

## OPTIMAL EXPANSIONS IN NON-INTEGER BASES

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ABSTRACT. For a given positive integer  $m$ , let  $A = \{0, 1, \dots, m\}$  and  $q \in (m, m + 1)$ . A sequence  $(c_i) = c_1 c_2 \dots$  consisting of elements in  $A$  is called an expansion of  $x$  if  $\sum_{i=1}^{\infty} c_i q^{-i} = x$ . It is known that almost every  $x$  belonging to the interval  $[0, m/(q - 1)]$  has uncountably many expansions. In this paper we study the existence of expansions  $(d_i)$  of  $x$  satisfying the inequalities  $\sum_{i=1}^n d_i q^{-i} \geq \sum_{i=1}^n c_i q^{-i}$ ,  $n = 1, 2, \dots$ , for each expansion  $(c_i)$  of  $x$ .

### 1. INTRODUCTION

Let  $x \in [0, 1)$ . The decimal expansion

$$x = \frac{b_1}{10} + \frac{b_2}{10^2} + \frac{b_3}{10^3} + \dots,$$

where we choose a finite expansion whenever it is possible, has a well-known “each-step” optimality property: for each  $k = 1, 2, \dots$ , among all finite sequences  $c_1 \dots c_k$  of integers with  $0 \leq c_i \leq 9$  for  $i = 1, \dots, k$ , satisfying the inequality  $\sum_{i=1}^k c_i 10^{-i} \leq x$ , the sum  $\sum_{i=1}^k b_i 10^{-i}$  is the closest to  $x$ . An analogous property holds for expansions in all integer bases  $2, 3, \dots$ .

In his celebrated paper [16], Rényi generalized these expansions to arbitrary real bases  $q > 1$  as follows. If  $b_1, \dots, b_{n-1}$  have already been defined for some  $n \geq 1$  (no condition for  $n = 1$ ), then let  $b_n$  be the largest integer satisfying the inequality

$$\frac{b_1}{q} + \dots + \frac{b_n}{q^n} \leq x.$$

One may readily verify that

$$\sum_{i=1}^{\infty} \frac{b_i}{q^i} = x;$$

it is called the *greedy* expansion of  $x$  in base  $q$ .

The purpose of this paper is to show that the natural analogue of the above optimality property fails for most non-integer bases, but it still holds for a particular

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countable set of bases, the smallest of them being the golden ratio  $q = (1 + \sqrt{5})/2 \approx 1.618$ . Before formulating our result precisely we will first introduce expansions of real numbers with respect to a more general set of digits.

Given a real number  $q > 1$  and a finite *alphabet* or *digit set*  $A = \{a_0, \dots, a_m\}$  consisting of real numbers satisfying  $a_0 < \dots < a_m$ , by an *expansion* of  $x$  (in *base*  $q$  with respect to  $A$ ) we mean a sequence  $(c_i)$  of *digits*  $c_i \in A$  satisfying

$$(1.1) \quad \sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

Pedicini [15] proved the following basic result on the existence of such expansions.

**Proposition 1.1.** *Each  $x \in J_{A,q} := [a_0/(q-1), a_m/(q-1)]$  has an expansion if and only if*

$$(1.2) \quad \max_{1 \leq j \leq m} (a_j - a_{j-1}) \leq \frac{a_m - a_0}{q - 1}.$$

For the convenience of the reader we provide an elementary proof of this proposition. Observe that  $(c_i)$  is an expansion of  $x$  in base  $q$  with respect to  $A$  if and only if  $(c_i - a_0) = (c_1 - a_0)(c_2 - a_0) \dots$  is an expansion of  $x - a_0/(q-1)$  in base  $q$  with respect to the alphabet  $\{0, a_1 - a_0, \dots, a_m - a_0\}$ . Moreover, the inequality (1.2) holds if and only if the same inequality holds with  $a_j - a_0$  in place of  $a_j$ ,  $0 \leq j \leq m$ . Hence we may (and will) assume in the rest of this paper that  $a_0 = 0$ .

*Proof of Proposition 1.1.* First assume that the inequality (1.2) holds. We define recursively a sequence  $(b_i)$  with digits  $b_i$  belonging to  $A$  by applying the following *greedy algorithm*: if for some integer  $n \in \mathbb{N} := \{1, 2, \dots\}$  the digits  $b_i$  have already been defined for all  $1 \leq i < n$  (no condition for  $n = 1$ ), then let  $b_n$  be the largest digit in  $A$  satisfying the inequality  $\sum_{i=1}^n b_i q^{-i} \leq x$ . Note that this algorithm is well defined for each  $x \geq 0$ . We show that  $(b_i)$  is an expansion of  $x$  for each  $x$  belonging to  $J_{A,q}$ .

