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QUASI SURE LOCAL CONVERGENCE RATE OF A BROWNIAN MOTION IN THE HÖLDER NORM

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ABSTRACT. We estimate the local convergence rate of Strassen type for a Brownian motion in the Hölder norm with respect to $C_{r,p}$-capacity on an abstract Wiener space. The local convergence rate for increments of a Brownian motion in the Hölder norm with respect to $C_{r,p}$-capacity is also derived.

1. INTRODUCTION

The quasi sure (q.s.) analysis is one of the important concepts in stochastic analysis. It studies the properties of stochastic processes which hold outside sets with capacity null. Capacity is a kind of set function which is much finer than probability.

Let $L^p = L^p(B,\mu)$ denote the real-valued function space on (B,μ) and let (B,H,μ) be an abstract Wiener space with the Ornstein-Uhlenbeck operator $\mathcal{L}$. We denote the Sobolev space $D^{r,p}$ by $D^{r,p} = (1-\mathcal{L})^{-\frac{r}{2}} L^p$ with norm

$$\|F\|_{r,p} = \|(1-\mathcal{L})^{r/2} F\|_p, \quad F \in D^{r,p},\ r>0,\ 1 \le p < \infty.$$

Let $\mathcal{C}$ denote the space of continuous functions from $[0,1]$ to $\mathbb{R}^d$ endowed with the usual norm $\|f\| := \sup_{0\le t\le 1} |f(t)|$. We denote $\mathcal{C}_x := \{f \in \mathcal{C} : f(0) = x\}$ and denote

$$\mathcal{H}^d := \left\{ f \in \mathcal{C}_0 \,:\, f(t) = \int_0^t \dot{f}(s)ds,\ \|f\|^2_{\mathcal{H}^d} := \int_0^1 |\dot{f}(t)|^2 dt < \infty \right\}.$$

It is clear that $\mathcal{H}^d$ is a Hilbert space with scalar product

$$\langle r_1, r_2\rangle_{\mathcal{H}^d} = \int_0^1 (\dot{r}_1(s), \dot{r}_2(s))ds.$$

Let the function $I : B \to [0,\infty]$ be defined by $I(f) = \frac{1}{2}\|f\|^2_{\mathcal{H}^d}$, if $f \in \mathcal{H}^d$ and let $I(f) = +\infty$, otherwise. Let μ be the Wiener measure. Then $(\mathcal{C}_0, \mathcal{H}^d, \mu)$ is an

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abstract Wiener space. We will consider two Banach spaces as follows:

$$\mathcal{C}^{\alpha} = \left\{ f \in \mathcal{C}_0 \,:\, \|f\|_{\alpha} = \sup_{s,\ t\in[0,1],\ s\neq t} \frac{|f(t)-f(s)|}{|t-s|^{\alpha}} < \infty \right\},$$

$$\mathcal{C}^{\alpha,\,0} = \left\{ f \in \mathcal{C}^{\alpha} \,:\, \lim_{\delta\to 0} \sup_{s,\ t\in[0,1],\ 0<|t-s|<\delta} \frac{|f(t)-f(s)|}{|t-s|^{\alpha}} = 0 \right\},$$

where $0 < \alpha < \frac{1}{2}$. Clearly, the space $(\mathcal{C}^{\alpha,0}, \mathcal{H}^d, \mu)$ is also an abstract Wiener space; see, e.g., [1].

The capacity is a set function on B with the property that it may take positive values even for μ-null sets. For $r > 0$ and $p > 1$, the (r,p)-capacity is defined by

$$C_{r,p}(O) = \inf \left\{ \|F\|_{r,p}^p \,:\, F \geq 1, \mu - a.s. \text{ on } O \right\}, \quad \text{for open sets } O \subset B,$$

and for any set $A \subset B$,

$$C_{r,p}(A) = \inf \left\{ C_{r,p}(O) \,:\, A \subset O \subset B,\ O \text{ is open} \right\}.$$

A large deviation principle for $C_{r,p}$ has been established by N. Yoshida [6]. The rate of clustering for Brownian motion in the Hölder norm was obtained by P. Baldi and B. Roynette [2]. In this paper, our aim is to estimate the quasi sure local convergence rate for a Brownian motion in the Hölder norm. We also investigate the quasi sure local convergence rate for increments of a Brownian motion in the Hölder norm.

2. Local functional convergence rate of Strassen type in the Hölder norm

The functional law of the iterated logarithm (LIL) of Strassen type in the Hölder norm was derived by P. Baldi, G. Ben Arous and G. Kerkyacharian in [1]. X. Chen and N. Balakrishnan obtained the functional LIL of Strassen type in the Hölder norm w.r.t. $C_{r,p}$-capacity [4]. We now present the local convergence rate of functional LIL of Strassen type for a Brownian motion in the Hölder norm with respect to $C_{r,p}$-capacity on an abstract Wiener space.

