THE DISK PROPERTY OF COVERINGS OF 1-CONVEX SURFACES

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Abstract. Let $X$ be a 1-convex surface and $p : \tilde{X} \to X$ an (unbranched) covering map. We prove that if $\tilde{X}$ does not contain an infinite Nori string of rational curves, then $\tilde{X}$ satisfies the discrete disk property.

1. Introduction

Let $X$ be a 1-convex surface, i.e. a two-dimensional complex manifold which is strongly pseudoconvex. Then $X$ is a proper modification of a 2-dimensional normal Stein space at a finite set of points. Let $p : \tilde{X} \to X$ be an (unbranched) covering map. In this paper we are interested in studying the convexity properties of $\tilde{X}$.

It was remarked in [1] that in general $\tilde{X}$ is not weakly 1-complete (i.e. it might be possible that there is no plurisubharmonic exhaustion function on $\tilde{X}$). This is due to the fact that $\tilde{X}$ might contain an infinite Nori string (necklace), that is, a 1-dimensional connected complex subspace which is non-compact and has infinitely many compact irreducible components. However the main result in [1] shows that $\tilde{X}$ can be exhausted by a sequence of relatively compact strongly pseudoconvex domains with smooth boundary ($\tilde{X}$ is $p_2$-convex in the terminology of [4]). In particular $\tilde{X}$ satisfies the continuous Kontinuitätssatz (the continuous disk property).

In this paper we investigate the discrete disk property for $\tilde{X}$. The main result can be stated as follows (see Theorem 3.4): If $\tilde{X}$ does not contain an infinite Nori string of rational curves, then $\tilde{X}$ satisfies the discrete disk property.

It should be noted that the discrete disk property is a much stronger condition than the continuous one.

2. Preliminaries

All complex spaces are assumed of bounded dimension and countable at infinity. Let $X$ be a complex manifold. We recall that $X$ is said to be 1-convex if there exists a $C^\infty$ function $\phi : X \to \mathbb{R}$ such that:

a) $\phi$ is an exhaustion function (i.e. $\{\phi < c\} \subseteq X$ for every $c \in \mathbb{R}$),
b) $\phi$ is strongly plurisubharmonic (its Levi form is positive definite) outside a compact subset of $X$.

It is known (see [9]) that this is equivalent to the following condition: there exists a proper surjective holomorphic map $\pi : X \to Y$ where $Y$ is a normal Stein space with finitely many singular points and there exists a finite subset of $Y$, $B$, such that $\pi$ induces a biholomorphism from $X \setminus \pi^{-1}(B)$ to $Y \setminus B$. $\pi^{-1}(B)$ is called the exceptional set of $X$.

A complex space $X$ is called holomorphically convex if for every discrete sequence $\{x_n\}$ in $X$ there exists a holomorphic function $f : X \to \mathbb{C}$ such that $\lim_{n \to \infty} |f(x_n)| = \infty$.

We denote by $\Delta$ the unit disk in $\mathbb{C}$, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and for $\epsilon > 0$ by $\Delta_{1+\epsilon}$ the disk $\Delta_{1+\epsilon} := \{z \in \mathbb{C} : |z| < 1 + \epsilon\}$.

**Definition 2.1.** Suppose that $X$ is a complex space. We say that $X$ satisfies the discrete disk property if whenever $f_n : U \to X$ is a sequence of holomorphic functions defined on an open neighborhood $U$ of $\Delta$ for which there exists an $\epsilon > 0$ and a continuous function $\gamma : S^1 = \{z \in \mathbb{C} : |z| = 1\} \to X$ such that $\Delta_{1+\epsilon} \subset U$, $\bigcup_{n \geq 1} f_n(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in $X$ and $f_n|_{S^1}$ converges uniformly to $\gamma$, we have that $\bigcup_{n \geq 1} f_n(\Delta)$ is relatively compact in $X$.

This definition of the disk property using coronae instead of the boundary $\partial \Delta$ of the unit disk is natural, as one sees by looking at the example $f_n : \mathbb{C} \to \mathbb{C}$, $f_n(z) = z^n$, which is a sequence that converges to $0$ if $|z| < 1$ and diverges otherwise.

