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NON-VANISHING OF DERIVATIVES OF $GL(3) \times GL(2)$ L-FUNCTIONS

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Abstract. Let f be a fixed self-dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ and let $\mathcal{B}_k$ be an orthogonal basis of holomorphic cusp forms of weight $k \equiv 2 (\operatorname{mod} 4)$ for $SL_2(\mathbb{Z})$. We prove an asymptotic formula for the first moment of the first derivative of $L(s, f \times g)$ at the central point $s = 1/2$, where g runs over $\mathcal{B}_k$, $K \le k \le 2K$, K large enough. This implies that for each K large enough there exists $g \in \mathcal{B}_k$ with $K \le k \le 2K$ such that $L'(1/2, f \times g) \neq 0$.

1. Introduction

The non-vanishing of L-functions at special points is an interesting subject and has been studied by many authors, for example, Bump, Friedberg and Hoffstein [2], Iwaniec and Sarnak [8], and Kowalski, Michel and Vanderkam [10], [11]. The non-vanishing problem for $GL(3) \times GL(2)$ L-functions was first studied by Li [12]. Let f be a fixed self-dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ and let $\mathcal{U} = \{u_j : j \ge 1\}$ be an orthogonal basis of even Hecke-Maass cusp forms for $SL_2(\mathbb{Z})$. Li considered the non-vanishing problem of Rankin-Selberg L-function $L(s, f \times u_j)$ times the L-function $L(s, u_j)$ at $s = 1/2$, where u_j runs over $\mathcal{U}$. In this paper, we will study the problem of non-vanishing of derivatives of $GL(3) \times GL(2)$ L-functions at the central point.

Let f be a fixed self-dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ and let $\mathcal{B}_k$ be an orthogonal basis of holomorphic cusp forms of weight $k \equiv 2 (\operatorname{mod} 4)$ for $SL_2(\mathbb{Z})$. Denote

$$\omega_g = \frac{k-1}{2\pi^2} L\left(1, \operatorname{sym}^2 g\right),$$

where $L\left(s, \operatorname{sym}^2 g\right)$ is the symmetric-square L-function associated to g. We first prove an asymptotic formula for the first moment of the first derivative of $L(s, f \times g)$ at $s = 1/2$.

Theorem 1.1. *Let f be a fixed self-dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ and let $\mathcal{B}_k$ be an orthogonal basis of holomorphic cusp forms of weight $k \equiv 2 (\operatorname{mod} 4)$ for*

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$SL_2(\mathbb{Z})$. Let h be a fixed positive valued, smooth function of compact support on $[1,2]$, with its derivatives satisfying $h^{(j)} \ll_j 1$. Then we have

$$\begin{aligned}&\sum_{k\equiv 2(\mathrm{mod}\,4)} h\left(\frac{k-1}{K}\right)\sum_{g\in\mathcal{B}_k}\omega_g^{-1}L'\left(\frac{1}{2},f\times g\right)\\&=\frac{3L(1,f)}{2}K\log K\widehat{h}(0)+\frac{2L'(1,f)+c_0L(1,f)}{2}K\widehat{h}(0)+O_{\epsilon,f}\left(K^{\epsilon}\right)\end{aligned}$$

for any $\epsilon>0$ and K large enough, where $c_0=-6\log 2-3\log\pi$.

By Jacquet and Shalika [9], $L(1,f)\neq 0$. Then for each large enough K, by Theorem 1.1, the first moment of $L'\left(\frac{1}{2},f\times g\right)$ is non-zero, and therefore not all values of $L'\left(\frac{1}{2},f\times g\right)$ can equal zero. We have the following result.

Corollary 1.2. Let f be a fixed self-dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$. For each K large enough, there exists $g\in\mathcal{B}_k$ with $K\le k\le 2K$ such that

$$L'\left(\frac{1}{2},f\times g\right)\neq 0.$$

Remark 1.3. Following Li [12], we can also study the first moment of the first derivative of the Rankin-Selberg L-function $L(s,f\times v_j)$ at the central point, where v_j runs over an orthogonal basis of odd Hecke-Maass cusp forms for $SL_2(\mathbb{Z})$. The proof starts from approximate functional equations and then the Kuznetsov formula instead of the Petersson formula, and can also be completed by applying the Voronoi formula of Goldfeld and Li [4]. For holomorphic cusp forms of weight $k\equiv 0(\mathrm{mod}\,4)$ for $SL_2(\mathbb{Z})$, a similar result has appeared in Liu [14].

Remark 1.4. When one tries to calculate asymptotic formulas for higher moments of $L'(s,f\times g)$ at the critical point, he encounters new difficulties. Even if we want to study upper bounds for the second moment, it seems out of reach at present due to large conductors of this family; see the interesting papers of Young [17], [18]. It is worth noting that by twisting by n^{it} with t almost as large as the spectral parameter, Young [17] proved an upper bound for the integrated second moment of $L(1/2+it,f\times u_j)$.

2. Holomorphic cusp forms for $SL_2(\mathbb{Z})$

Let $\mathcal{B}_k$ be as in Section 1. Any $g\in\mathcal{B}_k$ has the Fourier expansion

$$g(z)=\sum_{n\ge 1}\lambda_g(n)n^{(k-1)/2}e(nz)$$

for $\mathrm{Im}\,z>0$. Assume that g is a normalized Hecke eigenform so that $\lambda_g(1)=1$. Then Deligne's bound states that

$$|\lambda_g(n)|\le d(n),$$

where $d(n)$ is the divisor function.