If  $x = a_m/(q-1)$ , then the greedy algorithm provides  $b_i = a_m$  for all  $i \geq 1$ , whence  $(b_i)$  is indeed an expansion of  $x$ .

If  $0 \leq x < a_m/(q-1)$ , then there exists an index  $n$  such that  $b_n < a_m$ . If  $b_n < a_m$  for infinitely many  $n$ , then for each such  $n$  we have

$$0 \leq x - \sum_{i=1}^n \frac{b_i}{q^i} < \frac{\max_{1 \leq j \leq m} (a_j - a_{j-1})}{q^n}.$$

Letting  $n \rightarrow \infty$ , we see that  $(b_i)$  is an expansion of  $x$ . Next we show that there cannot be finitely many  $n$  such that  $b_n < a_m$ . Indeed, if there were a last index  $n$  with  $b_n = a_j < a_m$ , then

$$\left( \sum_{i=1}^n \frac{b_i}{q^i} \right) + \sum_{i=n+1}^{\infty} \frac{a_m}{q^i} \leq x < \left( \sum_{i=1}^n \frac{b_i}{q^i} \right) + \frac{a_{j+1} - a_j}{q^n}$$

or equivalently

$$\frac{a_m}{q-1} < a_{j+1} - a_j,$$

contradicting (1.2).

Finally, if condition (1.2) does not hold and  $a_\ell - a_{\ell-1} > a_m/(q-1)$  for some  $\ell \in \{1, \dots, m\}$ , then none of the numbers belonging to the non-empty interval

$$\left( \frac{a_{\ell-1}}{q} + \sum_{i=2}^{\infty} \frac{a_m}{q^i}, \frac{a_\ell}{q} \right) \subset J_{A,q}$$

has an expansion.  $\square$

The proof of Proposition 1.1 shows that if (1.2) holds, then each  $x \in J_{A,q}$  has a lexicographically largest expansion  $(b_i(x, A, q))$ , which we call the *greedy expansion* of  $x$ . The *normalized errors* of an arbitrary expansion  $(c_i)$  of  $x$  are defined by

$$\theta_n((c_i)) := q^n \left( x - \sum_{i=1}^n \frac{c_i}{q^i} \right), \quad n \in \mathbb{N}.$$

We call an expansion  $(d_i)$  of  $x$  *optimal* if  $\theta_n((d_i)) \leq \theta_n((c_i))$  for each  $n \in \mathbb{N}$  and each expansion  $(c_i)$  of  $x$ . It follows from the definitions that only the greedy expansion of a number  $x \in J_{A,q}$  can be optimal. The following example shows that the greedy expansion of a number  $x \in J_{A,q}$  is not always optimal. Other examples can be found in [3].

**Example 1.2.** Let  $A = \{0, 1\}$  and  $1 < q < (1 + \sqrt{5})/2$ . The sequence  $(c_i) := 011(0)^\infty$  is clearly an expansion of  $x := q^{-2} + q^{-3}$ . Applying the greedy algorithm we find that the first three digits of the greedy expansion  $(b_i) = (b_i(x, A, q))$  of  $x$  equal 100. Hence  $\theta_3((b_i)) > \theta_3((c_i)) = 0$ .

Let  $A = \{0, 1, \dots, m\}$  and  $q \in (m, m+1)$  for some positive integer  $m$ . Proposition 1.1 implies that in this case each  $x \in J_{A,q}$  has an expansion. Let  $P$  be the set consisting of those bases  $q \in (m, m+1)$  which satisfy one of the equalities

$$1 = \frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p}{q^{n+1}}, \quad n \in \mathbb{N} \text{ and } p \in \{1, \dots, m\}.$$

We have the following dichotomy:

**Theorem 1.3.**

- (i) If  $q \in P$ , then each  $x \in J_{A,q}$  has an optimal expansion.
- (ii) If  $q \in (m, m+1) \setminus P$ , then the set of numbers  $x \in J_{A,q}$  with an optimal expansion is nowhere dense and has Lebesgue measure zero.

In Section 2 we compare greedy expansions with respect to different alphabets. This gives us a characterization of optimal expansions which is essential to our proof of Theorem 1.3 in Section 3. In Section 4 we briefly discuss optimal expansions of real numbers in negative integer bases.

## 2. GREEDY EXPANSIONS

Consider an alphabet  $A = \{a_0, a_1, \dots, a_m\}$  ( $0 = a_0 < \dots < a_m$ ) and a base  $q$  satisfying the condition (1.2) as in the preceding section. Let the *greedy transformation*  $T : J_{A,q} \rightarrow J_{A,q}$  corresponding to  $(A, q)$  be given by

$$T(x) := \begin{cases} qx - a_j & \text{if } x \in C(a_j) := \left[ \frac{a_j}{q}, \frac{a_{j+1}}{q} \right), 0 \leq j < m, \\ qx - a_m & \text{if } x \in C(a_m) := \left[ \frac{a_m}{q}, \frac{a_m}{q-1} \right]. \end{cases}$$

Observe that  $b_i(x, A, q) = a_j$  if and only if  $T^{i-1}(x) \in C(a_j)$ ,  $i \geq 1$ .