For $\epsilon > 0$ we define $H_\epsilon = [0,1) \times \mathbb{C}$ as

$$H_\epsilon = \Delta_{1+\epsilon} \times [0,1) \cup \{z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon\} \times \{1\}.$$ 

Having in mind the definition of the continuity principle (see for example [7], page 47) we introduce the following:

**Definition 2.2.** A complex space $X$ is said to satisfy the continuous disk property if whenever $\epsilon$ is a positive number and $F : H_\epsilon \to X$ is a continuous function such that, for every $t \in [0,1)$, $F_t : \Delta_{1+\epsilon} \to X$, $F_t(z) = F(z,t)$ is holomorphic, we have that $F(H_\epsilon_t)$ is relatively compact in $X$ for any $0 < \epsilon_1 < \epsilon$.

Clearly the discrete disk property implies the continuous one, but as is shown by the example of Fornaess (see [3]) of an increasing union of Stein open subsets that does not satisfy the discrete disk property, the discrete disk property is a much stronger condition. (It is not difficult to see that an increasing union of open subsets of a complex space $X$, each one of them satisfying the continuous disk property, satisfies the continuous disk property. The proof is pretty much the same as the proof of Theorem 3.1, page 60, in [7].)

The following theorem was proved in [2] and [8].

**Theorem 2.3.** Let $\pi : X \to T$ be a proper holomorphic surjective map of complex spaces, let $t_0 \in T$ be any point, and denote by $X_{t_0} := \pi^{-1}(t_0)$ the fiber of $\pi$ at $t_0$. Assume that $\dim X_{t_0} = 1$. Let $\sigma : \hat{X} \to X$ be a covering space and let $\hat{X}_{t_0} = \sigma^{-1}(X_{t_0})$. If $\hat{X}_{t_0}$ is holomorphically convex, then there exists an open neighborhood $\Omega$ of $t_0$ such that $(\pi \circ \sigma)^{-1}(\Omega)$ is holomorphically convex.

The next result was proved in [1].

**Proposition 2.4.** Let $X$ be a 1-convex manifold with the exceptional set $S$ and let $p : \hat{X} \to X$ be any covering. Then there exists a strongly plurisubharmonic function
$\tilde{\phi}: \tilde{X} \to [-\infty, \infty)$ such that $p^{-1}(S) = \{\tilde{\phi} = -\infty\}$, and for any open neighborhood $U$ of $S$, the restriction $\tilde{\phi}|_{\tilde{X}\setminus p^{-1}(U)}$ is an exhaustion function on $\tilde{X}\setminus p^{-1}(U)$.

We recall that a topological space $X$ is called an ENR (Euclidean Neighborhood Retract) if it is homeomorphic to a closed subset $X_1$ of $\mathbb{R}^n$ for some $n$ such that there is an open neighborhood $V$ of $X_1$ and a continuous retract $p: V \to X_1$. For basic results regarding ENR’s we refer to [5]. It is proved there that a locally compact and locally contractible subset of $\mathbb{R}^n$ is and ENR and that if a Hausdorff topological space $X$ is covered by countable family of locally compact open subsets each one of them homeomorphic with a subset of fixed Euclidean space, then $X$ is an ENR. In particular, every complex space of bounded dimension is an ENR.

Lemma 2.5 was proved in [3].

**Lemma 2.5.** If $X$ is an ENR and $\{\gamma_n\}_{n \geq 1}$, $\gamma_n: S^1 \to X$, is a sequence of null-homotopic loops converging uniformly to $\gamma: S^1 \to X$, then $\gamma$ is null-homotopic. Moreover, given a covering $p: \tilde{X} \to X$ there exist liftings $\tilde{\gamma}_n$ for $\gamma_n$ and $\tilde{\gamma}$ for $\gamma$ such that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$.

This lemma can be applied in particular if $X$ is a complex space (by the above discussion).

**Definition 2.6.** Let $L$ be a connected 1-dimensional complex space and $\bigcup L_i$ be its decomposition into irreducible components. $L$ is called an infinite Nori string if all $L_i$ are compact and $L$ is not compact

It is clear that a 1-dimensional complex space is holomorphically convex if and only if it does not contain an infinite Nori string.

We recall that a compact complex curve is called rational if its normalization is $\mathbb{P}^1$.

### 3. The results

**Lemma 3.1.** If $L$ is a 1-dimensional complex space that does not contain an infinite Nori string of rational curves, then $L$ has a holomorphically convex covering space.