The Petersson formula states that (Theorem 3.8 in Iwaniec [6]) for any $m,\,n\ge 1$,

$$\sum_{g\in\mathcal{B}_k}\omega_g^{-1}\lambda_g(m)\lambda_g(n)=\delta(m,n)+2\pi i^k\sum_{c\ge 1}c^{-1}S(m,n;c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{2.1}$$

where $\delta(m,n)$ is the Krönecker symbol, $J_{k-1}(x)$ is the J-Bessel function and $S(m,n;c)$ is the classical Kloosterman sum defined by

$$S(m,n;c)=\sum_{d\overline{d}\equiv 1(\operatorname{mod} c)} e\left(\frac{md+n\overline{d}}{c}\right).$$

For $\operatorname{Re} s>1$, we define the L-function

$$L(s,g)=\sum_{n\geq 1}\lambda_g(n)n^{-s}$$

which has a homomorphic continuation to all $s\in\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s,g)=\pi^{-s}\Gamma\left(\frac{s+(k-1)/2}{2}\right)\Gamma\left(\frac{s+(k+1)/2}{2}\right)L(s,g)=-\Lambda(1-s,g).$$

3. Maass cusp forms for $SL_3(\mathbb{Z})$

We will follow the notations in Goldfeld's book [3] (see also Bump [1]). Let f be a Maass cusp form of type (ν_1,ν_2) for $SL_3(\mathbb{Z})$, normalized so that the first Fourier coefficient is 1. Then f has a Fourier-Whittaker expansion

$$f(z)=\sum_{\gamma\in U_2(\mathbb{Z})\backslash SL_2(\mathbb{Z})}\sum_{m_1\geq 1}\sum_{m_2\neq 0}\frac{A(m_1,m_2)}{m_1|m_2|}W_J\left(M\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}z,\nu,\psi_{1,1}\right),$$

where $U_2(\mathbb{Z})$ is the group of 2×2 upper triangular matrices with integer entries and ones on the diagonal, $W_J(z,\nu,\psi_{1,1})$ is the Jacquet-Whittaker function and $M=\operatorname{diag}(m_1|m_2|,m_1,1)$. The Fourier coefficients $A(m_1,m_2)$ satisfy

$$\sum_{m_2\leq N}|A(m_1,m_2)|\ll_f N|m_1|. \tag{3.1}$$

Let $\widetilde{f}$ denote the dual Maass form of f. Then $\widetilde{f}$ is of type (ν_2,ν_1) and the (m_1,m_2)-th Fourier coefficient of $\widetilde{f}$ is $A(m_2,m_1)$, which is the (m_2,m_1)-th Fourier coefficient of f. If f is self-dual, then $f=\widetilde{f}$, and so $A(m_1,m_2)=A(m_2,m_1)$.

The Godement-Jacquet L-function associated to f is defined by

$$L(s,f)=\sum_{n\geq 1}A(1,n)n^{-s}, \tag{3.2}$$

which has a holomorphic continuation to all $s\in\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s,f)=\Lambda(1-s,\widetilde{f}),$$

where

$$\Lambda(s,f)=\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s-\beta}{2}\right)\Gamma\left(\frac{s-\gamma}{2}\right)L(s,f),$$
$$\Lambda(s,\widetilde{f})=\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+\alpha}{2}\right)\Gamma\left(\frac{s+\beta}{2}\right)\Gamma\left(\frac{s+\gamma}{2}\right)L(s,\widetilde{f})$$

with

$$\alpha=-\nu_1-2\nu_2+1,\qquad \beta=-\nu_1+\nu_2,\qquad \gamma=2\nu_1+\nu_2-1. \tag{3.3}$$

Here $L(s,\widetilde{f})$ is the L-function associated to the dual Maass form $\widetilde{f}$. Luo, Rudnick and Sarnak [15] proved that

$$|\operatorname{Re}\alpha|, |\operatorname{Re}\beta|, |\operatorname{Re}\gamma| \le \frac{1}{2} - \frac{1}{10}.$$

The Voronoi formula for $GL(3)$ was obtained by Miller and Schmid [16]. In this paper, we will use the Voronoi formula in Goldfeld and Li [4]. Precisely, let α,β and γ be as in (3.3). Let $\psi(x)$ be a smooth function compactly supported on $(0,\infty)$ and denote by $\widetilde{\psi}(s)$ the Mellin transform of $\psi(x)$. For $k=0,1$, we set
(3.4)

$$\Psi_k(x) := \int_{\operatorname{Re} s=\sigma} (\pi^3 x)^{-s} \frac{\Gamma\left(\frac{1+s+2k+\alpha}{2}\right)\Gamma\left(\frac{1+s+2k+\beta}{2}\right)\Gamma\left(\frac{1+s+2k+\gamma}{2}\right)}{\Gamma\left(\frac{-s-\alpha}{2}\right)\Gamma\left(\frac{-s-\beta}{2}\right)\Gamma\left(\frac{-s-\gamma}{2}\right)} \widetilde{\psi}(-s-k)\mathrm{d}s$$

with $\sigma > \max\{-1-\operatorname{Re}\alpha, -1-\operatorname{Re}\beta, -1-\operatorname{Re}\gamma\}$,

$$\Psi_{0,1}^0(x) = \Psi_0(x) + \frac{\pi^{-3}c^3 m}{n_1^2 n_2 i}\Psi_1(x), \tag{3.5}$$

and

$$\Psi_{0,1}^1(x) = \Psi_0(x) - \frac{\pi^{-3}c^3 m}{n_1^2 n_2 i}\Psi_1(x). \tag{3.6}$$

Lemma 3.1. *Let $\psi(x) \in C_c^\infty(0,\infty)$. Let $d,\overline{d},c \in \mathbb{Z}$ with $c \neq 0$, $(d,c)=1$, and $d\overline{d} \equiv 1 (\operatorname{mod} c)$. Then*

$$\begin{aligned}
&\sum_{n>0} A(m,n) e\left(\frac{n\overline{d}}{c}\right)\psi(n) \\
&\quad = \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1 | cm} \sum_{n_2>0} \frac{A(n_2,n_1)}{n_1 n_2} S(md, n_2; mcn_1^{-1}) \Psi_{0,1}^0\left(\frac{n_2 n_1^2}{c^3 m}\right) \\
&\quad + \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{n_1 | cm} \sum_{n_2>0} \frac{A(n_2,n_1)}{n_1 n_2} S(md, -n_2; mcn_1^{-1}) \Psi_{0,1}^1\left(\frac{n_2 n_1^2}{c^3 m}\right).
\end{aligned}$$