For any fixed positive integer  $k$ , the equation (1.1) can be rewritten in the form

$$\frac{d_1}{q^k} + \frac{d_2}{q^{2k}} + \dots = x$$

by setting

$$d_i := \sum_{j=0}^{k-1} c_{ik-j} q^j, \quad i = 1, 2, \dots$$

In other words, every expansion in base  $q$  with respect to the alphabet  $A$  can also be considered as an expansion in base  $q^k$  with respect to the alphabet

$$A_k := \{c_1 q^{k-1} + \dots + c_k : c_1, \dots, c_k \in A\}.$$

(For  $k = 1$  this reduces to the original case.) In particular we have

$$J_{A_k, q^k} = J_{A, q}$$

for every  $k$ . We may therefore compare the greedy transformation  $T_k$  corresponding to  $(A_k, q^k)$  with the  $k$ -th iteration  $T^k$  of the map  $T$  corresponding to  $(A, q)$ . It is easily seen that  $T_k(x) \leq T^k(x)$  for each  $x \in J_{A, q}$ , but in general we do not have equality here.

Given  $(A, q)$  and a positive integer  $k$ , we denote by  $S_{A, q, k}$  the set of sequences  $(c_1, \dots, c_k) \in A^k$  satisfying the following condition: if  $(d_1, \dots, d_k) \in A^k$  and  $(d_1, \dots, d_k) > (c_1, \dots, c_k)$ , then

$$\sum_{i=1}^k \frac{d_i}{q^i} \neq \sum_{i=1}^k \frac{c_i}{q^i}.$$

For each  $x \in J_{A, q}$ , the sequence  $b_1(x, A, q) \dots b_k(x, A, q) 0^\infty$  is the greedy expansion in base  $q$  with respect to  $A$  of the number

$$\sum_{i=1}^k \frac{b_i(x, A, q)}{q^i}$$

as follows from the definition of the greedy algorithm. Hence

$$S_{A, q, k} \supset \{(b_1(x, A, q), \dots, b_k(x, A, q)) : x \in J_{A, q}\}.$$

Let the injective map  $f : S_{A, q, k} \rightarrow J_{A, q}$  be given by

$$(2.1) \quad f((c_1, \dots, c_k)) = \frac{c_1}{q} + \dots + \frac{c_k}{q^k}, \quad (c_1, \dots, c_k) \in S_{A, q, k}.$$

**Proposition 2.1.** *The following statements are equivalent.*

- (i) *The map  $f$  is increasing.*
- (ii)  $T_k = T^k$ .
- (iii)  $S_{A, q, k} = \{(b_1(x, A, q), \dots, b_k(x, A, q)) : x \in J_{A, q}\}$ .

*Proof.* (i)  $\Rightarrow$  (ii). Given any  $x \in J_{A, q}$ , let  $(c_1, \dots, c_k)$  be the lexicographically largest sequence in  $A^k$  satisfying

$$s := \frac{c_1}{q} + \dots + \frac{c_k}{q^k} \leq x.$$

Then  $(c_1, \dots, c_k) \in S_{A, q, k}$ , and (i) implies that  $T_k(x) = q^k(x - s)$ . On the other hand, we also have  $T^k(x) = q^k(x - s)$  by definition of the greedy expansion.

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<sup>1</sup>Other aspects of expansions with respect to alphabets of the form  $A_k$  are studied in [4], [11].

(ii)  $\Rightarrow$  (iii). Assume that  $(c_1, \dots, c_k) \in S_{A,q,k}$ , and let

$$x' := \sum_{i=1}^k \frac{c_i}{q^i}.$$

If we had  $(c_1, \dots, c_k) \notin \{(b_1(x, A, q), \dots, b_k(x, A, q)) : x \in J_{A,q}\}$ , then there would exist an index  $m > k$  such that  $b_m(x', A, q) \neq 0$ , whence  $T_k(x') = 0 < T^k(x')$ , contradicting (ii).

