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THE DISK PROPERTY OF COVERINGS OF 1-CONVEX SURFACES

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Abstract. Let X be a 1-convex surface and $p : \tilde{X} \to X$ an (unbranched) covering map. We prove that if $\tilde{X}$ does not contain an infinite Nori string of rational curves, then $\tilde{X}$ satisfies the discrete disk property.

1. Introduction

Let X be a 1-convex surface, i.e. a two-dimensional complex manifold which is strongly pseudoconvex. Then X is a proper modification of a 2-dimensional normal Stein space at a finite set of points. Let $p : \tilde{X} \to X$ be an (unbranched) covering map. In this paper we are interested in studying the convexity properties of $\tilde{X}$.

It was remarked in [1] that in general $\tilde{X}$ is not weakly 1-complete (i.e. it might be possible that there is no plurisubharmonic exhaustion function on $\tilde{X}$). This is due to the fact that $\tilde{X}$ might contain an infinite Nori string (necklace), that is, a 1-dimensional connected complex subspace which is non-compact and has infinitely many compact irreducible components. However the main result in [1] shows that $\tilde{X}$ can be exhausted by a sequence of relatively compact strongly pseudoconvex domains with smooth boundary ($\tilde{X}$ is p_3-convex in the terminology of [4]). In particular $\tilde{X}$ satisfies the continuous Kontinuitätssatz (the continuous disk property).

In this paper we investigate the discrete disk property for $\tilde{X}$. The main result can be stated as follows (see Theorem 3.4): *If $\tilde{X}$ does not contain an infinite Nori string of rational curves, then $\tilde{X}$ satisfies the discrete disk property.*

It should be noted that the discrete disk property is a much stronger condition than the continuous one.

2. Preliminaries

All complex spaces are assumed of bounded dimension and countable at infinity.

Let X be a complex manifold. We recall that X is said to be 1-convex if there exists a $\mathcal{C}^\infty$ function $\phi : X \to \mathbb{R}$ such that:

a) ϕ is an exhaustion function (i.e. $\{\phi < c\} \Subset X$ for every $c \in \mathbb{R}$),

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b) ϕ is strongly plurisubharmonic (its Levi form is positive definite) outside a compact subset of X.

It is known (see [9]) that this is equivalent to the following condition: there exists a proper surjective holomorphic map $\pi : X \to Y$ where Y is a normal Stein space with finitely many singular points and there exists a finite subset of Y, B, such that π induces a biholomorphism from $X \setminus \pi^{-1}(B)$ to $Y \setminus B$. $\pi^{-1}(B)$ is called the exceptional set of X.

A complex space X is called holomorphically convex if for every discrete sequence $\{x_n\}$ in X there exists a holomorphic function $f : X \to \mathbb{C}$ such that $\lim_{n\to\infty} |f(x)| = \infty$.

We denote by Δ the unit disk in $\mathbb{C}$, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and for $\epsilon > 0$ by $\Delta_{1+\epsilon}$ the disk $\Delta_{1+\epsilon} := \{z \in \mathbb{C} : |z| < 1 + \epsilon\}$.

Definition 2.1. Suppose that X is a complex space. We say that X satisfies the discrete disk property if whenever $f_n : U \to X$ is a sequence of holomorphic functions defined on an open neighborhood U of $\overline{\Delta}$ for which there exists an $\epsilon > 0$ and a continuous function $\gamma : S^1 = \{z \in \mathbb{C} : |z| = 1\} \to X$ such that $\Delta_{1+\epsilon} \subset U$, $\bigcup_{n\geq 1} f_n(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in X and $f_{n|S^1}$ converges uniformly to γ, we have that $\bigcup_{n\geq 1} f_n(\overline{\Delta})$ is relatively compact in X.

This definition of the disk property using coronae instead of the boundary $\partial\Delta$ of the unit disk is natural, as one sees by looking at the example $f_n : \mathbb{C} \to \mathbb{C}$, $f_n(z) = z^n$, which is a sequence that converges to 0 if $|z| < 1$ and diverges otherwise.