As pointed out in Li [13], $x^{-1}\Psi_1(x)$ has similar asymptotic behavior as that of $\Psi_0(x)$. In fact, let $\sigma > \max\{-1-\operatorname{Re}\alpha, -1-\operatorname{Re}\beta, -1-\operatorname{Re}\gamma\}$, which is fixed. By changing variables $s+1 \to s$ in the definition of $\Psi_1(x)$ in (3.4), we have

$$\Psi_1(x) = \int_{\operatorname{Re} s=\sigma+1} (\pi^3 x)^{-s+1} \frac{\Gamma\left(\frac{2+s+\alpha}{2}\right)\Gamma\left(\frac{2+s+\beta}{2}\right)\Gamma\left(\frac{2+s+\gamma}{2}\right)}{\Gamma\left(\frac{-s+1-\alpha}{2}\right)\Gamma\left(\frac{-s+1-\beta}{2}\right)\Gamma\left(\frac{-s+1-\gamma}{2}\right)} \widetilde{\psi}(-s)\mathrm{d}s.$$

By Stirling's formula, we have

$$\begin{aligned}
&e^{i\frac{2\pi}{3}} \left(\pi^3 x\right)^{-1} \Psi_1(x) \\
&= \int_{\operatorname{Re} s=\sigma+1} (\pi^3 x)^{-s} \frac{\Gamma\left(\frac{1+s+\alpha}{2}\right)\Gamma\left(\frac{1+s+\beta}{2}\right)\Gamma\left(\frac{1+s+\gamma}{2}\right)}{\Gamma\left(\frac{-s-\alpha}{2}\right)\Gamma\left(\frac{-s-\beta}{2}\right)\Gamma\left(\frac{-s-\gamma}{2}\right)} \\
&\times \widetilde{\psi}(-s) \left\{ \sum_{j=1}^{M} \frac{a_j}{s^j} + O\left(\frac{1}{|s|^{M+1}}\right) \right\} \mathrm{d}s,
\end{aligned}$$

where a_j are constants depending only on f. On the other hand, moving the line of integration in the definition of $\Psi_0(x)$ to $\mathrm{Re}s=\sigma+1$, we have

$$\Psi_0(x)=\int\limits_{\mathrm{Re}s=\sigma+1}(\pi^3x)^{-s}\frac{\Gamma\left(\frac{1+s+\alpha}{2}\right)\Gamma\left(\frac{1+s+\beta}{2}\right)\Gamma\left(\frac{1+s+\gamma}{2}\right)}{\Gamma\left(\frac{-s-\alpha}{2}\right)\Gamma\left(\frac{-s-\beta}{2}\right)\Gamma\left(\frac{-s-\gamma}{2}\right)}\widetilde{\psi}(-s)\mathrm{d}s.$$

Therefore, we only need to consider $\Psi_0(x)$, since other terms can be treated similarly.

The following result is Lemma 6.1 in Li [12]. For $\alpha=\beta=\gamma=0$, this was proved by Ivić [5].

Lemma 3.2. *Suppose $\psi(x)$ is a smooth function compactly supported on $[X,2X]$. Let $\Psi_0(x)$ be defined as in (3.3). Then for any fixed integer $M\geq 1$ and $xX\gg 1$, we have*

$$\Psi_0(x)=2\pi^4xi\int\limits_0^\infty\psi(y)\sum_{j=1}^M\frac{c_j\cos\left(6\pi x^{\frac13}y^{\frac13}\right)+d_j\sin\left(6\pi x^{\frac13}y^{\frac13}\right)}{(\pi^3xy)^{\frac{j}{3}}}\mathrm{d}y+O\left((xX)^{\frac{-M+2}{3}}\right),$$

where c_j and d_j are constants depending on α, β and γ, in particular, $c_1=0$, $d_1=-\frac{2}{\sqrt{3\pi}}$.

4. Rankin-Selberg L-functions

Let f be a self-dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ and let $\mathcal{B}_k$ be an orthogonal basis of holomorphic cusp forms of weight $k\equiv 2(\mathrm{mod}\,4)$ for $SL_2(\mathbb{Z})$. The Rankin-Selberg L-function of f and $g\in\mathcal{B}_k$ defined by

$$L(s,f\times g)=\sum_{m\geq 1}\sum_{n\geq 1}\frac{\lambda_g(n)A(m,n)}{(m^2n)^s} \tag{4.1}$$

is entire and satisfies the functional equation

$$\Lambda(s,f\times g)=-\Lambda(1-s,f\times g), \tag{4.2}$$

where

$$\Lambda(s,f\times g)=\gamma(s,k)L(s,f\times g), \tag{4.3}$$

and for $\alpha=-3\nu+1$,

(4.4)
$$\gamma(s,k)=\pi^{-3s}\Gamma\left(\frac{s+(k-1)/2-\alpha}{2}\right)\Gamma\left(\frac{s+(k-1)/2}{2}\right)\Gamma\left(\frac{s+(k-1)/2+\alpha}{2}\right)$$
$$\times\Gamma\left(\frac{s+(k+1)/2-\alpha}{2}\right)\Gamma\left(\frac{s+(k+1)/2}{2}\right)\Gamma\left(\frac{s+(k+1)/2+\alpha}{2}\right).$$

It follows that

$$L\left(\frac12,f\times g\right)=\Lambda\left(\frac12,f\times g\right)=0. \tag{4.5}$$

Set $G(u)=e^{u^2}$. We define

$$V(y,k)=\frac{1}{2\pi i}\int\limits_{(3)}y^{-u}G(u)\frac{\gamma\left(\frac12+u,k\right)}{\gamma\left(\frac12,k\right)}\frac{\mathrm{d}u}{u^2}. \tag{4.6}$$

One has the following approximate functional equation for $L'(1/2,f\times g)$.

Lemma 4.1. *We have*

$$L'\left(\frac{1}{2}, f\times g\right) = 2\sum_{m\geq 1}\sum_{n\geq 1}\frac{\lambda_g(n)A(m,n)}{(m^2n)^{1/2}}V\left(m^2n,k\right),$$

where $V(y,k)$ *is defined in* (4.6).