(iii)  $\Rightarrow$  (i). As already observed, the sequence  $b_1(x, A, q) \dots b_k(x, A, q)0^\infty$  is the greedy expansion of the number

$$\sum_{i=1}^k \frac{b_i(x, A, q)}{q^i}.$$

It remains to note that  $x < y$  if and only if  $(b_i(x, A, q)) < (b_i(y, A, q))$  for numbers  $x$  and  $y$  belonging to  $J_{A,q}$ .  $\square$

*Remarks 2.2.*

- (i) Observe that the maps  $T_k$  and  $T^k$  are continuous from the right. Hence if  $T_k \neq T^k$ , then the maps  $T_k$  and  $T^k$  differ on a whole interval.
- (ii) If  $T_k \neq T^k$ , then  $T_n \neq T^n$  for all  $n \geq k$ . In order to prove this, it is sufficient to show that  $T_{k+1} \neq T^{k+1}$ . By Proposition 2.1 there exist two sequences  $(b_1, \dots, b_k)$ ,  $(c_1, \dots, c_k)$  both belonging to  $S_{A,q,k}$  such that  $(b_1, \dots, b_k) < (c_1, \dots, c_k)$  and

$$\sum_{i=1}^k \frac{b_i}{q^i} > \sum_{i=1}^k \frac{c_i}{q^i}.$$

Note that the sequences  $(a_m, b_1, \dots, b_k)$  and  $(a_m, c_1, \dots, c_k)$  both belong to  $S_{A,q,k+1}$ , and

$$\frac{a_m}{q} + \sum_{i=1}^k \frac{b_i}{q^{i+1}} > \frac{a_m}{q} + \sum_{i=1}^k \frac{c_i}{q^{i+1}}.$$

Applying Proposition 2.1 once more, we reach the desired conclusion.

### 3. PROOF OF THEOREM 1.3

Let  $m$  be a given positive integer. Throughout this section we consider expansions with respect to the alphabet  $A = \{0, 1, \dots, m\}$  in a base  $q$  belonging to  $(m, m+1)$ . For any integers  $n \geq 1$  and  $0 \leq p \leq m$  we denote by  $q_{m,n,p}$  the positive solution of the equation

$$1 = \frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p}{q^{n+1}}.$$

We have

$$m = q_{m,1,0} < \dots < q_{m,1,m} = q_{m,2,0} < \dots < q_{m,2,m} = q_{m,3,0} < \dots$$

and

$$q_{m,n,p} \rightarrow m+1 \quad \text{if } n \rightarrow \infty.$$

Recall that the set  $P$  introduced in Section 1 consists of the numbers  $q_{m,n,p}$  with  $n \geq 1$  and  $1 \leq p \leq m$ .

**Proposition 3.1.** *Let  $n \geq 1$  and  $1 \leq p \leq m$ .*

- (i) If  $q = q_{m,n,p}$ , then  $T_k = T^k$  for all  $k \geq 1$ .
- (ii) If  $q_{m,n,p-1} < q < q_{m,n,p}$ , then  $T_k = T^k$  if and only if  $k \leq n + 1$ .
- (iii) If  $q \in (m, m + 1) \setminus P$ , then there exists a positive integer  $k = k(q)$  such that the maps  $T_k$  and  $T^k$  differ on an interval contained in  $[0, 1)$ .

*Proof.* (i) By Proposition 2.1 it is sufficient to prove that if

$$(c_1, \dots, c_k), (d_1, \dots, d_k) \in S_{A,q,k} \quad \text{and} \quad (c_1, \dots, c_k) > (d_1, \dots, d_k),$$

then

$$(3.1) \quad \sum_{i=1}^k \frac{c_i}{q^i} > \sum_{i=1}^k \frac{d_i}{q^i}.$$

Let  $j$  be the first index such that  $c_j > d_j$ . Since  $q = q_{m,n,p}$ , the elements of  $S_{A,q,k}$  do not contain any block of the form  $am^nb$  with  $a < m$  and  $b \geq p$ . Indeed, the sum corresponding to such a block is the same as the sum corresponding to the lexicographically larger block  $(a + 1)0^n(b - p)$ . Therefore, since  $d_j < m$ , a block of the form  $m^nb$  with  $b \geq p$  cannot occur in  $(d_{j+1}, \dots, d_k)$ . This implies that if  $d_{\ell+1} \dots d_{\ell+n+1}$  is a block of length  $n + 1$  that is contained in  $(d_{j+1}, \dots, d_k)$ , then

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{d_{\ell+i}}{q^i} &\leq \max \left\{ \frac{m}{q} + \dots + \frac{m}{q^{n-1}} + \frac{m-1}{q^n} + \frac{m}{q^{n+1}}, \frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p-1}{q^{n+1}} \right\} \\ &= \frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p-1}{q^{n+1}}. \end{aligned}$$

Therefore

$$\sum_{i=j+1}^k \frac{d_i}{q^i} < \frac{1}{q^j} \sum_{k=0}^{\infty} \left( \frac{1}{q^{n+1}} \right)^k \left( \frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p-1}{q^{n+1}} \right) = \frac{1}{q^j},$$

which implies (3.1).

