligned} d_x f(u) &= d_x\langle \nabla_x u_x, J u_x\rangle \\ &= \langle \nabla_x^2 u_x, J u_x\rangle + \langle \nabla_x u_x, J\nabla_x u_x\rangle \\ &= -\frac{1}{2}\langle R(u_x, Ju_x)Ju_x, Ju_x\rangle \\ &= 0. \end{aligned}$$

□

In fact, on some manifolds, KdV curves are just geodesics.

Theorem 1.2. *If the holomorphic sectional curvature k of N is non-positive, the KdV curves are geodesics. In particular, if k is strictly negative, then the KdV curves are just constant maps.*

Proof. On manifolds with non-positive holomorphic sectional curvature, we have

$$k(X) = \frac{R(X, JX, JX, X)}{|X|^4} \leq -\delta, \quad \delta \geq 0.$$

For any KdV curve u which satisfies equation (1.3), multiplying by u_x and integrating by parts, we have

$$\begin{aligned} 0 &= \int \langle \nabla_x^2 u_x, u_x \rangle + \frac{1}{2} \int R(u_x, Ju_x, Ju_x, u_x) \\ &= -\int |\nabla_x u_x|^2 + \frac{1}{2} \int k(u_x)|u_x|^4 \\ &\leq -\int |\nabla_x u_x|^2 - \frac{1}{2}\delta \int |u_x|^4. \end{aligned}$$

This implies that $\nabla_x u_x = 0$, which means that u is a geodesic. Moreover, if $\delta > 0$, then $u_x \equiv 0$, which means that u is a constant map. □

2. Schrödinger-Airy curve

In [5], the authors also addressed the so-called Schrödinger-Airy flow, which can be regarded as a geometric generalization of the Hirota equation. Explicitly, the flow is defined as follows:

$$u_t = \alpha J \nabla_x u_x + \beta(\nabla_x^2 u_x + \frac{1}{2} R(u_x, Ju_x) Ju_x), \tag{2.1}$$

where α and β are two positive numbers. We call a stable solution to equation (2.1) a *Schrödinger-Airy curve*. Namely, a Schrödinger-Airy curve is a map $u \in C^\infty(S^1, N)$ which satisfies the equation

$$\nabla_x^2 u_x + \frac{1}{2} R(u_x, Ju_x) Ju_x = \lambda J \nabla_x u_x, \tag{2.2}$$

where λ is a function on S^1. It's a natural extension of the KdV curve. Particularly, a KdV curve satisfies equation (2.2) for $\lambda \equiv 0$. If we regard λ as a Lagrange multiplier, then the Schrödinger-Airy curve is a critical point of $F(u)$ under the constraint $E(u) = const.$

Note that both the KdV curve and the Schrödinger-Airy curve can be defined weakly in $W^{2,2}(S^1, N)$. In fact, we may assume that the compact manifold N is embedded in a Euclidean space $\mathbb{R}^K$. Denote the inner product on $\mathbb{R}^K$ by $(\cdot, \cdot)$ and the second fundamental form of N by A. Then we have

$$\nabla_x u_x = u_{xx} - A(u)(u_x, u_x).$$

It follows that

$$\begin{aligned} \nabla_x^2 u_x &= (\nabla_x u_x)_x - A(u)(u_x, \nabla_x u_x) \\ &= u_{xxx} - [A(u)(u_x, u_x)]_x - A(u)(u_x, \nabla_x u_x) \\ &= u_{xxx} - \nabla A(u)(u_x, u_x, u_x) - 3A(u)(u_x, \nabla_x u_x). \end{aligned} \tag{2.3}$$

Therefore, we may call a map $u \in W^{2,2}(S^1, N)$ a weak KdV curve if u is a $W^{2,2}$-weak solution of equation (1.3). Namely, u is a weak KdV curve if for any $v \in C^\infty(S^1, \mathbb{R}^K)$, it follows that

$$\int_{S^1} (\nabla_x u_x, v_x)dx + \int_{S^1} (A(u)(u_x, \nabla_x u_x), v)dx - \frac{1}{2}\int_{S^1} (R(u_x, Ju_x)Ju_x, v) = 0.$$

Similarly a weak Schrödinger-Airy curve is defined as a $W^{2,2}$-weak solution to equation (2.2). The following theorem shows that a weak Schrödinger-Airy (or KdV) curve is actually smooth.