Proof. We consider the integral

$$\mathcal{I} = \frac{1}{2\pi i}\int_{(3)}\Lambda\left(\frac{1}{2}+u, f\times g\right)G(u)\frac{\mathrm{d}u}{u^2}.$$

By (4.5), we know that $\Lambda\left(\frac{1}{2}+u, f\times g\right)G(u)u^{-2}$ has a simple pole at $u=0$. Moving the line of integration to $\Re u = -3$, picking up a simple pole at $u=0$, we have

$$\mathcal{I} = \frac{1}{2\pi i}\int_{(-3)}\Lambda\left(\frac{1}{2}+u, f\times g\right)G(u)\frac{\mathrm{d}u}{u^2} + \gamma\left(\frac{1}{2},k\right)L'\left(\frac{1}{2}, f\times g\right), \tag{4.7}$$

where we have used (4.3), (4.5) and the fact that $G(0)=1$, $G'(0)=0$. By the functional equation in (4.2), we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{(-3)}\Lambda\left(\frac{1}{2}+u, f\times g\right)G(u)\frac{\mathrm{d}u}{u^2} &= -\frac{1}{2\pi i}\int_{(-3)}\Lambda\left(\frac{1}{2}-u, f\times g\right)G(u)\frac{\mathrm{d}u}{u^2}\\ &= -\frac{1}{2\pi i}\int_{(3)}\Lambda\left(\frac{1}{2}+v, f\times g\right)G(v)\frac{\mathrm{d}v}{v}\\ &= -\mathcal{I}.\end{aligned} \tag{4.8}$$

By (4.7) and (4.8), we have

$$\gamma\left(\frac{1}{2},k\right)L'\left(\frac{1}{2}, f\times g\right) = 2\mathcal{I}. \tag{4.9}$$

On the other hand, by (4.1) and (4.3) we have

$$\mathcal{I} = \gamma\left(\frac{1}{2},k\right)\sum_{m\geq 1}\sum_{n\geq 1}\frac{\lambda_g(n)A(m,n)}{(m^2n)^{1/2}}V\left(m^2n,k\right), \tag{4.10}$$

where $V(y,k)$ is defined in (4.6). By (4.9) and (4.10), Lemma 4.1 follows. □

$V(y,k)$ has the following properties.

Lemma 4.2. *For* $y>0$ *and* k *large enough, we have*

$$V(y,k) \ll_{f,A} \left(\frac{k^3}{y}\right)^A \tag{4.11}$$

and

$$V(y,k) = \log\left(k^3/y\right) + c_0 + O_f\left(\frac{y}{k^3} + k^{-1}\right), \tag{4.12}$$

where $c_0 = -6\log 2 - 3\log\pi$.

Proof. Moving the line of integration in (4.6) to $\Re u=-1$, passing a double pole at $u=0$, by residue theorem, we have

(4.13)
$$V(y,k)=\operatorname{res}_{u=0}\left(y^{-u}\frac{G(u)}{u^2}\frac{\gamma\left(\frac12+u,k\right)}{\gamma\left(\frac12,k\right)}\right)+\frac{1}{2\pi i}\int\limits_{(-1)}y^{-u}G(u)\frac{\gamma\left(\frac12+u,k\right)}{\gamma\left(\frac12,k\right)}\frac{\mathrm{d}u}{u^2}.$$

By duplication formula, $\gamma(s,k)$ in (4.4) is

$$\gamma(s,k)=\pi^{\frac32-3s}2^{3-3\left(s+\frac{k-1}{2}\right)}\Gamma\left(s+\frac{k-1}{2}-\alpha\right)\Gamma\left(s+\frac{k-1}{2}\right)\Gamma\left(s+\frac{k-1}{2}+\alpha\right).$$

Thus

$$\frac{\gamma\left(\frac12+u,k\right)}{\gamma\left(\frac12,k\right)}=(2\pi)^{-3u}\frac{\Gamma\left(u+\frac k2-\alpha\right)\Gamma\left(u+\frac k2\right)\Gamma\left(u+\frac k2+\alpha\right)}{\Gamma\left(\frac k2-\alpha\right)\Gamma\left(\frac k2\right)\Gamma\left(\frac k2+\alpha\right)} \tag{4.14}$$

and

$$\lim_{u\to0}\frac{\mathrm{d}}{\mathrm{d}u}\frac{\gamma\left(\frac12+u,k\right)}{\gamma\left(\frac12,k\right)}=-3\log(2\pi)+\frac{\Gamma'\left(\frac k2-\alpha\right)}{\Gamma\left(\frac k2-\alpha\right)}+\frac{\Gamma'\left(\frac k2\right)}{\Gamma\left(\frac k2\right)}+\frac{\Gamma'\left(\frac k2+\alpha\right)}{\Gamma\left(\frac k2+\alpha\right)}.$$

By Stirling's formula, for $|\arg z|\le\pi-\delta$, $\delta>0$,

$$\frac{\Gamma'(z)}{\Gamma(z)}=\log z-\frac{1}{2z}+O_\delta\left(\frac{1}{|z|^2}\right).$$

Thus

$$\lim_{u\to0}\frac{\mathrm{d}}{\mathrm{d}u}\frac{\gamma\left(\frac12+u,k\right)}{\gamma\left(\frac12,k\right)}=-3\log(2\pi)+3\log(k/2)+O_f(k^{-1}) \tag{4.15}$$

and

$$\operatorname{res}_{u=0}\left(y^{-u}\frac{G(u)}{u^2}\frac{\gamma\left(\frac12+u,k\right)}{\gamma\left(\frac12,k\right)}\right)=\log\left(k^3/y\right)+c_0+O_f(k^{-1}), \tag{4.16}$$

with $c_0=-6\log2-3\log\pi$.