For $\epsilon > 0$ we define $H_\epsilon \subset \mathbb{C} \times \mathbb{R}$ as

$$H_\epsilon = \Delta_{1+\epsilon} \times [0, 1) \cup \{z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon\} \times \{1\}.$$

Having in mind the definition of the continuity principle (see for example [7], page 47) we introduce the following:

Definition 2.2. A complex space X is said to satisfy the continuous disk property if whenever ϵ is a positive number and $F : H_\epsilon \to X$ is a continuous function such that, for every $t \in [0, 1)$, $F_t : \Delta_{1+\epsilon} \to X$, $F_t(Z) = F(z, t)$ is holomorphic, we have that $F(H_{\epsilon_1})$ is relatively compact in X for any $0 < \epsilon_1 < \epsilon$.

Clearly the discrete disk property implies the continuous one, but as is shown by the example of Fornaess (see [6]) of an increasing union of Stein open subsets that does not satisfy the discrete disk property, the discrete disk property is a much stronger condition. (It is not difficult to see that an increasing union of open subsets of a complex space X, each one of them satisfying the continuous disk property, satisfies the continuous disk property. The proof is pretty much the same as the proof of Theorem 3.1, page 60, in [7].)

The following theorem was proved in [2] and [8].

Theorem 2.3. *Let $\pi : X \to T$ be a proper holomorphic surjective map of complex spaces, let $t_0 \in T$ be any point, and denote by $X_{t_0} := \pi^{-1}(t_0)$ the fiber of π at t_0. Assume that $\dim X_{t_0} = 1$. Let $\sigma : \tilde{X} \to X$ be a covering space and let $\tilde{X}_{t_0} = \sigma^{-1}(X_{t_0})$. If $\tilde{X}_{t_0}$ is holomorphically convex, then there exists an open neighborhood Ω of t_0 such that $(\pi \circ \sigma)^{-1}(\Omega)$ is holomorphically convex.*

The next result was proved in [1].

Proposition 2.4. *Let X be a 1-convex manifold with the exceptional set S and let $p : \tilde{X} \to X$ be any covering. Then there exists a strongly plurisubharmonic function*

$\tilde{\phi} : \tilde{X} \to [-\infty, \infty)$ such that $p^{-1}(S) = \{\tilde{\phi} = -\infty\}$, and for any open neighborhood U of S, the restriction $\tilde{\phi}_{|\tilde{X}\setminus p^{-1}(U)}$ is an exhaustion function on $\tilde{X} \setminus p^{-1}(U)$.

We recall that a topological space X is called an ENR (Euclidean Neighborhood Retract) if it is homeomorphic to a closed subset X_1 of $\mathbb{R}^n$ for some n such that there is an open neighborhood V of X_1 and a continuous retract $\rho : V \to X_1$. For basic results regarding ENR's we refer to [5]. It is proved there that a locally compact and locally contractible subset of $\mathbb{R}^n$ is and ENR and that if a Hausdorff topological space X is covered by countable family of locally compact open subsets each one of them homeomorphic with a subset of fixed Euclidean space, then X is an ENR. In particular, every complex space of bounded dimension is an ENR.

Lemma 2.5 was proved in [3].

Lemma 2.5. *If X is an ENR and $\{\gamma_n\}_{n\geq 1}$, $\gamma_n : S^1 :\to X$, is a sequence of null-homotopic loops converging uniformly to $\gamma : S^1 \to X$, then γ is null-homotopic. Moreover, given a covering $p : \tilde{X} \to X$ there exist liftings $\tilde{\gamma}_n$ for γ_n and $\tilde{\gamma}$ for γ such that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$.*

This lemma can be applied in particular if X is a complex space (by the above discussion).

Definition 2.6. Let L be a connected 1-dimensional complex space and $\bigcup L_i$ be its decomposition into irreducible components. L is called an infinite Nori string if all L_i are compact and L is not compact

It is clear that a 1-dimensional complex space is holomorphically convex if and only if it does not contain an infinite Nori string.

We recall that a compact complex curve is called rational if its normalization is $\mathbb{P}^1$.