Theorem 2.1. *A $W^{2,2}$-weak Schrödinger-Airy curve is smooth.*

Proof. By (2.3), equation (2.2) can be rewritten as

$$u_{xxx} = f(u_x, u_{xx}), \tag{2.4}$$

where

$$\begin{aligned} f(u_x, u_{xx}) = {} & \nabla A(u)(u_x, u_x, u_x) + 3A(u)(u_x, \nabla_x u_x) \\ & - \frac{1}{2}R(u_x, Ju_x)Ju_x + \lambda J\nabla_x u_x. \end{aligned}$$

It is obvious that equation (2.4) is an elliptic equation for u_x, and u_x is a $W^{1,2}$-weak solution to (2.4) if u is a $W^{2,2}$-weak solution to (2.2). By Sobolev embedding theorems, $u \in W^{2,2}(S^1, N)$ implies $u_x \in C^\alpha$ for some $\alpha \in (0, 1)$. Hence $f(u_x, u_{xx}) \in L^2$. Now by the standard L^2-estimate, it follows that $u_x \in W^{2,2}$ from equation (2.4). A bootstrapping argument then proves the theorem. □

In the remaining part of this section, we always suppose that N is a closed Riemann surface. In this case, we shall show that Schrödinger-Airy curves satisfy some nice properties. In particular, all these properties hold for KdV curves.

First we observe that the functional F is not a geometric invariant. That is, it depends on the choice of the parameter of the curve u. Let $s = \int_0^x |u_x|dx$ be the arc length parameter of u. Let $\mathbf{t} = u_s$ be the unit tangent vector and $\mathbf{n}$ be the unit normal vector orthogonal to $\mathbf{t}$. We denote $\nabla_s = \nabla_{u_s}$. Then

$$\nabla_s u_s = k_g \mathbf{n},$$

where k_g is the geodesic curvature. It follows that

$$\langle \nabla_s u_s, Ju_s\rangle = \langle k_g \mathbf{n}, J\mathbf{t}\rangle = k_g.$$

Note that k_g in the last equality may vary from the usual definition of geodesic curvature by a minus sign, depending on the complex structure J. But here we only care about the absolute value of k_g and ignore the slight difference of sign. On the other hand, $ds = |u_x|dx$. Therefore, we get

$$\langle \nabla_x u_x, Ju_x\rangle = |u_x|^3\langle \nabla_s u_s, Ju_s\rangle + |u_x|^2\nabla_s|u_x|\langle u_s, Ju_s\rangle = |u_x|^3 k_g. \tag{2.5}$$

Theorem 2.2. *Suppose N is a Riemann surface and u is a Schrödinger-Airy curve. If u has constant speed $|u_x| = c$, then u has constant geodesic curvature.*

Proof. Multiplying equation (2.2) by Ju_x, we get

$$\langle \nabla_x^2 u_x, Ju_x\rangle = -\lambda\langle \nabla_x u_x, u_x\rangle. \tag{2.6}$$

By (2.5), the geodesic curvature k_g satisfies

$$\langle \nabla_x u_x, Ju_x\rangle = |u_x|^3 k_g.$$

Note that

$$\nabla_x\langle \nabla_x u_x, Ju_x\rangle = \langle \nabla_x^2 u_x, Ju_x\rangle.$$

Hence equality (2.6) is equivalent to

$$\nabla_x(|u_x|^3 k_g) = -\frac{\lambda}{2}\nabla_x |u_x|^2.$$

It is easy to see that if $|u_x| = c$ is a constant, then $\nabla_x k_g = 0$, as desired. □

Next we suppose in addition that N has constant curvature. We show that in this case the geodesic curvature can be computed.