By Stirling's formula, for $|\arg z|\le\pi-\delta$, $\delta>0$,

$$\log\Gamma(z)=\left(z-\frac12\right)\log z-z+\frac12\log(2\pi)+O_\delta\left(\frac{1}{|z|}\right).$$

Thus, for $u=-1+iv$, we have

(4.17)
$$\begin{aligned}\log\frac{\Gamma\left(u+\frac k2-\alpha\right)}{\Gamma\left(\frac k2-\alpha\right)}&=\left(-\frac32+\frac k2-\operatorname{Re}\alpha\right)\log\left(\frac{k^2}{4}+v^2\right)^{\frac12}-(v-\operatorname{Im}\alpha)\arctan\left(\frac{2v}{k}\right)\\&\quad-\left(-\frac12+\frac k2-\operatorname{Re}\alpha\right)\log\left(\frac k2\right)+i\theta+O_f(1),\end{aligned}$$

where

$$\theta=(v-\operatorname{Im}\alpha)\log\left(\frac{k^2}{4}+v^2\right)^{\frac12}+\left(-\frac32+\frac k2-\operatorname{Re}\alpha\right)\arctan\left(\frac{2v}{k}\right)-(\operatorname{Im}\alpha)\log\left(\frac k2\right)-v.$$

Here we have used the estimate $\log(1+x) = O(|x|)$ for $|x| < 1$. By (4.17),

$$\left|\frac{\Gamma\left(u+\frac{k}{2}-\alpha\right)}{\Gamma\left(\frac{k}{2}-\alpha\right)}\right| \ll_f \frac{\left(\frac{k}{2}+|v|\right)^{-\frac{3}{2}+\frac{k}{2}-\operatorname{Re}\alpha} e^{\frac{\pi}{2}(|v|+|\operatorname{Im}\alpha|)}}{\left(\frac{k}{2}\right)^{-\frac{1}{2}+\frac{k}{2}-\operatorname{Re}\alpha}} \ll_f k^{-1}\left(1+\frac{2|v|}{k}\right)^{-\frac{3}{2}+\frac{k}{2}-\operatorname{Re}\alpha} e^{\frac{\pi|v|}{2}}. \tag{4.18}$$

Similarly,

$$\left|\frac{\Gamma\left(u+\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right| \ll k^{-1}\left(1+\frac{2|v|}{k}\right)^{-\frac{3}{2}+\frac{k}{2}} e^{\frac{\pi|v|}{2}} \tag{4.19}$$

and

$$\left|\frac{\Gamma\left(u+\frac{k}{2}+\alpha\right)}{\Gamma\left(\frac{k}{2}+\alpha\right)}\right| \ll_f k^{-1}\left(1+\frac{2|v|}{k}\right)^{-\frac{3}{2}+\frac{k}{2}+\operatorname{Re}\alpha} e^{\frac{\pi|v|}{2}}. \tag{4.20}$$

By (4.14), (4.18)-(4.20), we have

$$\left|\frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)}\right| \ll_f k^{-3}\left(1+\frac{2|v|}{k}\right)^{-\frac{9}{2}+\frac{3k}{2}} e^{\frac{3\pi|v|}{2}}.$$

It follows that the integral in (4.13) is

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(-1)} y^{-u}G(u)\frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)}\frac{\mathrm{d}u}{u^2} \\
&\ll_f yk^{-3}\int_{-\infty}^{\infty}\left(1+\frac{2|v|}{k}\right)^{-\frac{9}{2}+\frac{3k}{2}} e^{\frac{3\pi|v|}{2}}e^{-v^2}\frac{\mathrm{d}v}{1+|v|} \\
&\ll_f yk^{-3}\int_{|v|\le\frac{k}{4}}\exp\left\{\frac{3k}{2}\log\left(1+\frac{2|v|}{k}\right)\right\}e^{\frac{3\pi|v|}{2}}e^{-v^2}\frac{\mathrm{d}v}{1+|v|} \\
&+yk^{-3}\int_{|v|\ge\frac{k}{4}}\left(1+\frac{2|v|}{k}\right)^{\frac{3k}{2}} e^{\frac{3\pi|v|}{2}}e^{-v^2}\frac{\mathrm{d}v}{1+|v|} \\
&\ll_f yk^{-3}\int_{|v|\le\frac{k}{4}} e^{3|v|}e^{\frac{3\pi|v|}{2}}e^{-v^2}\frac{\mathrm{d}v}{1+|v|} + yk^{-3}\int_{|v|\ge\frac{k}{4}}\left(\frac{6|v|}{k}\right)^{\frac{3k}{2}} e^{-v^2+6v}\frac{\mathrm{d}v}{|v|} \\
&\ll_f yk^{-3}.
\end{aligned} \tag{4.21}$$

By (4.13), (4.16) and (4.21), we conclude that

$$V(y,k) = \log\left(k^3/y\right) + c_0 + O_f\left(\frac{y}{k^3}+k^{-1}\right),$$

where $c_0 = -6\log 2 - 3\log\pi$. This proves (4.12).

To prove (4.11), we move the line of integration in (4.6) to $\mathrm{Re}\, u = A$ to get

$$V(y,k) = \frac{1}{2\pi i}\int_{(A)} y^{-u} G(u) \frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)} \frac{\mathrm{d}u}{u^2}.$$

By repeating the above arguments, we get

$$\left|\frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)}\right| \ll_f k^{3A}\left(1+\frac{2|v|}{k}\right)^{-\frac{9}{2}+\frac{3k}{2}} e^{\frac{3\pi|v|}{2}}$$

and

$$V(y,k) \ll_f \left(\frac{k^3}{y}\right)^{A}. \qquad \square$$

5. An application of the Petersson formula

Applying Lemma 4.1 and the Petersson formula in (2.1) we obtain

$$\begin{aligned} &\sum_{k\equiv 2 (\mathrm{mod}\, 4)} h\left(\frac{k-1}{K}\right) \sum_{g\in\mathcal{B}_k} \omega_g^{-1} L'\left(\frac{1}{2}, f\times g\right) \\ &= \sum_{k\equiv 2 (\mathrm{mod}\, 4)} h\left(\frac{k-1}{K}\right) \sum_{g\in\mathcal{B}_k} \omega_g^{-1} \left\{ 2\sum_{m\geq 1}\sum_{n\geq 1} \frac{\lambda_g(n)A(m,n)}{(m^2n)^{1/2}} V\left(m^2n,k\right)\right\} \\ &= 2\sum_{k\equiv 2 (\mathrm{mod}\, 4)} h\left(\frac{k-1}{K}\right) \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{(m^2n)^{1/2}} V\left(m^2n,k\right)\left\{\sum_{g\in\mathcal{B}_k}\omega_g^{-1}\lambda_g(n)\right\} \\ &= \mathcal{D} + \mathcal{ND} \end{aligned}$$