**Proof.** *Step 1.* We assume that $L$ is connected and all its irreducible components are compact not rational. Let $L = \bigcup_{i \in I} L_i$ be the decomposition of $L$ into irreducible components and $A = \{x \in L : \exists i_1, i_2 \in I, i_1 \neq i_2, \text{ such that } x \in L_{i_1} \cap L_{i_2}\}$. Let $p_i: \tilde{L}_i \to L_i$ be the universal cover of $L_i$. Since $L_i$ is not rational for all $i \in I$ we have that $\tilde{L}_i$ is Stein. For each $x \in A \cap L_i$, we choose a bijection $\tau(x, i): \mathbb{Z} \to p_i^{-1}(x)$ and we write $\tau(x, i)(k) = (k, x, i)$.

On the disjoint union $S := \bigcup_{i \in I} \tilde{L}_i$ we define the projection $\tilde{p}: S \to L$ induced by $p_i$, $i \in I$. We also define an equivalence relation on $S$ by defining the equivalence classes as follows: if $s \in S$ is such that $\tilde{p}(s) \notin A$, then $\tilde{s} = \{s\}$. Note that on $\tilde{p}^{-1}(A)$ we have a map $\zeta: \tilde{p}^{-1}(A) \to \mathbb{Z}$ which is nothing other than the projection on the first component. Then if $\tilde{p}(s) \in A$ and $\zeta(s) = k$ we set $\tilde{s} = \{(k, x, i) : x \in L_i, i \in I, \tilde{p}(s) = x\}$. We let $\hat{L}$ be the quotient space of $S$ and $p: \hat{L} \to L$ the application induced by $\tilde{p}$. It is not difficult to see that $p$ is a covering and $\hat{L}$ is Stein.

*Step 2.* All irreducible components of $L$ are compact and not rational. In this case we apply the above procedure to each connected component.

*Step 3.* The general case. Let $L'$ be the union of all irreducible components of $L$ that are compact and not rational and let $\{C_j\}_{j \in J}$ be the connected components
of $L'$. For each $j \in J$ let $p_j : \tilde{C}_j \to C_j$ be the Stein coverings obtained at the first step. Also, let $L''$ be the union of the other irreducible components of $L$. We set $\Gamma_j = L'' \cap C_j$. (Note that $\Gamma_j$ might be empty.) If $\Gamma_j \neq \emptyset$ we consider in $\tilde{C}_j$ infinitely many disjoint copies $\{\Gamma_j^\alpha : \alpha \in \mathbb{Z}\}$ of $\Gamma_j$. (Each $\Gamma_j^\alpha$ is in bijection via $p_j$ with $\Gamma_j$.) We now glue infinitely many copies $\{L_j'' : \alpha \in \mathbb{Z}\}$ of $L''$ to $\tilde{C}_j$ on $\{\Gamma_j^\alpha : \alpha \in \mathbb{Z}\}$, and we obtain in this way a holomorphically convex covering of $L$.

**Lemma 2.5.** Suppose that $\tilde{X}$ and $X$ are Hausdorff topological spaces and $p : \tilde{X} \to X$ is a covering. Let $\tilde{\gamma}_n : S^1 \to \tilde{X}$, $\gamma_n : S^1 \to X$, $n \geq 1$, $\tilde{\gamma} : S^1 \to \tilde{X}$, $\gamma : S^1 \to X$ be continuous functions such that $\gamma_n = p \circ \tilde{\gamma}_n$, $\gamma = p \circ \tilde{\gamma}$ and $a \in S^1$ be a fixed point. If $\gamma_n$ converges uniformly to $\gamma$ and $\tilde{\gamma}_n(a)$ converges to $\tilde{\gamma}(a)$, then $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$.