3. The results

Lemma 3.1. *If L is a 1-dimensional complex space that does not contain an infinite Nori string of rational curves, then L has a holomorphically convex covering space.*

Proof. Step 1. We assume that L is connected and all its irreducible components are compact not rational. Let $L = \bigcup_{i\in I} L_i$ be the decomposition of L into irreducible components and $A = \{x \in L : \exists i_1, i_2 \in I,\ i_1 \neq i_2,$ such that $x \in L_{i_1} \cap L_{i_2}\}$. Let $p_i : \tilde{L}_i \to L_i$ be the universal cover of L_i. Since L_i is not rational for all $i \in I$ we have that $\tilde{L}_i$ is Stein. For each $x \in A \cap L_i$, we choose a bijection $\tau(x, i) : \mathbb{Z} \to p_i^{-1}(x)$ and we write $\tau(x, i)(k) = (k, x, i)$.

On the disjoint union $S := \bigsqcup_{i\in I} \tilde{L}_i$ we define the projection $\tilde{p} : S \to L$ induced by p_i, $i \in I$. We also define an equivalence relation on S by defining the equivalence classes as follows: if $s \in S$ is such that $\tilde{p}(s) \notin A$, then $\hat{s} = \{s\}$. Note that on $\tilde{p}^{-1}(A)$ we have a map $\zeta : \tilde{p}^{-1}(A) \to \mathbb{Z}$ which is nothing other than the projection on the first component. Then if $\tilde{p}(s) \in A$ and $\zeta(s) = k$ we set $\hat{s} = \{(k, x, i) : x \in L_i, i \in I, \tilde{p}(s) = x\}$. We let $\tilde{L}$ be the quotient space of S and $p : \tilde{L} \to L$ the application induced by $\tilde{p}$. It is not difficult to see that p is a covering and $\tilde{L}$ is Stein.

Step 2. All irreducible components of L are compact and not rational. In this case we apply the above procedure to each connected component.

Step 3. The general case. Let L' be the union of all irreducible components of L that are compact and not rational and let $\{C_j\}_{j\in J}$ be the connected components

of L'. For each $j \in J$ let $p_j : \tilde{C}_j \to C_j$ be the Stein coverings obtained at the first step. Also, let L'' be the union of the other irreducible components of L. We set $\Gamma_j = L'' \cap C_j$. (Note that Γ_j might be empty.) If $\Gamma_j \neq \emptyset$ we consider in $\tilde{C}_j$ infinitely many disjoint copies $\{\Gamma_j^\alpha : \alpha \in \mathbb{Z}\}$ of Γ_j. (Each Γ_j^α is in bijection via p_j with Γ_j.) We now glue infinitely many copies $\{L''_\alpha : \alpha \in \mathbb{Z}\}$ of L'' to $\tilde{C}_j$ on $\{\Gamma_j^\alpha : \alpha \in \mathbb{Z}\}$, and we obtain in this way a holomorphically convex covering of L. □

Lemma 3.2. *Suppose that $\tilde{X}$ and X are Hausdorff topological spaces and $p : \tilde{X} \to X$ is a covering. Let $\tilde{\gamma}_n : S^1 \to \tilde{X}$, $\gamma_n : S^1 \to X$, $n \geq 1$, $\tilde{\gamma} : S^1 \to \tilde{X}$, $\gamma : S^1 \to X$ be continuous functions such that $\gamma_n = p \circ \tilde{\gamma}_n$, $\gamma = p \circ \tilde{\gamma}$ and $a \in S^1$ be a fixed point. If γ_n converges uniformly to γ and $\tilde{\gamma}_n(a)$ converges to $\tilde{\gamma}(a)$, then $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$.*

Proof. Let $\Omega = \{t \in S^1 : \lim_{n\to\infty} \tilde{\gamma}_n(t) = \tilde{\gamma}(t)\}$. We will show first that $\Omega = S^1$. As $a \in \Omega$ we have that $\Omega \neq \emptyset$. We prove that Ω is open in S^1. Let $t_0 \in \Omega$ and let V_0 a neighborhood of $\gamma(t_0)$ which is evenly covered by $\bigcup \tilde{V}_k$. Let Ω_0 be a connected open neighborhood of t_0 such that for n large enough $\gamma_n(\Omega_0) \subset V_0$ and $\gamma(\Omega_0) \subset V_0$. It follows that $\tilde{\gamma}_n(\Omega_0) \subset \bigcup \tilde{V}_k$. As $\tilde{\gamma}_n(\Omega_0)$ and $\tilde{\gamma}(\Omega_0)$ are connected we have that for each n there exists k_n such that $\tilde{\gamma}_n(\Omega_0) \subset \tilde{V}_{k_n}$ and there exists k_0 such that $\tilde{\gamma}(\Omega_0) \subset \tilde{V}_{k_0}$. However, we assumed that $t_0 \in \Omega$, and therefore for n large enough $\tilde{\gamma}_n(t_0) \in \tilde{V}_{k_0}$, hence $V_{k_n} = V_{k_0}$. Since $p : V_{k_0} \to V_0$ is a homeomorphism we conclude that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on Ω_0. This shows that $\Omega_0 \subset \Omega$, and hence Ω is open.