Theorem 2.3. *Suppose N is a Riemann surface with constant curvature k and u is a non-trivial Schrödinger-Airy curve. Assume u has constant speed $|u_x| = c$. Then the geodesic curvature k_g is determined by λ.*

Proof. For a constant curvature surface N, we have

$$R(u_x, Ju_x)Ju_x = k|u_x|^2 u_x. \tag{2.7}$$

Since u has constant speed, we have

$$0 = \nabla_x |u_x|^2 = \langle \nabla_x u_x, u_x\rangle.$$

Thus we may suppose $\nabla_x u_x = \alpha J u_x$ for some function $\alpha(x)$. It follows from (2.5) that

$$\langle \nabla_x u_x, Ju_x\rangle = -|u_x|^2\alpha = |u_x|^3 k_g.$$

Hence $\alpha = -ck_g$, which is constant by Theorem 2.2. Consequently, we have

$$\nabla_x u_x = -ck_g J u_x, \quad \nabla_x^2 u_x = -c^2 k_g^2 u_x.$$

Then using equations (2.2) and (2.7), we get

$$\begin{aligned}\nabla_x^2 u_x + \frac{1}{2}R(u_x, Ju_x)Ju_x &= -c^2 k_g^2 u_x + \frac{1}{2}c^2 k u_x\\ &= \lambda J\nabla_x u_x = \lambda c k_g u_x.\end{aligned}$$

This simplifies to

$$k_g^2 + \frac{\lambda}{c}k_g - \frac{1}{2}k = 0.$$

This is a second order equation for k_g, and the solution is given by

$$k_g = \frac{\lambda \pm c\sqrt{\lambda^2 + 2kc^2}}{2c}, \tag{2.8}$$

provided $\lambda^2 + 2kc^2 \geq 0$. □

3. Examples

In this section let's see some examples of Schrödinger-Airy curves on Riemann surfaces.

Example 3.1. On the Euclidean plane, the round circles are Schrödinger-Airy curves for appropriate λ.

Example 3.2. On the standard sphere S^2, except the great circle, any non-trivial intersection of a plane in $\mathbb{R}^3$ and the sphere S^2 gives rise to a Schrödinger-Airy curve.

For example, let $u : S^1 \to S^2$ be given by

$$u(x) = (\cos x \cos\phi, \sin x \cos\phi, \sin\phi), \quad x \in [0, 2\pi).$$

Here we suppose $\phi \in (0, \pi/2) \cup (\pi/2, \pi)$. That is, we exclude trivial maps and the geodesics. Then a direct computation yields:

$$\begin{aligned} u_x(x) &= (-\sin x, \cos x, 0)\cos\phi, \\ \nabla_x u_x(x) &= (-\cos x \sin\phi, -\sin x \sin\phi, \cos\phi)\sin\phi\cos\phi, \\ \nabla_x^2 u_x(x) &= -(\sin^2\phi)u_x. \end{aligned}$$

Since the sphere has constant curvature 1, it follows that

$$R(u_x, Ju_x)Ju_x = |u_x|^2 u_x = (\cos^2\phi)u_x.$$

On the other hand, the complex structure on S^2 is given by $J(u) = u\times$. Thus

$$J(u)\nabla_x u_x = -(\sin x, -\cos x, 0)\sin\phi\cos\phi = -(\sin\phi)u_x.$$

Therefore, u is a Schrödinger-Airy curve with

$$\lambda = \frac{\sin^2\phi - \frac{1}{2}\cos^2\phi}{\sin\phi}.$$

Particularly, if we choose $\phi = \arccos\sqrt{\frac{2}{3}}$, we get a non-trivial KdV curve.