(ii) It follows from our assumption on  $q$  that

$$(3.2) \quad \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p-1}{q^{n+2}} < \frac{1}{q} < \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}.$$

First we show that  $T_k = T^k$  for every  $k \leq n + 1$ . Let  $(c_1, \dots, c_k)$  and  $(d_1, \dots, d_k)$  be sequences in  $A^k$  satisfying  $(c_1, \dots, c_k) > (d_1, \dots, d_k)$ , and let  $j$  be the smallest positive integer such that  $c_j > d_j$ . Then we have

$$\begin{aligned} \sum_{i=1}^k \frac{c_i - d_i}{q^i} &\geq \frac{1}{q^{j-1}} \left( \frac{1}{q} - \frac{m}{q^2} - \dots - \frac{m}{q^{k+1-j}} \right) \\ &\geq \frac{1}{q^{j-1}} \left( \frac{1}{q} - \frac{m}{q^2} - \dots - \frac{m}{q^{n+1}} \right) \\ &> 0 \end{aligned}$$

by using (3.2) in the last step.

Due to a remark following the proof of Proposition 2.1 it remains to show that  $T_{n+2} \neq T^{n+2}$ . The sequence  $10^{n+1}$  clearly belongs to  $S_{A,q,n+2}$ . In order to show that  $0m^np$  belongs to  $S_{A,q,n+2}$  as well, we must prove that

$$\sum_{i=1}^{n+2} \frac{c_i}{q^i} \neq \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}$$

for every sequence  $c_1 \dots c_{n+2} \in A^{n+2}$  satisfying  $c_1 \dots c_{n+2} > 0m^n p$ .

If  $c_1 = 0$ , this is clear. If  $c_1 \dots c_{n+2} = 10^{n+1}$ , then

$$(3.3) \quad \sum_{i=1}^{n+2} \frac{c_i}{q^i} = \frac{1}{q} < \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}$$

by (3.2). In the remaining cases we have  $c_1 \geq 1$  and  $c_1 + \dots + c_{n+2} \geq 2$ , so that

$$(3.4) \quad \sum_{i=1}^{n+2} \frac{c_i}{q^i} \geq \frac{1}{q} + \frac{1}{q^{n+2}} > \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}$$

by (3.2) again.

Since  $10^{n+1}, 0m^n p \in S_{A,q,n+2}$  and  $10^{n+1} > 0m^n p$ , the inequality (3.3) shows that the map (2.1) with  $k = n + 2$  is not increasing.

(iii) As in part (ii), suppose that  $q_{m,n,p-1} < q < q_{m,n,p}$  for some  $n, p \geq 1$ . It follows from (3.3) and (3.4) that if  $x$  belongs to the non-empty interval

$$D := \left[ \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}, \frac{1}{q} + \frac{1}{q^{n+2}} \right),$$

then

$$\sum_{i=1}^{n+2} \frac{b_i(x, A, q)}{q^i} = \frac{1}{q} < \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}} = \frac{b_1(x, A_{n+2}, q^{n+2})}{q^{n+2}},$$

i.e.,

$$T_{n+2}(x) = q^{n+2} \left( x - \frac{m}{q^2} - \dots - \frac{m}{q^{n+1}} - \frac{p}{q^{n+2}} \right) < q^{n+2} \left( x - \frac{1}{q} \right) = T^{n+2}(x).$$

If  $(m, n, p) \neq (1, 1, 1)$ , then the interval  $D$  is contained in  $[0, 1)$ . If  $(m, n, p) = (1, 1, 1)$  and  $1 > q^{-2} + q^{-3}$ , then  $D \cap [0, 1)$  is non-empty. Therefore, also in this case the maps  $T_{n+2}$  and  $T^{n+2}$  differ on an interval contained in  $[0, 1)$ . It remains to consider those values of  $q$  that satisfy  $1 \leq q^{-2} + q^{-3}$ .

If  $1 \leq q^{-2} + q^{-3}$ , then let  $\ell \geq 3$  be the (unique) positive integer satisfying

$$(3.5) \quad \frac{1}{q^\ell} + \frac{1}{q^{\ell+1}} < 1 \leq \frac{1}{q^{\ell-1}} + \frac{1}{q^\ell}.$$

If the latter inequality in (3.5) is strict, then for each  $x$  belonging to the non-empty interval

$$\left[ \frac{1}{q^\ell} + \frac{1}{q^{\ell+1}}, \min \left\{ 1, \frac{1}{q} + \frac{1}{q^{\ell+1}} \right\} \right),$$

we have  $b_1(x, A, q) \dots b_{\ell+1}(x, A, q) = 10^\ell$  and

$$T_{\ell+1}(x) \leq q^{\ell+1} \left( x - \frac{1}{q^\ell} - \frac{1}{q^{\ell+1}} \right) < q^{\ell+1} \left( x - \frac{1}{q} \right) = T^{\ell+1}(x).$$

If the latter inequality in (3.5) is in fact an equality, then we consider the non-empty interval

$$\left[ \frac{1}{q^{\ell-1}} + \frac{1}{q^{\ell+1}}, \min \left\{ 1, \frac{1}{q} + \frac{1}{q^{\ell+1}} \right\} \right).$$