Next we show in a similar fashion that Ω is closed. Let $t_0 \in S^1 \setminus \Omega$. As before we choose V_1 to be a neighborhood of $\gamma(t_0)$ which is evenly covered by $\bigcup \tilde{V}_k$, Ω_0 a connected open neighborhood of t_0 such that for n large enough $\gamma_n(\Omega_0) \subset V_0$, k_n and k_0 such that $\tilde{\gamma}_n(\Omega_0) \subset \tilde{V}_{k_n}$ and $\tilde{\gamma}(\Omega_0) \subset \tilde{V}_{k_0}$. If $V_{k_n} = V_{k_0}$ for every n we would have that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on Ω_0 and hence at t_0, and this would contradict $t_0 \notin \Omega$. There then exists a subsequence $\tilde{\gamma}_{n_p}$ such that $\tilde{\gamma}_{n_p}(\Omega_0) \cap \tilde{V}_{k_0} = \emptyset$, and from here we get that $\Omega_0 \subset S^1 \setminus \Omega$. Hence Ω is closed, and therefore $\Omega = S^1$.

Note that when we proved that Ω is open we actually proved that each $t_0 \in \Omega = S^1$ has an open neighborhood Ω_0 such that on this neighborhood $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$. The compactness of S^1 then implies that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on S^1. □

Lemma 3.3. *Suppose that S is a complex space that has a holomorphically convex covering $p : \tilde{S} \to S$. If $\Omega \subset \mathbb{C}$ is an open neighborhood of $\overline{\Delta}$ and $f_n : \Omega \to S$ is a sequence of holomorphic functions which converges uniformly on $\{z \in \mathbb{C} : |z| = 1\}$, then $\bigcup_n f_n(\overline{\Delta})$ is relatively compact in S. (In particular, S satisfies the discrete disk property.)*

Proof. Let $\gamma : \{z \in \mathbb{C} : |z| = 1\} \to S$ be the limit of $\{f_n\}$ on $\{z \in \mathbb{C} : |z| = 1\}$. It follows from Lemma 2.5 that γ is null-homotopic. We choose a point $a \in \{z \in \mathbb{C} : |z| = 1\}$. Let ϵ be a positive number such that $\Delta_{1+\epsilon} \subset \Omega$, $\tilde{f}_n : \Delta_{1+\epsilon} \to \tilde{S}$ and let $\tilde{\gamma} : \{z \in \mathbb{C} : |z| = 1\} \to \tilde{S}$ be liftings of f_n and γ respectively. We choose $\tilde{f}_n$ and $\tilde{\gamma}$ such that $\tilde{f}_n$ converges uniformly to $\tilde{\gamma}$ on $\{z \in \mathbb{C} : |z| = 1\}$. This is possible by Lemma 2.5. As $\tilde{S}$ is holomorphically convex this implies that $\bigcup_n \tilde{f}_n(\overline{\Delta})$ is relatively compact in $\tilde{S}$ and therefore $\bigcup_n f_n(\overline{\Delta})$ is relatively compact in S. □

Theorem 3.4. *Let X be a 1-convex surface and $p : \tilde{X} \to X$ be a covering map. If $\tilde{X}$ does not contain an infinite Nori string of rational curves, then $\tilde{X}$ satisfies the discrete disk property.*

Proof. Let L be the exceptional set of X. Without loss of generality we may assume that L is connected. Let Y be a normal Stein complex space of dimension 2 and let $\pi : X \to Y$ be a proper surjective holomorphic map such that $\pi(L) = \{y_0\}$ and $\pi : X \setminus L \to Y \setminus \{y_0\}$ is a biholomorphism. We may assume that Y is an analytic closed subset of $\mathbb{C}^N$.