Suppose $w \in \mathcal{C}^{\alpha,0}$ and let $K = \{f \in \mathcal{H}^d \,:\, I(f) \leq 1\}$. In [2], the authors proved that there exists a constant $k(\alpha) > 0$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(1-2\alpha)} \log P\{\|w\|_{\alpha} \leq \varepsilon\} = -k(\alpha). \tag{2.1}$$

Moreover, for every $f \in K$ and $\gamma = (1-2\alpha)/2$,

$$\lim_{\varepsilon\to 0} \varepsilon^{1/\gamma} \log P\left(\left\| w - \frac{f}{\varepsilon^{1/(2\gamma)}} \right\|_{\alpha} \leq r\varepsilon \right) = -I(f) - \frac{k(\alpha)}{r^{1/\gamma}}. \tag{2.2}$$

Let us use the notation $LL(t) := \log\log t$ in the sequel. We state the main result of this section as follows.

Theorem 2.1. *Let $0 < \alpha < 1, \gamma = \frac{1}{2} - \alpha$ and let $f \in K$. If f satisfies $I(f) < 1$, then we have*

$$\liminf_{t\to 0} \left(LL(t^{-1}) \right)^{1-\alpha} \left\| \frac{w(t\cdot)}{\sqrt{tLL(t^{-1})}} - f \right\|_{\alpha} = \left(\frac{k(\alpha)}{1-I(f)} \right)^{\gamma}, \qquad C_{r,p} - q.s.$$

where $k(\alpha) > 0$ is defined as in (2.1).

The theorem follows from Lemmas 2.6 and 2.9, which we will state and prove in the rest of this section.

Lemma 2.2. *Let k be a natural number and let $q_1, q_2 \in (1,\infty)$ satisfy $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, there exists a constant $c = c(k, p, q_1, q_2) > 0$ such that for any $\delta \in (0,1)$,*

$$C_{k,p}\left(\bigcap_{i=1}^{n}\left\{z\ :\ a_i < \tilde{F}_i(z) < b_i\right\}\right)^{1/p}$$
$$\leq c\left(\frac{n}{\delta}\right)^k\left(1 + \max_{1\leq i\leq n}\|F_i\|_{k,kq_1}\right)^k \mu\left(\bigcap_{i=1}^{n}\{a_i - \delta < F_i(z) < b_i + \delta\}\right)^{1/q_2}$$

for any $F_i \in D^{k,kq_1}$ and $-\infty < a_i < b_i < \infty$ holds. Here $\tilde{F}_i$ denotes a quasi continuous modification of F_i.

Proof. Set

$$\chi(z) = \begin{cases} 1, & \text{if } \bigcap_{i=1}^{n}\{z\ :\ a_i - \delta < F_i(z) < b_i + \delta\}, \\ 0, & \text{otherwise.}\end{cases}$$

We see that for $m \leq k$, $n_1 + \cdots + n_m = l \leq k$, $i_1, \cdots, i_m \leq n$,

$$\begin{aligned}\|\chi D^{n_1}F_{i_1} \otimes \cdots \otimes D^{n_m}F_{i_m}\|_p &\leq \|\chi\|_{q_2}\prod_{a=1}^{m}\|D^{n_a}F_{i_a}\|_{mq_1} \\ &\leq c_{k,q_1}\left(\max_{1\leq i\leq n}\|F_i\|_{k,mq_1} + 1\right)^m\|\chi\|_{q_2}.\end{aligned}$$

The rest of the proof is the same as that of Proposition 3.1 in [6]. □

Lemma 2.3. *Let k, p, q_1, q_2 be defined as in Lemma 2.2. For any $f \in K$ and $\varepsilon > 0$, we set*

$$F_\varepsilon^{(i)}(w) = \left\|\varepsilon\left(\frac{w(t_i + \cdot h_i) - w(t_i)}{\sqrt{h_i}}\right) - f\right\|_\alpha, \quad 0 \leq t_i < \infty,\ h_i > 0,\ i = 1, 2, \cdots, n.$$

Then there exists a constant $c = c(k, p, q_1, f, d) > 0$ such that for any $\delta \in (0,1], \varepsilon \in (0,1]$, we have

$$C_{k,p}\left(\bigcap_{i=1}^{n}\{z : a_i < F_\varepsilon^{(i)}(z) < b_i\}\right)^{\frac{1}{p}}$$
$$\leq\ c\,\delta^{-2k^2-k}n^k\mu\left(\bigcap_{i=1}^{n}\{z : a_i - \delta < F_\varepsilon^{(i)}(z) < b_i + \delta\}\right)^{\frac{1}{q_2}}.$$

Proof. Let $F : B \to R$ be a Borel function which represents an element $\bar{F}$ in L^p. We set

$$(T_tF)(z) = \int_B F\left(e^{-t}z + \sqrt{1 - e^{-2t}}y\right)\mu(dy).$$

Then T_tF is a quasi continuous version of $e^{t\mathcal{L}}\bar{F}$. Here $(e^{t\mathcal{L}})_{t>0}$ is the Ornstein-Uhlenbeck semigroup acting on L^p.