where $\mathcal{D}$ is the contribution from the diagonal term with

$$\mathcal{D} = 2\sum_{k\equiv 2 (\mathrm{mod}\, 4)} h\left(\frac{k-1}{K}\right)\sum_{m\geq 1}\frac{A(m,1)}{m} V\left(m^2,k\right), \tag{5.1}$$

and $\mathcal{ND}$ is the contribution from the non-diagonal term with

$$\begin{aligned} \mathcal{ND} = -4\pi &\sum_{m\geq 1}\sum_{n\geq 1}\frac{A(m,n)}{(m^2n)^{\frac{1}{2}}}\sum_{c\geq 1} c^{-1} S(n,1;c) \\ &\times \sum_{k\equiv 2 (\mathrm{mod}\, 4)} h\left(\frac{k-1}{K}\right) V\left(m^2n,k\right) J_{k-1}\left(\frac{4\pi\sqrt{n}}{c}\right). \end{aligned} \tag{5.2}$$

Then Theorem 1.1 follows from

$$\mathcal{D} = \frac{3L(1,f)}{2} K\log K\widehat{h}(0) + \frac{2L'(1,f)+c_0L(1,f)}{2}K\widehat{h}(0) + O_f(1) \tag{5.3}$$

and

$$\mathcal{ND} = O_{\epsilon,f}\left(K^{\epsilon}\right). \tag{5.4}$$

We will establish (5.3) in Section 6 and prove (5.4) in Sections 7 and 8.

6. The diagonal term

In this section, we will estimate $\mathcal{D}$ in (5.1). We have

$$\mathcal{D} = 2 \sum_{k\equiv 2 (\operatorname{mod} 4)} h\left(\frac{k-1}{K}\right) \triangle(k), \tag{6.1}$$

where

$$\triangle(k) = \sum_{m\geq 1} \frac{A(m,1)}{m} V\left(m^2, k\right). \tag{6.2}$$

We first compute $\triangle(k)$ in (6.2). By the definition of $V(y,k)$ in (4.6), we have

$$\begin{aligned}\triangle(k) &= \frac{1}{2\pi i}\int_{(3)} \left(\sum_{m\geq 1} \frac{A(m,1)}{m^{1+2u}}\right) G(u) \frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)} \frac{\mathrm{d}u}{u^2} \\ &= \frac{1}{2\pi i}\int_{(3)} L\left(1+2u,f\right) G(u) \frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)} \frac{\mathrm{d}u}{u^2},\end{aligned}$$

where $L(s,f)$ is defined in (3.2). Moving the line of integration to $\operatorname{Re} u = -1/2$, picking up a double pole at $u=0$, by the residue theorem we have

$$\begin{aligned}\triangle(k) = {} & \operatorname{res}_{u=0}\left(L\left(1+2u,f\right) \frac{G(u)}{u^2} \frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)}\right) \\ & + \frac{1}{2\pi i}\int_{\left(-\frac{1}{2}\right)} L\left(1+2u,f\right) G(u) \frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)} \frac{\mathrm{d}u}{u^2}.\end{aligned} \tag{6.3}$$

By (4.15) we have

$$\begin{aligned}&\operatorname{res}_{u=0}\left(L\left(1+2u,f\right) \frac{G(u)}{u^2} \frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)}\right) \\ &= \lim_{u\to 0} \frac{\mathrm{d}}{\mathrm{d}u}\left(L\left(1+2u,f\right) G(u) \frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)}\right) \\ &= 3L(1,f)\log k + 2L'(1,f) + c_0 L(1,f) + O_f(k^{-1}),\end{aligned} \tag{6.4}$$

where $c_0 = -6\log 2 - 3\log \pi$. By the convexity bound for $L(s,f)$,

$$L(\sigma+it,f) \ll_f (1+|t|)^{\frac{3(1-\sigma)}{2}}, \qquad 0\leq \sigma \leq 1,$$

the integral in (6.3) is

$$\frac{1}{2\pi i}\int_{\left(-\frac{1}{2}\right)} L\left(1+2u,f\right) G(u) \frac{\gamma\left(\frac{1}{2}+u,k\right)}{\gamma\left(\frac{1}{2},k\right)} \frac{\mathrm{d}u}{u^2} \ll_f k^{-\frac{3}{2}} \int_{-\infty}^{\infty} (1+|t|)^{\frac{3}{2}} e^{-t^2} \mathrm{d}t \ll_f k^{-\frac{3}{2}}. \tag{6.5}$$

By (6.3)-(6.5), we have

$$\triangle(k) = 3L(1,f)\log k + 2L'(1,f) + c_0 L(1,f) + O_f(k^{-1}). \tag{6.6}$$

By (6.1) and (6.6), we have

$$\begin{aligned}\mathcal{D} &= 2\sum_{k\equiv 2(\bmod 4)} h\left(\frac{k-1}{K}\right)\left(3L(1,f)\log k + 2L'(1,f) + c_0L(1,f) + O_f(k^{-1})\right)\\ &= \frac{3L(1,f)}{2}K\log K\widehat{h}(0) + \frac{2L'(1,f)+c_0L(1,f)}{2}K\widehat{h}(0) + O_f\left(1\right).\end{aligned}$$

Here we have used the fact that (see p. 88, (5.78) in Iwaniec [6])

$$4\sum_{k\equiv 2(\bmod 4)} h\left(\frac{k-1}{K}\right) = K\widehat{h}(0) + O\left(K^{-A}\right)$$

for any $A>0$.

7. Averaging of J-Bessel functions

In this section, we begin to estimate $\mathcal{ND}$ in (5.2). Let $\omega(x)$ be a smooth function of compact support on $[1,2]$. Then by Lemma 4.1, we only need to estimate

$$\begin{aligned}\mathcal{R} = &\sum_{m\geq 1}\sum_{n\geq 1}\frac{A(m,n)}{(m^2n)^{\frac{1}{2}}}\omega\left(\frac{m^2n}{N}\right)\sum_{c\geq 1}c^{-1}S(n,1;c)\\ &\times\sum_{k\equiv 2(\bmod 4)} h\left(\frac{k-1}{K}\right)V\left(m^2n,k\right)J_{k-1}\left(\frac{4\pi\sqrt{n}}{c}\right),\end{aligned}\tag{7.1}$$

with $N\leq K^{3+\epsilon}$ for any $\epsilon>0$.