We put $\tilde{L} := p^{-1}(L)$. Since we assumed that $\tilde{L}$ does not contain an infinite Nori string of rational curves, it follows from Lemma 3.1 that $\tilde{L}$ has a holomorphically convex covering $\hat{p} : \hat{L} \to \tilde{L}$. We choose U_1 an open neighborhood of L in X that has a continuous deformation retract on L. We let $\tilde{U}_1 = p^{-1}(U_1)$, and then $\tilde{U}_1$ has a continuous deformation retract on $\tilde{L}$, $\rho : \tilde{U}_1 \to \tilde{L}$. By considering the fiber product of ρ and $\hat{p}$ we obtain a covering $\hat{U}_1$ of $\tilde{U}$ such that the pull-back of $\tilde{L}$ is $\hat{L}$. We then apply Theorem 2.3 and we deduce that there exists $U \subset Y$ an open neighborhood of y_0 such that $(\pi \circ p)^{-1}(U)$ has a holomorphically convex covering.

Let $f_n : \Delta_{1+\epsilon} \to \tilde{X}$, $n \geq 1$, be a sequence of holomorphic functions such that $\bigcup_{n\geq 1} f_n(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in $\tilde{X}$ and $f_{n|S^1}$ is uniformly convergent. We argue by contradiction, and we assume that $\bigcup_{n\geq 1} f_n(\overline{\Delta})$ is not relatively compact in $\tilde{X}$ and hence that $\bigcup_{n\geq 1} f_n(\Delta_{1+\epsilon})$ is not relatively compact. By passing to a subsequence we may assume that $\cup_{k\geq 1} f_{n_k}(\Delta_{1+\epsilon})$ is not relatively compact in $\tilde{X}$ for every subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$.

Since $\bigcup(\pi \circ p \circ f_n)(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in Y, and hence in $\mathbb{C}^N$, by the maximum modulus principle we have that $\bigcup(\pi \circ p \circ f_n)(\Delta_{1+\epsilon})$ is relatively compact in Y. Then it follows that there exists a subsequence of $\{f_n\}_n$, $\{f_{n_k}\}_k$ such that $\pi \circ p \circ f_{n_k}$ converges uniformly on compacts to a holomorphic function $g : D_\epsilon \to Y$. Without loss of generality we can assume that $\pi \circ p \circ f_n$ converges uniformly on compacts to g. We distinguish three cases.

Case 1. $g \equiv y_0$. Then $(\pi \circ p \circ f_n)(\overline{\Delta}_{1+\epsilon_1}) \subset U$ for $\epsilon_1 \in (0, \epsilon)$ and n large enough. Therefore we can apply Lemma 3.3.

Case 2. $g(z) \neq y_0$ for every $z \in \Delta_{1+\epsilon}$. Then there exists a neighborhood V of y_0 and $\epsilon_1 \in (0, \epsilon)$ such that $g(\overline{\Delta}_{1+\epsilon_1}) \cap \overline{V} = \emptyset$. Then for n large enough we get that $\pi \circ p \circ f_n(\overline{\Delta}_{1+\epsilon_1}) \cap \overline{V} = \emptyset$ and hence $f_n(\overline{\Delta}_{1+\epsilon_1}) \cap (\pi \circ p)^{-1}(V) = \emptyset$. We consider now a plurisubharmonic function $\tilde{\phi}$ on $\tilde{X}$ with the properties given in Proposition 2.4. In particular, its restriction to $\tilde{X} \setminus (\pi \circ p)^{-1}(V)$ is an exhaustion. Applying the maximum principle to $\tilde{\phi} \circ f_n$ we obtain immediately that $\bigcup f_n(\overline{\Delta})$ is relatively compact.