For any $q_1 > 0$, $F_\varepsilon^{(i)} \in L^{q_1}, i = 1, 2, \cdots, n, \varepsilon > 0$, we know that $T_t F_\varepsilon^{(i)}$ represents an element in $\bigcap_{l=1}^{\infty} D^{l,kq_1}$. We also have the following estimate:

$$\left\| T_t F_\varepsilon^{(i)} \right\|_{2k,kq_1} = e^t \left\| (1-\mathcal{L})^k e^{-t(1-\mathcal{L})} F_\varepsilon^{(i)} \right\|_{kq_1} \le c_1 e^t t^{-k} \left\| F_\varepsilon^{(i)} \right\|_{kq_1},$$

where $c_1 = c_1(k, q_1) > 0$. By the definition of $F_\varepsilon^{(i)}$, we have

$$\begin{aligned} F_\varepsilon^{(i)}(e^{-t}z + \sqrt{1-e^{-2t}}y) \le e^{-t} F_\varepsilon^{(i)}(z) + \sqrt{1-e^{-2t}} F_\varepsilon^{(i)}(y) \\ + |e^{-t} + \sqrt{1-e^{-2t}} - 1| \cdot \|f\|_\alpha. \end{aligned}$$

Therefore, we deduce that

$$\begin{aligned} & T_t F_\varepsilon^{(i)}(z) \\ = & \int_{\mathcal{C}^{\alpha,0}} F_\varepsilon^{(i)}(e^{-t}z + \sqrt{1-e^{-2t}}y)\mu(dy) \\ \le & e^{-t} F_\varepsilon^{(i)}(z) + \sqrt{1-e^{-2t}} \int_{\mathcal{C}^{\alpha,0}} F_\varepsilon^{(i)}(y)\mu(dy) + |e^{-t} + \sqrt{1-e^{-2t}} - 1| \cdot \|f\|_\alpha \\ \le & e^{-t} F_\varepsilon^{(i)}(z) + e^t \sqrt{1-e^{-2t}} \int_{\mathcal{C}^{\alpha,0}} F_\varepsilon^{(i)}(y)\mu(dy) + e^t |e^{-t} + \sqrt{1-e^{-2t}} - 1| \cdot \|f\|_\alpha \\ = & e^{-t} F_\varepsilon^{(i)}(z) + C_t, \end{aligned}$$

where $C_t = e^t \sqrt{1-e^{-2t}} \int_{\mathcal{C}^{\alpha,0}} F_\varepsilon^{(i)}(y)\mu(dy) + e^t |e^{-t} + \sqrt{1-e^{-2t}} - 1| \, \|f\|_\alpha$. Similarly, we can also derive $F_\varepsilon^{(i)}(z) \le e^t T_t F_\varepsilon^{(i)}(z) + C_t$. By applying Lemma 2.2, similarly to the proof of Lemma 2.3 in [7], we complete the proof of Lemma 2.3. □

Lemma 2.4. *Let the constant $k(\alpha) > 0$ be defined as in* (2.1). *For any $f \in K$, $\tau > 0$, $h > 0$, we have*

$$\begin{aligned} & \lim_{\varepsilon \to 0} \varepsilon^{1/\gamma} \log C_{r,p} \left(\left\| \frac{w(h\cdot)}{\sqrt{h}} - \frac{f}{\varepsilon^{1/(2\gamma)}} \right\|_\alpha \le \varepsilon\tau \right) \\ = & \lim_{\varepsilon \to 0} \varepsilon^{1/\gamma} \log \mu \left(\left\| \frac{w(h\cdot)}{\sqrt{h}} - \frac{f}{\varepsilon^{1/(2\gamma)}} \right\|_\alpha \le \varepsilon\tau \right) \\ = & -\frac{k(\alpha)}{\tau^{1/\gamma}} - I(f). \end{aligned}$$