**Proof.** Let $\Omega = \{t \in S^1 : \lim_{n \to \infty} \tilde{\gamma}_n(t) = \tilde{\gamma}(t)\}$. We will show first that $\Omega = S^1$. As $a \in \Omega$ we have that $\Omega \neq \emptyset$. We prove that $\Omega$ is open in $S^1$. Let $t_0 \in \Omega$ and let $V_0$ a neighborhood of $\gamma(t_0)$ which is evenly covered by $\bigcup V_k$. Let $\Omega_0$ be a connected open neighborhood of $t_0$ such that for $n$ large enough $\gamma_n(\Omega_0) \subset V_0$ and $\gamma(\Omega_0) \subset V_0$. It follows that $\tilde{\gamma}_n(\Omega_0) \subset \bigcup V_k$. As $\gamma_n(\Omega_0)$ and $\tilde{\gamma}_n(\Omega_0)$ are connected we have that for each $n$ there exists $k_n$ such that $\gamma_n(\Omega_0) \subset V_{k_n}$ and there exists $k_0$ such that $\tilde{\gamma}(\Omega_0) \subset V_{k_0}$. However, we assumed that $t_0 \in \Omega$, and therefore for $n$ large enough $\tilde{\gamma}_n(t_0) \in V_{k_0}$. Hence $V_{k_n} = V_{k_0}$. Since $p : V_{k_0} \to V_0$ is a homeomorphism we conclude that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on $\Omega_0$. This shows that $\Omega_0 \subset \Omega$, and hence $\Omega$ is open.

Next we show in a similar fashion that $\Omega$ is closed. Let $t_0 \in S^1 \setminus \Omega$. As before we choose $V_1$ to be a neighborhood of $\gamma(t_0)$ which is evenly covered by $\bigcup V_k$, $\Omega_0$ a connected open neighborhood of $t_0$ such that for $n$ large enough $\gamma_n(\Omega_0) \subset V_0$, $k_n$ and $k_0$ such that $\gamma_n(\Omega_0) \subset V_{k_n}$ and $\gamma(\Omega_0) \subset V_{k_0}$. If $V_{k_n} = V_{k_0}$ for every $n$ we would have that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on $\Omega_0$ and hence at $t_0$, and this would contradict $t_0 \not\in \Omega$. There then exists a subsequence $\tilde{\gamma}_{n_0}$ such that $\tilde{\gamma}_{n_0}(\Omega_0) \cap \tilde{V}_{k_0} = \emptyset$, and from here we get that $\Omega_0 \subset S^1 \setminus \Omega$. Hence $\Omega$ is closed, and therefore $\Omega = S^1$.

Note that when we proved that $\Omega$ is open we actually proved that each $t_0 \in \Omega = S^1$ has an open neighborhood $\Omega_0$ such that on this neighborhood $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$. The compactness of $S^1$ then implies that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on $S^1$.

**Lemma 3.3.** Suppose that $S$ is a complex space that has a holomorphically convex covering $p : \tilde{S} \to S$. If $\Omega \subset \mathbb{C}$ is an open neighborhood of $\Delta$ and $f_n : \Omega \to S$ is a sequence of holomorphic functions which converges uniformly on $\{z \in \mathbb{C} : |z| = 1\}$, then $\bigcup f_n(\Delta)$ is relatively compact in $S$. (In particular, $S$ satisfies the discrete disk property.)

**Proof.** Let $\gamma : \{z \in \mathbb{C} : |z| = 1\} \to S$ be the limit of $\{f_n\}$ on $\{z \in \mathbb{C} : |z| = 1\}$. It follows from Lemma 2.5 that $\gamma$ is null-homotopic. We choose a point $a \in \{z \in \mathbb{C} : |z| = 1\}$. Let $\epsilon$ be a positive number such that $\Delta_{1+\epsilon} \subset \Omega$, $\tilde{f}_n : \Delta_{1+\epsilon} \to \tilde{S}$ and let $\tilde{\gamma} : \{z \in \mathbb{C} : |z| = 1\} \to \tilde{S}$ be liftings of $f_n$ and $\gamma$ respectively. We choose $\tilde{f}_n$ and $\tilde{\gamma}$ such that $f_n$ converges uniformly to $\tilde{\gamma}$ on $\{z \in \mathbb{C} : |z| = 1\}$. This is possible by Lemma 2.5. As $\tilde{S}$ is holomorphically convex this implies that $\bigcup f_n(\Delta)$ is relatively compact in $\tilde{S}$ and therefore $\bigcup f_n(\Delta)$ is relatively compact in $S$. 

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Theorem 3.4. Let \( X \) be a 1-convex surface and \( p : \tilde{X} \to X \) be a covering map. If \( \tilde{X} \) does not contain an infinite Nori string of rational curves, then \( \tilde{X} \) satisfies the discrete disk property.