Let us now turn our attention to the hyperbolic plane.

Example 3.3. Let $g_{-1} = dr^2 + (\sinh r)^2 d\theta^2$ be the hyperbolic metric of constant curvature -1 on $\mathbb{R}^2$. We consider the curve $u : S^1 \to \mathbb{R}^2$ given by $u(x) = (r, x)$. It is clear that $|u_x| = \sinh r$ and

$$\nabla_x u_x = -(\sinh r)(\cosh r)\frac{\partial}{\partial r} = (\cosh r)Ju_x.$$

Using this equation together with (2.2), we observe that

$$\begin{aligned} -\lambda(\cosh r)u_x = \lambda J\nabla_x u_x &= \nabla_{u_x}^2 u_x + \frac{1}{2}R(u_x, Ju_x)Ju_x \\ &= \nabla_x[(\cosh r)Ju_x] - \frac{1}{2}|u_x|^2 u_x \\ &= [-(\cosh r)^2 - \frac{1}{2}(\sinh r)^2]u_x. \end{aligned}$$

It follows that

$$\lambda = \frac{(\cosh r)^2 + \frac{1}{2}(\sinh r)^2}{\cosh r}.$$

Note that u has constant geodesic curvature $\cosh r/\sinh r$. Consequently, there are no non-trivial KdV curves on a hyperbolic plane.

4. Perturbed KdV curves on T^{2n}

By Theorem 1.2, the KdV curves on non-positively curved manifolds are trivial, so it is natural to consider the perturbed system. Namely, we shall consider the following system:

$$\nabla_x^2 u_x + \frac{1}{2}R(u_x, Ju_x)Ju_x = -J(u)\nabla_u H(x,u), \tag{4.1}$$

where $H(x,u)$ is a Hamiltonian function defined on $S^1 \times N$. Obviously, solutions to the above system are critical points of the perturbed functional

$$F_H(u) = F(u) + \int_{S^1} H(x,u)dx.$$

This is a strongly indefinite functional which has no lower bound. In general it is very hard to find critical points of F_H. Fortunately, in the special case of the torus, this problem can be significantly simplified.

In the following context, we suppose that $N = T^{2n}$ is the $2n$-dimensional flat torus and use t to denote the variable in S^1 instead of x. Then the system (4.1) is reduced to

$$\frac{d^3u}{dt^3} = -J\nabla H_t(u), \quad t \in S^1. \tag{4.2}$$

System (4.2) is similar to the usual Hamiltonian system

$$\frac{du}{dt} = -J\nabla H_t(u). \tag{4.3}$$

It is well known that the solution to (4.3) corresponds to the fixed point of symplectomorphisms on T^{2n}, and the number of solutions is related to the famous Arnold conjecture, which is solved by Conley and Zehnder [3]. The method they used relies on a saddle point reduction, which is due to Amann [1]. Namely, finding periodic solutions to the infinite dimensional system (4.3) shifts to a finite dimensional variational problem by a Lyapunov-Schmidt reduction, and the Morse theory can be applied.

The same saddle point reduction can be applied to our system (4.2). But here we follow a simplified version which appears in Chang's book [2]. Our main result is the following:

Theorem 4.1. *Suppose $H_t(u) = H(t,u) \in C^2(S^1 \times T^{2n}, \mathbb{R}^1)$. Then the system* (4.2) *has at least $2n+1$ solutions. If in addition all solutions are non-degenerate, then there are 2^{2n} solutions to* (4.2).