Proof. Let k, p, q_1, q_2 be as in Lemma 2.2. Since $C_{r,p}(\cdot) \ge \mu(\cdot)$, it suffices to prove that

$$\begin{aligned} & \lim_{\varepsilon \to 0} \varepsilon^{1/\gamma} \log C_{r,p} \left(\left\| \frac{w(h\cdot)}{\sqrt{h}} - \frac{f}{\varepsilon^{1/(2\gamma)}} \right\|_\alpha \le \varepsilon\tau \right) \\ \le & \lim_{\varepsilon \to 0} \varepsilon^{1/\gamma} \log \mu \left(\left\| \frac{w(h\cdot)}{\sqrt{h}} - \frac{f}{\varepsilon^{1/(2\gamma)}} \right\|_\alpha \le \varepsilon\tau \right). \end{aligned}$$

Set $k = [r] + 1$. By Lemma 2.3, for any $1 > \delta > 0$ and $c_0 > 0$, we have

$$\begin{aligned} & C_{r,p}\left(\left\|\frac{w(h\cdot)}{\sqrt{h}} - \frac{f}{\varepsilon^{1/(2\gamma)}}\right\|_\alpha \le \varepsilon\tau\right)^{1/p} \\ = \quad & C_{r,p}\left(\left\|\varepsilon^{1/(2\gamma)}\frac{w(h\cdot)}{\sqrt{h}} - f\right\|_\alpha \le \varepsilon^{1/(2\gamma)+1}\tau\right)^{1/p} \\ \le \quad & c_0(\varepsilon^{1/(2\gamma)+1}\delta)^{-2k^2-k}\mu\left(\left\|\varepsilon^{1/(2\gamma)}\frac{w(h\cdot)}{\sqrt{h}} - f\right\|_\alpha \le \varepsilon^{1/(2\gamma)+1}(\tau+\delta)\right)^{1/q_2} \\ = \quad & c_0(\varepsilon^{1/(2\gamma)+1}\delta)^{-2k^2-k}\mu\left(\left\|w(\cdot) - \frac{f}{\varepsilon^{1/(2\gamma)}}\right\|_\alpha \le \varepsilon(\tau+\delta)\right)^{1/q_2}. \end{aligned}$$

By (2.2), we have

$$\begin{aligned} & \lim_{\varepsilon\to 0}\varepsilon^{1/\gamma}\log C_{r,p}\left(\left\|\frac{w(h\cdot)}{\sqrt{h}} - \frac{f}{\varepsilon^{1/(2\gamma)}}\right\|_\alpha \le \varepsilon\tau\right) \\ \le \quad & \frac{p}{q_2}\lim_{\varepsilon\to 0}\varepsilon^{1/\gamma}\log\mu\left(\left\|w(\cdot) - \frac{f}{\varepsilon^{1/(2\gamma)}}\right\|_\alpha \le \varepsilon(\tau+\delta)\right) \\ = \quad & \frac{p}{q_2}(-k(\alpha)(\tau+\delta)^{-\frac{1}{\gamma}} - I(f)). \end{aligned}$$

Letting $\delta \to 0$ and $q_2 \to p$, we end the proof of Lemma 2.4. □

Lemma 2.5. *For $f \in K$ with $I(f) < 1$, we have*

$$\liminf_{n\to\infty}(LL(t_n^{-1}))^{1-\alpha}\left\|\frac{w(t_n\cdot)}{\sqrt{t_nLL(t_n^{-1})}} - f\right\|_\alpha \ge \left(\frac{k(\alpha)}{1-I(f)}\right)^\gamma, \qquad C_{r,p} - q.s. \tag{2.3}$$

where $t_n = \exp\left(-\frac{n}{(\log n)^a}\right)$, $a > 0$.

Proof. For any $\varepsilon \in (0,1), f \in K$, since $I(f) < 1$, choose $\delta_1 > 0$ such that $\eta_0 = I(f) + \frac{1-I(f)}{(1-\varepsilon)^{1/\gamma}} - \delta_1 > 1$. By Lemma 2.4, for n large enough, we have

$$\begin{aligned} & C_{r,p}\left((LL(t_n^{-1}))^{1-\alpha}\left\|\frac{w(t_n\cdot)}{\sqrt{t_nLL(t_n^{-1})}} - f\right\|_\alpha \le (1-\varepsilon)\left(\frac{k(\alpha)}{1-I(f)}\right)^\gamma\right) \\ = \quad & C_{r,p}\left(\left\|\frac{w(t_n\cdot)}{\sqrt{t_n}} - (LL(t_n^{-1}))^{1/2}f\right\|_\alpha \le (LL(t_n^{-1}))^{-\frac{1}{2}+\alpha}(1-\varepsilon)\left(\frac{k(\alpha)}{1-I(f)}\right)^\gamma\right) \\ \le \quad & \exp\left\{(LL(t_n^{-1}))(-\frac{1-I(f)}{(1-\varepsilon)^{1/\gamma}} - I(f) + \delta_1)\right\} = \left(\frac{1}{\log t_n^{-1}}\right)^{\eta_0}. \end{aligned}$$