For each  $x$  belonging to this interval we have  $b_1(x, A, q) \dots b_{\ell+1}(x, A, q) = 10^\ell$  and

$$T_{\ell+1}(x) \leq q^{\ell+1} \left( x - \frac{1}{q^{\ell-1}} - \frac{1}{q^{\ell+1}} \right) < q^{\ell+1} \left( x - \frac{1}{q} \right) = T^{\ell+1}(x).$$

For each  $q \in (m, m + 1) \setminus P$  we now have constructed an interval  $I \subset [0, 1)$  and a positive integer  $k$  such that  $T_k < T^k$  on  $I$ . □

*Remarks 3.2.*

- (i) It follows from the above proof that if  $q_{m,n,p-1} < q < q_{m,n,p}$  ( $n, p \geq 1$ ) and  $(m, n, p) \neq (1, 1, 1)$ , then one may take  $k = n + 2$  in the statement of Proposition 3.1(iii).
- (ii) If  $T_k(x) \neq T^k(x)$  for some  $x \in [0, 1)$ , then the first digit of any expansion of  $xq^{-1}$  in base  $q$  with respect to  $A$  must be zero, whence

$$T_{k+1}\left(\frac{x}{q}\right) = T_k(x) < T^k(x) = T^{k+1}\left(\frac{x}{q}\right).$$

Hence if  $T_k \neq T^k$  on a subinterval of  $[0, 1)$ , then  $T_n \neq T^n$  on a subinterval of  $[0, 1)$  for each integer  $n \geq k$ .

*Proof of Theorem 1.3.* (i) Let  $q \in P$ . Note that the greedy expansion of  $x \in J_{A,q}$  is optimal if and only if  $T_k(x) = T^k(x)$  for each  $k \geq 1$ . Hence each  $x \in J_{A,q}$  has an optimal expansion by Proposition 3.1(i).

(ii) Let  $q \in (m, m + 1) \setminus P$ . It is well known (see, e.g., [14], [16]) that the map  $T$  is ergodic with respect to a unique normalized absolutely continuous  $T$ -invariant measure  $\mu$  with a density which is positive on the interval  $[0, 1)$ . According to Proposition 3.1(iii) there exists an interval  $I \subset [0, 1)$  and a number  $k = k(q)$  such that  $T_k < T^k$  on  $I$ . An application of Birkhoff’s ergodic theorem yields that for almost every  $x \in [0, 1)$  there exists a positive integer  $\ell = \ell(x)$  such that  $T^\ell(x) \in I$ . For each such  $x$  the greedy expansion of  $x$  is not optimal because the greedy expansion  $b_{\ell+1}(x, A, q)b_{\ell+2}(x, A, q) \dots$  of  $T^\ell(x)$  is not optimal. Since the map  $T$  is non-singular<sup>2</sup> and since for each  $x \in [1, m/(q - 1))$  there exists a positive integer  $n = n(x)$  such that  $T^n(x) \in [0, 1)$ , we may conclude that  $x$  has no optimal expansion for almost every  $x \in J_{A,q}$ .

It remains to show that the set of numbers with an optimal expansion is nowhere dense. We call an expansion  $(d_i)$  of a number  $x \in J_{A,q}$  *infinite* if  $d_n > 0$  for infinitely many  $n \in \mathbb{N}$ . Otherwise it is called *finite*. Let  $x \in J_{A,q}$  be a number with no optimal and no finite expansion, and let  $(b_i) = (b_i(x, A, q))$ . Then there exists an expansion  $(c_i)$  of  $x$  and a number  $n \in \mathbb{N}$  such that the inequalities

$$\sum_{i=1}^n \frac{b_i}{q^i} < \sum_{i=1}^n \frac{c_i}{q^i} < x$$

hold. Hence the number  $x$  belongs to the interior of the interval

$$E := \left[ \sum_{i=1}^n \frac{c_i}{q^i}, \left( \sum_{i=1}^n \frac{c_i}{q^i} \right) + \sum_{i=n+1}^{\infty} \frac{m}{q^i} \right].$$

It follows from Proposition 1.1 that the set  $E$  consists precisely of those numbers in  $J_{A,q}$  that have an expansion starting with  $c_1 \dots c_n$ . Since  $(b_i)$  is infinite by hypothesis, there exists a number  $\delta = \delta(x) > 0$  such that  $(x - \delta, x + \delta) \subset E$  and such that the greedy expansion of each number belonging to  $(x - \delta, x + \delta)$  starts with  $b_1 \dots b_n$  (this follows for instance from Lemmas 3.1 and 3.2 in [5]). Hence none of the numbers in  $(x - \delta, x + \delta)$  has an optimal expansion. Denoting by  $\mathcal{O}_q$  the set of numbers in  $J_{A,q}$  with an optimal expansion and its closure by  $\overline{\mathcal{O}_q}$ , we may