Before proving the theorem, let us first observe some basic facts. The operator

$$A := J\frac{d^3}{dt^3}$$

is defined on a dense subspace $D(A)$ of the Hilbert space

$$L := L^2(S^1, T^{2n}).$$

Any function in L can be viewed as a periodic function in $L^2(S^1, \mathbb{R}^{2n})$. By Fourier series theory, a function $u \in L^2(S^1, \mathbb{R}^{2n})$ can always be decomposed into

$$u(t) = \sum_{j=1}^{n} \sum_{m=-\infty}^{+\infty} c_{mj} e^{-imt}\phi_j,$$

where $\{\phi_1, \cdots, \phi_n\}$ is a basis of $\mathbb{C}^n = \mathbb{R}^{2n}$ and c_{mj} satisfies

$$\sum_{m=-\infty}^{+\infty} |c_{mj}|^2 < \infty, j = 1, \cdots, 2n.$$

Since

$$J\frac{d^3}{dt^3}(e^{-imt}\phi_j) = -m^3 e^{-imt}\phi_j,$$

it is obvious that A is a selfadjoint operator which only has point spectrum $\{-m^3; m \in \mathbb{Z}\}$, and the eigenspace of the eigenvalue $\lambda = -m^3$ is

$$\Lambda(-m^3) = \operatorname{span}\{e^{-imt}\phi_1, \cdots, e^{-imt}\phi_n\}. \tag{4.4}$$

We shall work on the space $V = D(|A|^{\frac{1}{2}})$ defined by

$$V := \{v \in L\colon \sum_{j=1}^{n} \sum_{m=-\infty}^{+\infty} (1 + |m|^3)|c_{mj}|^2 < \infty\}.$$

There is a well-defined functional on V given by

$$a(v) = \frac{1}{2}\int_{S^1} \langle Av, v\rangle dt - \int_{S^1} H_t(v)dt, v \in V. \tag{4.5}$$

Just as in the theory of the Hamiltonian system (4.3), it easy to verify that the Euler-Lagrange equation of a is exactly the system (4.2). A solution v is called non-degenerate if the Hessian $d^2a(v)$ is an isomorphism.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Since the solutions to equation (4.2) are just critical points of the functional $a \in C^2(V, \mathbb{R}^1)$, we only have to investigate the number of critical points of a. We prove this theorem in two steps. First, we show that the required critical points of a are in one-one correspondence with a function h, which is defined on a finite dimensional space. Then we show that h satisfies the P.S. condition, so that the standard Morse theory yields the desired conclusion.

As previously noted, the selfadjoint operator A only has a discrete spectrum. So there exists $\epsilon > 0$ small, such that $-\epsilon$ is not in the spectrum and $A_\epsilon := \epsilon I + A$ is invertible.

On the other hand, since the Hamiltonian $H(t, u) \in C^2(S^1 \times T^{2n})$ is defined on a compact space, we may suppose that there exists a real number $B > 0$ such that

$$\|\nabla_u^2 H_t\| \le B, \forall t \in S^1.$$

The following step is to decompose the domain space $V = D(|A|^{\frac{1}{2}})$. First, for each eigenvalue λ of A, let $P(\lambda)$ be the projection from L to the eigenspace $\Lambda(\lambda)$. Let

$$P_0 = \sum_{-B\le\lambda\le B} P(\lambda), P_+ = \sum_{\lambda<-B} P(\lambda), P_- = \sum_{\lambda>B} P(\lambda),$$

and let $L_0 = P_0L, L_\pm = P_\pm L$. We also denote

$$E_+ = \sum_{\lambda>0} P(\lambda), E_- = \sum_{\lambda<0} P(\lambda).$$

Next, using the invertible operator A_ϵ, we may decompose the space $V = D(|A|^{\frac{1}{2}})$ into

$$V = V_0 \oplus V_+ \oplus V_-,$$

where $V_0 = |A_\epsilon|^{-\frac{1}{2}} L_0, V_\pm = |A_\epsilon|^{-\frac{1}{2}} L_\pm$. Define the graph norm on V by

$$\|v\|_V^2 := \||A|^{\frac{1}{2}} v\|_L^2 + \epsilon^2 \|v\|_L^2.$$

Then for any $u \in L$ and $v = |A_\epsilon|^{-\frac{1}{2}} u \in V$, we have $\|u\|_L = \|v\|_V$. Moreover, we have the following decomposition:

$$u = u_+ + u_- + u_0, \quad v = v_+ + v_- + v_0,$$

where $u_0 = P_0 u, u_\pm = P_\pm u$, and $v_0 = |A_\epsilon|^{-\frac{1}{2}} u_0, v_\pm = |A_\epsilon|^{-\frac{1}{2}} u_\pm$.