Proof. Let \( L \) be the exceptional set of \( X \). Without loss of generality we may assume that \( L \) is connected. Let \( Y \) be a normal Stein complex space of dimension 2 and let \( \pi : X \to Y \) be a proper surjective holomorphic map such that \( \pi(L) = \{y_0\} \) and \( \pi : X \setminus L \to Y \setminus \{y_0\} \) is a biholomorphism. We may assume that \( Y \) is an analytic closed subset of \( \mathbb{C}^N \).

We put \( \tilde{L} := p^{-1}(L) \). Since we assumed that \( \tilde{L} \) does not contain an infinite Nori string of rational curves, it follows from Lemma 3.1 that \( \tilde{L} \) has a homolorphically convex covering \( \tilde{p} : \tilde{L} \to \tilde{L} \). We choose \( U_1 \) an open neighborhood of \( L \) in \( X \) that has a continuous deformation retract on \( L \). We let \( \tilde{U}_1 = p^{-1}(U_1) \), and then \( \tilde{U}_1 \) has a continuous deformation retract on \( \tilde{L}, \rho : \tilde{U}_1 \to \tilde{L} \). By considering the fiber product of \( \rho \) and \( \tilde{p} \) we obtain a covering \( \tilde{U}_1 \) of \( \tilde{U} \) such that the pull-back of \( \tilde{L} \) is \( \tilde{L} \). We then apply Theorem 2.3 and we deduce that there exists \( \epsilon \) such that \( (\pi \circ \rho)^{-1}(U) \) has a homolorphically convex covering.

Let \( f_n : \Delta_{1+r} \to \tilde{X}, n \geq 1 \), be a sequence of holomorphic functions such that \( \bigcup_{n \geq 1} f_n(\Delta_{1+r}) \) is relatively compact in \( \tilde{X} \) and \( f_n|_{\mathbb{S}} \) is uniformly convergent. We argue by contradiction, and we assume that \( \bigcup_{n \geq 1} f_n(\Delta) \) is not relatively compact in \( \tilde{X} \) and hence that \( \bigcup_{n \geq 1} f_n(\Delta_{1+r}) \) is not relatively compact. By passing to a subsequence we may assume that \( \bigcup_{k \geq 1} f_{n_k}(\Delta_{1+r}) \) is not relatively compact in \( \tilde{X} \) for every subsequence \( \{f_{n_k}\}_{k} \) of \( \{f_n\}_n \).

Since \( \bigcup_{n \geq 1} f_n(\Delta_{1+r}) \) is relatively compact in \( Y \), and hence in \( \mathbb{C}^N \), by the maximum modulus principle we have that \( \bigcup_{n \geq 1} (\pi \circ f_n)(\Delta_{1+r}) \) is relatively compact in \( Y \). Then it follows that there exists a subsequence of \( \{f_n\}_n, \{f_{n_k}\}_k \) such that \( \pi \circ f_{n_k} \) converges uniformly on compacts to a holomorphic function \( g : D \to Y \). Without loss of generality we can assume that \( \pi \circ f_n \) converges uniformly on compacts to \( g \). We distinguish three cases.

Case 1. \( g \equiv y_0 \). Then \( (\pi \circ f_n)(\Delta_{1+r}) \subset U \) for \( \epsilon_1 \in (0, \epsilon) \) and \( n \) large enough. Therefore we can apply Lemma 3.3.

Case 2. \( g(z) \neq y_0 \) for every \( z \in \Delta_{1+r} \). Then there exists a neighborhood \( V \) of \( y_0 \) and \( \epsilon_1 \in (0, \epsilon) \) such that \( g(\Delta_{1+r}) \cap V = \emptyset \). Then for \( n \) large enough we get that \( \pi \circ f_n(\Delta_{1+r}) \cap V = \emptyset \) and hence \( f_n(\Delta_{1+r}) \cap (\pi \circ p)^{-1}(V) = \emptyset \). We consider now a plurisubharmonic function \( \tilde{\phi} \) on \( \tilde{X} \) with the properties given in Proposition 2.4.

In particular, its restriction to \( \tilde{X} \setminus (\pi \circ p)^{-1}(V) \) is an exhaustion. Applying the maximum principle to \( \tilde{\phi} \circ f_n \) we obtain immediately that \( \bigcup f_n(\Delta) \) is relatively compact.