<sup>2</sup>Non-singularity of  $T$  means that  $T^{-1}(B)$  is a null set whenever  $B \subset J_{A,q}$  is a null set.



thus conclude that numbers belonging to  $\overline{\mathcal{O}_q} \setminus \mathcal{O}_q$  have a finite expansion, whence  $\overline{\mathcal{O}_q} \setminus \mathcal{O}_q$  is at most countable. This implies in particular that the set  $\overline{\mathcal{O}_q}$  is also a null set and therefore has no interior points.  $\square$

For each positive integer  $k$ , the map  $T_k$  is also ergodic with respect to a unique normalized absolutely continuous  $T_k$ -invariant measure  $\mu_k$  as follows from Theorem 4 in [13]. Since  $T_1 = T$ , the measure  $\mu$  introduced in the proof of Theorem 1.3 equals  $\mu_1$ . Methods to construct an explicit formula for (a version of) the density of the measure  $\mu_k$  can be found in [12] (see also [9], [2]).

**Corollary 3.3.**  $q \in P$  if and only if  $\mu_1 = \mu_k$  for each  $k \geq 1$ .

*Proof.* Proposition 3.1(i) implies that  $\mu_1 = \mu_2 = \dots$  if  $q$  belongs to  $P$ . Conversely, suppose that  $q \in (m, m+1) \setminus P$  and let  $I \subset [0, 1)$  be an interval such that  $T_k < T^k$  on  $I$  for some positive integer  $k$ . Since the maps  $T_k$  and  $T^k$  are continuous from the right, there exists a subinterval  $J \subset I$  and a number  $t > 0$  such that  $T_k < t < T^k$  on  $J$ . Note that  $T^{-k}([0, t]) \subset T_k^{-1}([0, t])$  because  $T_k \leq T^k$  on  $J_{A,q}$ . If we had  $\mu_k = \mu_1$ , then  $\mu_1$  would also be  $T_k$ -invariant, whence

$$0 = \mu_1(T_k^{-1}[0, t]) - \mu_1(T^{-k}[0, t]) \geq \mu_1(J),$$

which contradicts the fact that the density of  $\mu_1$  is positive on the interval  $[0, 1)$ .  $\square$

*Remarks 3.4.*

- (i) For each  $q \in (m, m+1)$ , almost every  $x \in J_{A,q}$  has uncountably many expansions (see [17], [1]). It follows from Theorem 1.3(i) that a number with an optimal expansion may have uncountably many expansions. We do not know whether the greedy expansion of a number with at most countably many expansions is always optimal.
- (ii) It has been shown in [8] (see also [5], [6]) that if  $q \in (m, m+1)$  is close enough to  $m+1$ , then the set  $\mathcal{U}_q$  of numbers in  $J_{A,q}$  with a unique expansion is uncountable. Moreover, the Hausdorff dimension of  $\mathcal{U}_q$  tends to one if  $q \rightarrow m+1$ . Since a unique expansion is clearly optimal, the same properties hold for the set of numbers belonging to  $J_{A,q}$  with an optimal expansion.
- (iii) Let  $\mathcal{U}$  be the set of bases  $q \in (m, m+1)$  such that the number  $1 \in J_{A,q}$  has a unique expansion. The set  $\mathcal{U}$  has been extensively studied in [7], [10], [5]. For instance it has been shown in [5] that  $\mathcal{U}_q$  is closed if and only if  $q \in (m, m+1) \setminus \overline{\mathcal{U}}$ , where  $\overline{\mathcal{U}}$  is the closure of  $\mathcal{U}$ . It follows from the proof of Theorem 1.3 in [5] that each number  $x$  belonging to the closure  $\overline{\mathcal{U}_q}$  of the set  $\mathcal{U}_q$  has an optimal expansion for each  $q \in (m, m+1)$ . We conclude this section with an example showing that the set  $\mathcal{O}_q$  of numbers with an optimal expansion properly contains  $\overline{\mathcal{U}_q}$  for all  $q \in (m, m+1)$ .