By Borel-Cantelli's Lemma, we have

$$\liminf_{n\to\infty}(LL(t_n^{-1}))^{1-\alpha}\left\|\frac{w(t_n\cdot)}{\sqrt{t_nLL(t_n^{-1})}} - f\right\|_\alpha \ge \left(\frac{k(\alpha)}{1-I(f)}\right)^\gamma, \qquad C_{r,p} - q.s. \tag{2.4}$$

□

Lemma 2.6. *For any $f \in K$ with $I(f) < 1$, we have*

$$\liminf_{t\to 0}(LL(t^{-1}))^{1-\alpha}\left\|\frac{w(t\cdot)}{\sqrt{tLL(t^{-1})}}-f\right\|_\alpha \geq \left(\frac{k(\alpha)}{1-I(f)}\right)^\gamma, \quad C_{r,p}-q.s. \tag{2.5}$$

Proof. Let $\psi_t(s)=\frac{w(ts)}{\sqrt{tLL(t^{-1})}}$, $s\in[0,1], t\in[0,1]$. Let t_n be as in Lemma 2.5. For $t_{n+1}<t\leq t_n$, we set $X(t)=(LL(t^{-1}))^{1-\alpha}\|\psi_t(\cdot)-f\|_\alpha$ and $X_n=\inf_{t_{n+1}<t\leq t_n}X(t)$. For any $\varepsilon>0$, by the definition of the infimum, there exists $T_n\in(t_{n+1},t_n]$ such that $X_n\geq X(T_n)-\varepsilon$. For any $u,v\in[0,1]$, let $x=\frac{ut_{n+1}}{T_n}, y=\frac{vt_{n+1}}{T_n}$; then $0\leq x,y\leq\frac{t_{n+1}}{T_n}\leq 1$. We have

$$\begin{aligned}
&\|\psi_{t_{n+1}}(\cdot)-f\|_\alpha\\
=&\sup_{0\leq u<v\leq 1}\frac{|(\psi_{t_{n+1}}(u)-f(u))-(\psi_{t_{n+1}}(v)-f(v))|}{|u-v|^\alpha}\\
\leq&\sup_{0\leq x<y\leq\frac{t_{n+1}}{T_n}}\frac{|(\psi_{t_{n+1}}(\frac{T_nx}{t_{n+1}})-f(\frac{T_nx}{t_{n+1}}))-(\psi_{t_{n+1}}(\frac{T_ny}{t_{n+1}})-f(\frac{T_ny}{t_{n+1}}))|}{|x-y|^\alpha}\\
\leq&\left\|\gamma_n\psi_{T_n}(\cdot)-f(\frac{T_n}{t_{n+1}}\cdot)\right\|_\alpha\\
\leq&\gamma_n\|\psi_{T_n}(\cdot)-f(\cdot)\|_\alpha+|\gamma_n-1|\,\|f(\cdot)\|_\alpha+\|f(\cdot)-f(\frac{T_n}{t_{n+1}}\cdot)\|_\alpha,
\end{aligned} \tag{2.6}$$

where we use the notation $\gamma_n=\frac{\sqrt{T_nLL(T_n^{-1})}}{\sqrt{t_{n+1}LL(t_{n+1}^{-1})}}$.

By (2.6), we have

$$X(T_n)\geq\tilde{\gamma}_n(LLt_{n+1}^{-1})^{1-\alpha}\Big(\|\psi_{t_{n+1}}(\cdot)-f\|_\alpha-|\gamma_n-1|\,\|f(\cdot)\|_\alpha-\|f(\cdot)-f(\frac{T_n}{t_{n+1}}\cdot)\|_\alpha\Big), \tag{2.7}$$

where $\tilde{\gamma}_n=\sqrt{\frac{t_n}{t_{n+1}}}\left(\frac{LL(t_n^{-1})}{LL(t_{n+1}^{-1})}\right)^{\frac{1}{2}-\alpha}$.