The following result is Proposition 8.1 in Iwaniec, Luo and Sarnak [7].

Lemma 7.1. *Fix a real valued function $h\in\mathcal{C}_0^\infty(\mathbb{R}^+)$ and $K\geq 1$. Then*

$$4\sum_{k\equiv 2(\bmod 4)} h\left(\frac{k-1}{K}\right)J_{k-1}(x) = h\left(\frac{x}{K}\right) + \frac{K}{\sqrt{x}}\mathrm{Im}\left(e^{ix-\frac{i\pi}{4}}\mathcal{H}\left(\frac{K^2}{2x}\right)\right) + O\left(\frac{x}{K^3}\right),$$

where

$$\mathcal{H}(v) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}e^{iuv}\mathrm{d}u.$$

Applying Lemma 7.1 for $x=\frac{4\pi\sqrt{n}}{c}$ we have

$$\begin{aligned}&4\sum_{k\equiv 2(\bmod 4)} h\left(\frac{k-1}{K}\right)V\left(m^2n,k\right)J_{k-1}(x)\\ &= h\left(\frac{x}{K}\right)V\left(m^2n,x+1\right) + \frac{K}{\sqrt{x}}\mathrm{Im}\left(e^{ix-\frac{i\pi}{4}}\mathcal{H}\left(\frac{K^2}{2x}\right)\right) + O_f\left(\frac{x}{K^3}\right),\end{aligned}\tag{7.2}$$

where

$$\mathcal{H}(v) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}V\left(m^2n,\sqrt{u}K+1\right)e^{iuv}\mathrm{d}u.$$

By partial integration many times, we have

$$\mathcal{H}(v) \ll_{f,A,B}\left(\frac{K^3}{m^2n}\right)^B\left(1+|v|\right)^{-A},\tag{7.3}$$

for any $A > 0$. By Weil's bound for the Kloosterman sum, $|S(n, 1; c)| \leq c^{1/2}\tau(c)$. Thus, by (3.1), the contribution from the error term in (7.2) to $\mathcal{R}$ in (7.1) is

$$\begin{aligned} &\sum_{m\geq 1}\sum_{n\geq 1}\frac{|A(m,n)|}{(m^2n)^{\frac{1}{2}}}\omega\left(\frac{m^2n}{N}\right)\sum_{c\geq 1}c^{-1}c^{\frac{1}{2}}\tau(c)K^{-3}\frac{4\pi\sqrt{n}}{c} \\ &\ll_f K^{-3}\sum_{m\geq 1}m^{-1}\sum_{n\geq 1}|A(m,n)|\,\omega\left(\frac{m^2n}{N}\right)\sum_{c\geq 1}c^{-\frac{3}{2}+\epsilon} \\ &\ll_f NK^{-3}\sum_{m\ll\sqrt{N}}m^{-2} \\ &\ll_f K^{\epsilon}, \end{aligned} \tag{7.4}$$

for any $\epsilon > 0$. Then by (7.1), (7.2) and (7.4), we are led to estimate

$$\mathcal{R}_1 = \sum_{m\geq 1}\sum_{n\geq 1}\frac{A(m,n)}{(m^2n)^{\frac{1}{2}}}\,\omega\left(\frac{m^2n}{N}\right)\sum_{c\geq 1}c^{-1}S(n,1;c) \times h\left(\frac{4\pi\sqrt{n}}{cK}\right)V\left(m^2n,\frac{4\pi\sqrt{n}}{c}+1\right), \tag{7.5}$$

$$\mathcal{R}_2 = \sum_{m\geq 1}\sum_{n\geq 1}\frac{A(m,n)}{(m^2n)^{\frac{1}{2}}}\,\omega\left(\frac{m^2n}{N}\right)\sum_{c\geq 1}c^{-1}S(n,1;c) \times \frac{K\sqrt{c}}{n^{\frac{1}{4}}}\mathrm{Im}\left(e^{i\frac{4\pi\sqrt{n}}{c}-\frac{i\pi}{4}}\mathcal{H}\left(\frac{K^2c}{8\pi\sqrt{n}}\right)\right), \tag{7.6}$$

where $N \leq K^{3+\epsilon}$ for any $\epsilon > 0$.

One can show that $\mathcal{R}_2$ in (7.6) is negligible. In fact,

$$\frac{K^2c}{8\pi\sqrt{n}} \gg K^2N^{-\frac{1}{2}} \gg K^{\frac{1}{2}-\epsilon}$$

for any $\epsilon > 0$; thus by (7.3), $\mathcal{R}_2$ is negligible. We will show in Section 8 that $\mathcal{R}_1$ is also negligible by using the Voronoi formula for $GL(3)$. Then (5.4) follows.

8. The non-diagonal term

Recall that the estimation of the non-diagonal term $\mathcal{ND}$ is reduced to bounding $\mathcal{R}_1$ in (7.5). Note that

$$1 \leq \frac{4\pi\sqrt{n}}{cK} \leq 2, \qquad 1 \leq \frac{m^2n}{N} \leq 2.$$

Thus

$$\frac{2\pi\sqrt{N}}{Km} \leq c \leq \frac{4\pi\sqrt{2N}}{Km}. \tag{8.1}$$

Opening the Kloosterman sum in (7.5), we have

$$\begin{aligned}\mathcal{R}_1 &= \sum_{m\geq 1}\sum_{n\geq 1}\frac{A(m,n)}{(m^2n)^{\frac{1}{2}}}\,\omega\left(\frac{m^2n}{N}\right)\sum_{c}c^{-1}\sum_{d\bar{d}\equiv 1(\mathrm{mod}\, c)}e\left(\frac{d+n\bar{d}}{c}\right)\\ &\quad\times h\left(\frac{4\pi\sqrt{n}}{cK}\right)V\left(m^2n,\frac{4\pi\sqrt{n}}{c}+1\right)\\ &= \sum_{m\geq 1}m^{-1}\sum_{c}c^{-1}\sum_{d\bar{d}\equiv 1(\mathrm{mod}\, c)}e\left(\frac{d}{c}\right)\left\{\sum_{n\geq 1}A(m,n)e\left(\frac{n\bar{d}}{c}\right)\psi(n)\right\},\end{aligned}\tag{8.2}$$

where

$$\psi(y)=y^{-\frac{1}{2}}\,\omega\left(\frac{m^2y}{N}\right)h\left(\frac{4\pi\sqrt{y}}{cK}\right)V\left(m^2y,\frac{4\pi\sqrt{y}}{c}+1\right).$$