**Example 3.5.** Fix  $q \in (m, m+1)$ . It is well known that each number  $x \in J_{A,q} \setminus \{0\}$  has a lexicographically largest infinite expansion  $(a_i(x))$  which coincides with its greedy expansion if and only if the latter is infinite. If the greedy expansion  $(b_i(x))$  of a number  $x \in J_{A,q} \setminus \{0\}$  is finite and  $b_n(x)$  is its last non-zero element, then  $(a_i(x)) = b_1(x) \dots b_{n-1}(x)(b_n(x) - 1)a_1(1)a_2(1) \dots$ . For convenience we set  $(a_i(0)) := 0^\infty$ . It is shown in [5] that  $\overline{\mathcal{U}_q} \subset \mathcal{V}_q$ , where  $\mathcal{V}_q$  is the set of numbers  $x \in J_{A,q}$  such that

$$(m - a_{n+1}(x))(m - a_{n+2}(x)) \dots \leq a_1(1)a_2(1) \dots \quad \text{whenever } a_n(x) > 0.$$

Let  $k$  be the largest positive integer satisfying the inequality  $\sum_{i=1}^k mq^{-i} < 1$ , and consider the number

$$x := \frac{1}{q} + \frac{1}{q^{k+2}}.$$

The greedy expansion  $(b_i(x))$  of  $x$  is clearly given by  $10^k 10^\infty$ . Our choice of  $k$  implies that  $(b_i(x))$  is optimal. However, the number  $x$  does not belong to  $\mathcal{V}_q$  because  $a_1(x) \dots a_{k+2}(x) = 10^{k+1}$  and  $a_1(1) \dots a_{k+1}(1) = m^k c$  with  $c < m$ .

#### 4. OPTIMAL EXPANSIONS IN NEGATIVE BASES

Given a positive integer  $m$  and a real number  $m < q \leq m + 1$ , by an expansion of a real number  $x$  in base  $-q$  we mean a sequence  $(c_i) = c_1 c_2 \dots$  of integers  $c_i \in A := \{0, 1, \dots, m\}$  satisfying

$$\sum_{i=1}^{\infty} \frac{c_i}{(-q)^i} = x.$$

One easily verifies that  $(c_i)$  is an expansion of a real number  $x$  in base  $-q$  if and only if  $(c'_i) := (m - c_1, c_2, m - c_3, c_4, \dots)$  is an expansion of  $x' := x + mq/(q^2 - 1)$  in base  $q$  (with respect to  $A$ ). It follows from Proposition 1.1 that each  $x$  belonging to the interval

$$J_{A,-q} := \left[ \frac{-mq}{q^2 - 1}, \frac{m}{q^2 - 1} \right]$$

has an expansion in base  $-q$ .

**Definition 4.1.** An expansion  $(d_i)$  of  $x$  in base  $-q$  is *optimal* if for any other expansion  $(c_i)$  of  $x$  in base  $-q$  we have

$$\left| x - \sum_{i=1}^n \frac{d_i}{(-q)^i} \right| \leq \left| x - \sum_{i=1}^n \frac{c_i}{(-q)^i} \right|$$

for all  $n = 1, 2, \dots$

We only consider here expansions in negative integer bases  $-2, -3, \dots$ . While in positive integer bases the greedy expansion is always optimal, in negative integer bases there are infinitely many numbers with no optimal expansion:

**Proposition 4.2.** *In negative integer bases only the unique expansions are optimal.*

*Proof.* Let  $q = m + 1$  for some positive integer  $m$ . If  $x \in J_{A,-q}$  has no unique expansion in base  $-q$ , then  $x$  has exactly two expansions  $(c_i)$  and  $(d_i)$  in base  $-q$  because  $(c'_i)$  and  $(d'_i)$  are the only expansions of  $x'$  in base  $q$ . Moreover, there exists a positive integer  $k$  such that  $c'_i = d'_i$  for  $1 \leq i \leq k - 1$  and such that the sequences  $(c'_k, c'_{k+1}, \dots)$  and  $(d'_k, d'_{k+1}, \dots)$  are equal to  $(p + 1)0^\infty$  or  $pm^\infty$  for some  $p \in \{0, \dots, m - 1\}$ . If necessary, interchange  $(c_i)$  and  $(d_i)$  so that  $(c'_i) > (d'_i)$ , and let  $n$  be a positive integer such that  $2n \geq k$ . Then

$$x = \left( \sum_{i=1}^{2n} \frac{c_i}{(-q)^i} \right) - \sum_{i=n}^{\infty} \frac{m}{q^{2i+1}} = \left( \sum_{i=1}^{2n} \frac{d_i}{(-q)^i} \right) + \sum_{i=n}^{\infty} \frac{m}{q^{2i+2}},$$

whence

$$\left| x - \sum_{i=1}^{2n+1} \frac{c_i}{(-q)^i} \right| = \frac{1}{q} \left| x - \sum_{i=1}^{2n+1} \frac{d_i}{(-q)^i} \right| < \left| x - \sum_{i=1}^{2n+1} \frac{d_i}{(-q)^i} \right|$$

and

$$\left| x - \sum_{i=1}^{2n} \frac{d_i}{(-q)^i} \right| = \frac{1}{q} \left| x - \sum_{i=1}^{2n} \frac{c_i}{(-q)^i} \right| < \left| x - \sum_{i=1}^{2n} \frac{c_i}{(-q)^i} \right|$$

so that the expansions  $(c_i)$  and  $(d_i)$  are not optimal.  $\square$

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