The inequality $\exp(-x)\geq 1-x$ yields

$$\frac{t_{n+1}}{t_n}\geq 1-\frac{n+1}{(\log(n+1))^a}+\frac{n}{(\log n)^a}.$$

From the inequality above we have

$$1-\frac{t_{n+1}}{t_n}\leq\frac{1}{(\log n)^a}. \tag{2.8}$$

Similar to the proof of (5.3) in [2], we have

$$\left\|f(\frac{T_n}{t_{n+1}}\cdot)-f(\cdot)\right\|_\alpha\leq 2\Big(\frac{T_n}{t_{n+1}}-1\Big)^{\frac{1}{2}-\alpha}\leq 2\Big(\frac{t_n}{t_{n+1}}-1\Big)^{\frac{1}{2}-\alpha}. \tag{2.9}$$

Since $t_n/t_{n+1}\to 1$, as $n\to\infty$, for n large enough, we have $t_n/t_{n+1}<2$. By (2.8) and (2.9), for n large enough, we have

$$\left\|f(\frac{T_n}{t_{n+1}}\cdot)-f(\cdot)\right\|_\alpha\leq 4\left(\frac{1}{(\log n)^a}\right)^{\frac{1}{2}-\alpha}. \tag{2.10}$$

Noting the above, for n large enough, we have

$$\left|\frac{(T_n LL(T_n^{-1}))^{1/2}}{(t_{n+1}LL(t_{n+1}^{-1}))^{1/2}} - 1\right| \le \left|\frac{t_n}{t_{n+1}} - 1\right| \le \frac{2}{(\log n)^a}. \tag{2.11}$$

Choosing a suitable a, by (2.7), (2.10), (2.11) and Lemma 2.5, we have

$$\liminf_{n\to\infty} X(T_n) \ge \left(\frac{k(\alpha)}{1-I(f)}\right)^{\gamma}, \quad C_{r,p} - q.s.$$

Since

$$\liminf_{t\to 0} X(t) \ge \liminf_{n\to\infty} X_n \ge \liminf_{n\to\infty} X(T_n) - \varepsilon,$$

we can conclude (2.5). □

Lemma 2.7. *For $u \ge 3$, we define*

$$\xi_u(t,w) = \frac{w(\frac{t}{u})}{\sqrt{\frac{2}{u}\log\log u}}, \qquad t\in[0,1],\ w\in\mathcal{C}^{\alpha,0}.$$

We have, for any $(r,p)\in(0,\infty)\times(1,\infty)$, for $C_{r,p} - q.s.$ w, $\{\xi_u(\cdot,w), u\ge 3\}$ is relatively compact in $\mathcal{C}^{\alpha,0}$, and the set of its limit points is a subset of $C := \{f\in\mathcal{H}^d\ :\ \|f\|^2_{\mathcal{H}^d}\le 1\}$ as $u\to\infty$.

Proof. Since C is compact in $\mathcal{C}^{\alpha,0}$, it is sufficient to prove that

$$\lim_{u\to\infty}\|\xi_u(\cdot,w) - C\|_\alpha = 0, \quad C_{r,p} - q.s.$$

Noting that

$$\begin{aligned}&\mu\left(\left\|\frac{w(\frac{\cdot}{u})}{\sqrt{\frac{2}{u}\log\log u}} - C\right\|_\alpha \ge \varepsilon\right)\\ =\ &\mu\left(\left\|\frac{1}{\sqrt{2\log\log u}}w(\cdot) - C\right\|_\alpha \ge\varepsilon\right) = \mu\left(\frac{1}{\sqrt{2\log\log u}}w\in A\right),\end{aligned}$$

where $A = \{f\in\mathcal{C}^{\alpha,0}\ :\ \|f(\cdot) - C\|_\alpha\ge\varepsilon\}$, it is clear that the set A is closed and for any $f\in A$,

$$\int_0^1 |\dot f(s)|^2 ds > 1.$$

Accordingly, there exists $\delta > 0$ such that $\inf_{f\in A} I(f) > \frac{1}{2} + \frac{\delta}{2}$. For $\theta > 1$, and letting $u_n = \theta^n$, for n large enough, by Theorem 2.1 in [4], we have

$$C_{r,p}\left(\left\|\frac{w(\frac{\cdot}{u_n})}{\sqrt{\frac{2}{u_n}\log\log u_n}} - C\right\|_\alpha \ge\varepsilon\right) \le \exp\left(-(1+\delta)\log\log u_n\right).$$

By Borel-Cantelli's Lemma,

$$\limsup_{n\to\infty}\|\xi_{u_n}(\cdot,w) - C\|_\alpha = 0, \quad C_{r,p} - q.s.$$

Thus, by Lemma 3.1 in [4], we have

$$\lim_{u\to\infty}\|\xi_u(\cdot,w) - C\|_\alpha = 0, \quad C_{r,p} - q.s.$$

□

Lemma 2.8. *Let $t_n = \frac{1}{n^n}$. For $\varphi \in K$ with $I(\varphi) < 1$, we have*

$$\liminf_{n\to\infty}(LL(t_n^{-1}))^{1-\alpha}\left\|\frac{w(t_n\cdot)}{\sqrt{t_nLL(t_n^{-1})}}-\varphi\right\|_\alpha \le \left(\frac{k(\alpha)}{1-I(\varphi)}\right)^\gamma, \quad C_{r,p}-q.s.$$