Case 3. $g \not\equiv y_0$ and $y_0 \in g(\Delta_{1+\epsilon})$. Then $g^{-1}(y_0)$ is a (non-empty) discrete subset of $\Delta_{1+\epsilon}$, and therefore there exists $\epsilon_1 \in (0, \epsilon)$ such that $g^{-1}(y_0) \cap \{z \in \mathbb{C} : |z| = 1 + \epsilon_1\} = \emptyset$. We set $g^{-1}(y_0) \cap \{z \in \mathbb{C} : |z| < 1 + \epsilon_1\} = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$. Let $r_1, \ldots, r_s$ be positive numbers such that the discs $\overline{\Delta}_j = \{z \in \mathbb{C} : |z - \lambda_j| \leq r_j\}$, $j = 1, 2, \ldots, s$, are pairwise disjoint, $\overline{\Delta}_j \subset \Delta_{1+\epsilon_1}$ and $g(\overline{\Delta}_j) \subset U$. Let $V \subset Y$ be an open neighborhood of y_0 such that $\overline{V} \subset U$ and

$$g(\{z \in \overline{\Delta}_{1+\epsilon_1} : |z - \lambda_j| \geq r_j \ \forall j = 1, \ldots, s\}) \cap \overline{V} = \emptyset.$$

Then for n large enough we have that

$$(\pi \circ p \circ f_n)(\{z \in \overline{\Delta}_{1+\epsilon_1} : |z - \lambda_j| \geq r_j \ \forall j = 1, .., s\}) \cap \overline{V} = \emptyset.$$

We again apply Proposition 2.4 and we obtain a plurisubharmonic function $\tilde{\phi} : \tilde{X} \to [-\infty, \infty)$ such that $\tilde{L} = \{\tilde{\phi} = -\infty\}$ $\tilde{\phi}_{|\tilde{X}\setminus(\pi\circ p)^{-1}(V)}$ is an exhaustion function on $\tilde{X}\setminus(\pi\circ p)^{-1}(V)$. Since $\bigcup f_n(\Delta_{1+\epsilon}\setminus\Delta)$ is relatively compact there exists a positive constant M such that $\tilde{\phi}\circ f_n(z) \leq M$ for every n and every $z \in \Delta_{1+\epsilon} \setminus \Delta$. From the plurisubharmonicity of $\tilde{\phi}$ we have that $\tilde{\phi} \circ f_n(z) \leq M$ for every $z \in \Delta_{1+\epsilon}$, hence in particular for $z \in \overline{\Delta}_{1+\epsilon_1} \setminus \bigcup_{j=1}^s \Delta_j$. As $f_n(\overline{\Delta}_{1+\epsilon_1} \setminus \bigcup_{j=1}^s \Delta_j) \subset \tilde{X} \setminus (\pi \circ p)^{-1}(V)$ and $\tilde{\phi}_{|\tilde{X}\setminus(\pi\circ p)^{-1}(V)}$ is an exhaustion, we deduce that $\bigcup f_n(\overline{\Delta}_{1+\epsilon_1} \setminus \bigcup_{j=1}^s \Delta_j)$ is relatively compact in $\tilde{X}$.

For $j = 1, \ldots, s$ we set $S_j = \{z \in \Delta_{1+\epsilon_1} : |z - \lambda_j| = r_j\}$ and we pick a point $a_j \in S_j$. As $\bigcup_{n\geq 1} f_n(S_j)$ is relatively compact, by passing to a subsequence we can assume that each sequence $\{f_n(a_j)\}_n$ is convergent and we denote by x_j its limit. We have that $(\pi\circ p\circ f_n)_{|S_j}$ converges uniformly to $g_{|S_j}$ and $(\pi\circ p\circ f_n)(S_j) \subset Y\setminus\{y_0\}$, $g(S_j) \subset Y \setminus \{y_0\}$. Because $\pi : X \setminus L \to Y \setminus \{y_0\}$ is a biholomorphism we deduce that $(p \circ f_n)_{|S_j}$ converges uniformly to $(\pi^{-1} \circ g)_{|S_j}$. Note now that $(p \circ f_n)_{|S_j}$ is in fact a null-homotopic loop. It follows from Lemma 2.5 that $(\pi^{-1} \circ g)_{|S_j}$ is null-homotopic as well, and therefore there exists a loop $\gamma_j : S_j \to \tilde{X}$ such that $p \circ \gamma_j = (\pi^{-1} \circ g)$ on S_j. From Lemma 3.2 we conclude that f_n converges uniformly to γ_j on S_j. Finally, Lemma 3.3 implies that $\bigcup f_n(\overline{\Delta}_j)$ is relatively compact for every $j = 1, \ldots, s$. $\square$

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