Applying the Voronoi formula in Lemma 3.1 for the n-sum, we have

$$\begin{aligned}&\sum_{n\geq 1}A(m,n)e\left(\frac{n\bar{d}}{c}\right)\psi(n)\\ &\quad=\frac{c\pi^{-\frac{5}{2}}}{4i}\sum_{n_1|cm}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1n_2}S(md,n_2;mcn_1^{-1})\Psi_{0,1}^0\left(\frac{n_2n_1^2}{c^3m}\right)\\ &\quad+\frac{c\pi^{-\frac{5}{2}}}{4i}\sum_{n_1|cm}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1n_2}S(md,-n_2;mcn_1^{-1})\Psi_{0,1}^1\left(\frac{n_2n_1^2}{c^3m}\right),\end{aligned}\tag{8.3}$$

where $\Psi_{0,1}^0(x)$ and $\Psi_{0,1}^1(x)$ are defined in (3.5) and (3.6), respectively. We only need to treat the terms involving $\Psi_0(x)$ since other terms can be treated similarly. By (8.2) and (8.3), we need to estimate

$$\begin{aligned}\mathcal{R}_{1,0}&=\frac{\pi^{-\frac{5}{2}}}{4i}\sum_{m\geq 1}m^{-1}\sum_{c}\sum_{d\bar{d}\equiv 1(\mathrm{mod}\, c)}e\left(\frac{d}{c}\right)\\ &\quad\times\sum_{n_1|cm}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1n_2}S(md,n_2;mcn_1^{-1})\Psi_0\left(\frac{n_2n_1^2}{c^3m}\right),\end{aligned}$$

where the summation over c is as in (8.1). Now

$$\frac{n_2n_1^2}{c^3m}\frac{N}{m^2}=N\frac{n_2n_1^2}{(cm)^3}\gg N\left(\frac{K}{\sqrt{N}}\right)^3\gg K^{3/2-\epsilon}$$

for any $\epsilon>0$, and by Lemma 3.2 for $x=\frac{n_2n_1^2}{c^3m}$,

$$\begin{aligned}\Psi_0(x)=&2\pi^4xi\int_0^\infty\psi(y)\sum_{j=1}^{M}\frac{c_j\cos\left(6\pi x^{\frac{1}{3}}y^{\frac{1}{3}}\right)+d_j\sin\left(6\pi x^{\frac{1}{3}}y^{\frac{1}{3}}\right)}{(\pi^3xy)^{\frac{j}{3}}}\mathrm{d}y\\ &+O\left(\left(\frac{n_2n_1^2}{c^3m}\frac{N}{m^2}\right)^{\frac{-M+2}{3}}\right),\end{aligned}$$

where c_j and d_j are constants depending only on f, in particular, $c_1 = 0$, $d_1 = -\frac{2}{\sqrt{3}\pi}$. Denote

$$\Psi_0^j(x) = 2\pi^4 xi \int_0^\infty \psi(y) \frac{c_j \cos\left(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}}\right) + d_j \sin\left(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}}\right)}{(\pi^3 xy)^{\frac{j}{3}}} \mathrm{d}y.$$

Then

$$\Psi_0(x) = \sum_{j=1}^{M} \Psi_0^j(x) + O\left(\left(\frac{n_2 n_1^2}{c^3 m}\frac{N}{m^2}\right)^{\frac{-M+2}{3}}\right),$$

taking $M = 8$. Then the contribution from the O-term above to $\mathcal{R}_{1,0}$ is negligible. Now we estimate $\Psi_0^1(x)$:

$$\begin{aligned} \Psi_0^1(x) &= 2\pi^4 xi \int_0^\infty \psi(y) \frac{d_1 \sin\left(6\pi x^{\frac{1}{3}} y^{\frac{1}{3}}\right)}{(\pi^3 xy)^{\frac{1}{3}}} \mathrm{d}y \\ &= \pi^3 x^{\frac{2}{3}} d_1 \int_0^\infty \psi(y) \left(e\left(3x^{\frac{1}{3}} y^{\frac{1}{3}}\right) - e\left(-3x^{\frac{1}{3}} y^{\frac{1}{3}}\right)\right) y^{-\frac{1}{3}} \mathrm{d}y \\ &= \pi^3 x^{\frac{2}{3}} d_1 \int_0^\infty e\left(b_1(y)\right) a(y) \mathrm{d}y - \pi^3 x^{\frac{2}{3}} d_1 \int_0^\infty e\left(b_2(y)\right) a(y) \mathrm{d}y, \end{aligned}$$

where

$$b_1(y) = 3x^{\frac{1}{3}} y^{\frac{1}{3}}, \qquad b_2(y) = -3x^{\frac{1}{3}} y^{\frac{1}{3}},$$

and

$$a(y) = y^{-\frac{5}{6}} \varpi\left(\frac{m^2 y}{N}\right) h\left(\frac{4\pi\sqrt{y}}{cK}\right) V\left(m^2 y, \frac{4\pi\sqrt{y}}{c} + 1\right).$$

Since

$$b_i'(y) y \gg K^{\frac{1}{2}-\epsilon}, \quad i = 1, 2,$$

the contribution from $\Psi_0^1(x)$ to $\mathcal{R}_{1,0}$ is negligible by partial integration many times. Repeating the above arguments to $\Psi_0^j(x)$, $j = 2, \ldots, M$, one shows that the other terms are also negligible. Thus $\mathcal{R}_1$ is negligible. This finishes the proof of Theorem 1.1.

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