Proof. We write $g_n(s) = \frac{w(t_ns)}{\sqrt{t_nLL(t_n^{-1})}} - \varphi(s)$,

$$\begin{aligned}
\|g_n(\cdot)\|_\alpha &= \sup_{0\le t<s\le 1}\frac{|g_n(s)-g_n(t)|}{|s-t|^\alpha}\\
&\le \sup_{0\le t<s\le \frac{t_{n+1}}{t_n}}\frac{|g_n(s)-g_n(t)|}{|s-t|^\alpha} + \sup_{\frac{t_{n+1}}{t_n}\le t<s\le 1}\frac{|g_n(s)-g_n(t)|}{|s-t|^\alpha}\\
&\le (\frac{t_n}{t_{n+1}})^\alpha\left\|\frac{w(t_{n+1}\cdot)}{\sqrt{t_nLL(t_n^{-1})}}-\varphi(\frac{t_{n+1}}{t_n}\cdot)\right\|_\alpha + \sup_{\frac{t_{n+1}}{t_n}\le t<s\le 1}\frac{|g_n(s)-g_n(t)|}{|s-t|^\alpha}. \qquad (2.12)
\end{aligned}$$

In the following, we prove that
(2.13)

$$\limsup_{n\to\infty}\left(LL(t_n^{-1})\right)^{1-\alpha}\left(\frac{t_n}{t_{n+1}}\right)^\alpha\left\|\frac{w(t_{n+1}\cdot)}{\sqrt{t_nLL(t_n^{-1})}}-\varphi(\frac{t_{n+1}}{t_n}\cdot)\right\|_\alpha = 0, \quad C_{r,p}-q.s.$$

In fact

$$\begin{aligned}
\left\|g_n\left(\frac{t_{n+1}}{t_n}\cdot\right)\right\|_\alpha &\le \left\|\frac{w(t_{n+1}\cdot)}{(t_nLL(t_n^{-1}))^{1/2}}\right\|_\alpha + \left\|\varphi(\frac{t_{n+1}}{t_n}\cdot)\right\|_\alpha\\
&= \frac{(t_{n+1}LL(t_{n+1}^{-1}))^{1/2}}{(t_nLL(t_n^{-1}))^{1/2}}\left\|\frac{w(t_{n+1}\cdot)}{\sqrt{t_{n+1}LL(t_{n+1}^{-1})}}\right\|_\alpha + \left\|\varphi(\frac{t_{n+1}}{t_n}\cdot)\right\|_\alpha.
\end{aligned}$$

Since $\left\|\frac{w(t_{n+1}\cdot)}{\sqrt{t_{n+1}LL(t_{n+1}^{-1})}}\right\|_\alpha$ is bounded by local Strassen's law (see Lemma 2.7) and

$$(LL(t_n^{-1}))^{1-\alpha}\left(\frac{t_n}{t_{n+1}}\right)^\alpha\frac{(t_{n+1}LL(t_{n+1}^{-1}))^{1/2}}{(t_nLL(t_n^{-1}))^{1/2}} \to 0, \text{ as } n\to\infty,$$

we get

$$\left(LL(t_n^{-1})\right)^{1-\alpha}\left(\frac{t_n}{t_{n+1}}\right)^\alpha\frac{(t_{n+1}LL(t_{n+1}^{-1}))^{1/2}}{(t_nLL(t_n^{-1}))^{1/2}}\left\|\frac{w(t_{n+1}\cdot)}{\sqrt{t_{n+1}LL(t_{n+1}^{-1})}}\right\|_\alpha \to 0, \text{ as } n\to\infty.$$

Since $f\in K$, for $\gamma\le 1$, we have $\|f(\gamma\cdot)\|_\alpha \le \|f(\gamma\cdot)\|_{\mathcal{H}^d} \le \gamma^{1/2}\|f\|_{\mathcal{H}^d}$ (see the proof of Theorem 5.1 in [2]). Therefore we deduce that

$$\begin{aligned}
&(LL(t_n^{-1}))^{1-\alpha}(\frac{t_n}{t_{n+1}})^\alpha\|\varphi(\frac{t_{n+1}}{t_n}\cdot)\|_\alpha\\
\le\ & 2(LL(t_n^{-1}))^{1-\alpha}(\frac{t_n}{t_{n+1}})^\alpha\sqrt{\frac{t_{n+1}}{t_n}}\\
=\ & 2(\log n + LL(n))^{1-\alpha}\left(\frac{n^n}{(n+1)^{(n+1)}}\right)^{1/2-\alpha} \to 0, \text{ as } n\to\infty.
\end{aligned}$$

So we obtain (2.13).