Case 3. \( g \neq y_0 \) and \( y_0 \in g(\Delta_{1+r}) \). Then \( g^{-1}(y_0) \) is a (non-empty) discrete subset of \( \Delta_{1+r} \), and there exists \( \epsilon_1 \in (0, \epsilon) \) such that \( g^{-1}(y_0) \cap \{z \in \mathbb{C} : |z| = 1 + \epsilon_1\} = \emptyset \). We set \( g^{-1}(y_0) \cap \{z \in \mathbb{C} : |z| < 1 + \epsilon_1\} = \{\lambda_1, \lambda_2, \ldots, \lambda_s\} \). Let \( r_1, \ldots, r_s \) be positive numbers such that the discs \( \Delta_j = \{z \in \mathbb{C} : |z - \lambda_j| \leq r_j\}, j = 1, 2, \ldots, s \), are pairwise disjoint, \( \Delta_j \subset \Delta_{1+r_1} \), and \( g(\Delta_j) \subset U \). Let \( V \subset Y \) be an open neighborhood of \( y_0 \) such that \( V \subset U \) and

\[
g(\{z \in \Delta_{1+r_1} : |z - \lambda_j| \geq r_j \forall j = 1, \ldots, s\}) \cap V = \emptyset.
\]
Then for \( n \) large enough we have that
\[
(\pi \circ p \circ f_n)((z \in \Delta_{1+\epsilon_j} : |z - \lambda_j| \geq r_j \ \forall j = 1, ..., s)) \cap V = \emptyset.
\]

We again apply Proposition 2.3 and we obtain a plurisubharmonic function \( \tilde{\phi} : \tilde{X} \to [-\infty, \infty) \) such that \( \tilde{L} = \{ \phi = -\infty \} \tilde{\phi}^{-1}(V) \) is an exhaustion function on \( \tilde{X} \backslash (\pi \circ p)^{-1}(V) \). Since \( \bigcup f_n(\Delta_{1+\epsilon}) \) is relatively compact there exists a positive constant \( M \) such that \( \tilde{\phi} \circ f_n(z) \leq M \) for every \( n \) and every \( z \in \bigcup \Delta_{1+\epsilon} \). From the plurisubharmonicity of \( \phi \), we have that \( \tilde{\phi} \circ f_n(z) \leq M \) for every \( z \in \Delta_{1+\epsilon} \), hence in particular for \( z \in \Delta_{1+\epsilon} \). As \( f_n(\bigcup_{j=1}^s \Delta_{\epsilon_j}) \subset \tilde{X} \backslash (\pi \circ p)^{-1}(V) \) and \( \tilde{\phi} \tilde{X} \backslash (\pi \circ p)^{-1}(V) \) is an exhaustion, we deduce that \( \bigcup f_n(\bigcup_{j=1}^s \Delta_{\epsilon_j}) \) is relatively compact in \( \tilde{X} \).

For \( j = 1, ..., s \) we set \( S_j = \{ z \in \Delta_{1+\epsilon_j} : |z - \lambda_j| = r_j \} \) and we pick a point \( a_j \in S_j \). As \( \bigcup_{n \geq 1} f_n(S_j) \) is relatively compact, by passing to a subsequence we can assume that each sequence \( \{ f_n(a_j) \} \) is convergent and we denote by \( x_j \) its limit. We have that \( (\pi \circ p \circ f_n)_{S_j} \) converges uniformly to \( g_{S_j} \) and \( (\pi \circ p \circ f_n)(S_j) \subset Y \bk \{ y_0 \} \). Because \( \pi : X \to Y \bk \{ y_0 \} \) is a biholomorphism we deduce that \( (p \circ f_n)_{S_j} \) converges uniformly to \( (\pi^{-1} \circ g)_{S_j} \). Note now that \( (p \circ f_n)_{S_j} \) is in fact a null-homotopic loop. It follows from Lemma 3.3 that \( (\pi^{-1} \circ g)_{S_j} \) is null-homotopic as well, and therefore there exists a loop \( \gamma_j : S_j \to \tilde{X} \) such that \( p \circ \gamma_j = (\pi^{-1} \circ g) \) on \( S_j \). From Lemma 3.2 we conclude that \( f_n \) converges uniformly to \( \gamma_j \) on \( S_j \). Finally, Lemma 3.3 implies that \( \bigcup f_n(\Delta_j) \) is relatively compact for every \( j = 1, ..., s \). □

References


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