Now we define a functional on L as follows:

$$b(u) = \frac{1}{2}(\|u_+\|_L^2 + \|E_+ u_0\|_L^2 - \|E_- u_0\|_L^2 - \|u_-\|_L^2) - \Phi_\epsilon(v),$$

where

$$\Phi_\epsilon(v) = \frac{\epsilon}{2}\|v\|_L^2 + \int_{S^1} H_t(v) dt.$$

Obviously, this functional b is actually the same as the functional a given by (4.5) such that $a(v) = b(u)$. Besides, u is a critical point of b if and only if v is a critical point of a. The Euler-Lagrange equation of b for $u_\pm$ is

$$u_\pm = \pm |A_\epsilon|^{-\frac{1}{2}} P_\pm F_\epsilon(v), \tag{4.6}$$

where

$$F_\epsilon = \epsilon I + \nabla H_t \in C^1(V, V).$$

However, equation (4.6) is equivalent to

$$v_\pm = A_\epsilon^{-1} P_\pm F_\epsilon(v_+ + v_- + v_0). \tag{4.7}$$

A direct computation shows that the operator

$$\mathcal{F}_\pm := A_\epsilon^{-1} P_\pm F_\epsilon \in C^1(V, V)$$

is a contraction. It follows by the implicit function theorem that there exists a solution $v_\pm(v_0)$ to equation (4.7) for fixed $v_0 \in V_0$. Thus we may define a functional on V_0 by

$$h(z) = a(v(z)) = a(v_+(z) + v_-(z) + z), z \in V_0.$$

One verifies readily that z is a critical point of h if and only if $v(z)$ is a critical point of a. As a consequence, the problem of finding critical points of a is now reduced to finding critical points of a C^2 functional h defined on a finite dimensional space V_0.

To proceed, we decompose V_0 into $V_0 = Z_\star \oplus Z_0$, where $Z_\star = E_+ V_0 \oplus E_- V_0$, and Z_0 is the kernel of A. Let $z = z_\star + z_0$, where $z_\star \in Z_\star, z_0 \in Z_0$. Because H_t is periodic in v and $v_\pm$ is periodic in z by equation (4.7), it follows that $h(z)$ is periodic in z_0. Note that $Z_0 = \Lambda(0)$ is a $2n$-dimensional space by (4.4). So $h(z)$ may be viewed as a functional defined on $Z_\star \times T^{2n}$. If we let

$$g(z) = \frac{1}{2}(A(v_+(z) + v_-(z)), v_+(z) + v_-(z)) - \int_{S^1} H_t(v(z)) dt,$$

then the functional h is in the following form:

$$h(z) = h(z_\star, z_0) = \frac{1}{2}(A z_\star, z_\star) + g(z_\star, z_0).$$

It is easy to see that $dg(z) = P_0 \nabla H_t(v(z))$ is bounded. Now by a standard argument in Morse theory (for example, see Theorem 5.3 in [2]), f satisfies the P.S. condition and Theorem 4.1 is proved, provided the cuplength of T^{2n} is $2n$ and the sum of the Betti numbers is 2^{2n}. □

Acknowledgements

The author would like to express his deep gratitude to Prof. Youde Wang for giving him this interesting problem and for his continuous support. He would also like to thank Prof. Jianguo Cao and Prof. Meiyue Jiang for inspiring discussions and helpful advice.

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