We set $w_n(s) = \frac{w(t_n s) - w(t_{n+1})}{\sqrt{t_n LL(t_n^{-1})}} - \varphi(s)$ for convenience. We now prove that

(2.14)
$$\liminf_{n\to\infty} (LL(t_n^{-1}))^{1-\alpha} \sup_{\frac{t_{n+1}}{t_n} \le t < s \le 1} \frac{|w_n(s) - w_n(t)|}{|s-t|^\alpha} \le \left(\frac{k(\alpha)}{1 - I(\varphi)}\right)^\gamma, \quad C_{r,p} - q.s.$$

In order to complete the proof, we set

$$\tilde{w}_n(s) = \frac{w((t_n - t_{n+1})s + t_{n+1}) - w(t_{n+1})}{\sqrt{t_n - t_{n+1}}}, \quad s \ge 0,$$
$$f(s_1) = \left(\frac{t_n}{t_n - t_{n+1}}\right)^{\frac{1}{2}} \left(\varphi\left(\frac{t_{n+1} + s_1(t_n - t_{n+1})}{t_n}\right) - \varphi\left(\frac{t_{n+1}}{t_n}\right)\right), \quad s_1 \in [0,1].$$

Then $\tilde{w}_n = \{\tilde{w}_n(s) \ : \ s \ge 0\}$ is again a standard Brownian motion and $f \in K$ with $I(f) \le I(\varphi)$. It is clear that $\frac{t_n}{t_n - t_{n+1}} \to 1$, as $n \to \infty$. For any $\varepsilon > 0$, choose $\delta > 0$ such that $\sigma := \frac{1 - I(f)}{(1+\varepsilon)^{1/\gamma}} + I(f) + \delta < 1$. By Lemma 2.3,

$$C_{r,p}\left(\bigcap_{n=n_0}^{N} \Big(LL(t_n^{-1})^{1-\alpha} \sup_{\frac{t_{n+1}}{t_n} \le t < s \le 1} \frac{|w_n(s) - w_n(t)|}{|s-t|^\alpha} \ge (\frac{k(\alpha)}{1 - I(\varphi)})^\gamma (1 + 2\varepsilon)\Big)\right)^{\frac{1}{p}}$$
$$= C_{r,p}\left\{\bigcap_{n=n_0}^{N} \Big(W^\alpha(n) \ge (\frac{k(\alpha)}{1 - I(\varphi)})^\gamma \frac{1 + 2\varepsilon}{(LL(t_n^{-1}))^{\frac{1}{2}-\alpha}}\Big)\right\}^{1/p}$$
$$\le cN^k (LL(t_N^{-1})^{\frac{1}{2}-\alpha})^{2k^2+k} \mu\left\{\bigcap_{n=n_0}^{N} W^\alpha(n) \ge \left(\frac{k(\alpha)}{1 - I(f)}\right)^\gamma \frac{(1+\varepsilon)}{(LL(t_n^{-1}))^{\frac{1}{2}-\alpha}}\right\}^{1/q_2},$$

where $W^\alpha(n) = \left(\frac{t_n}{t_n - t_{n+1}}\right)^{\alpha - \frac{1}{2}} \left\|\tilde{w}_n(\cdot) - \sqrt{LL(t_n^{-1})} f(\cdot)\right\|_\alpha$. For n large enough, by a small deviation,

$$\mu\left\{\bigcap_{n=n_0}^{N} W^\alpha(n) \ge \left(\frac{k(\alpha)}{1 - I(f)}\right)^\gamma \frac{(1+\varepsilon)}{(LL(t_n^{-1}))^{\frac{1}{2}-\alpha}}\right\}^{1/q_2} \le \prod_{n=n_0}^{N} (1 - \exp(-\sigma LL(t_n^{-1})))^{1/q_2}.$$

We further deduce that

$$C_{r,p}\left(\bigcap_{n=n_0}^{N} \Big(LL(t_n^{-1})^{1-\alpha} \sup_{\frac{t_{n+1}}{t_n} \le t < s \le 1} \frac{|w_n(s) - w_n(t)|}{|s-t|^\alpha} \ge (\frac{k(\alpha)}{1 - I(\varphi)})^\gamma (1 + 2\varepsilon)\Big)\right)^{\frac{1}{p}}$$
$$\le cN^k (LL(t_N^{-1})^{\frac{1}{2}-\alpha})^{2k^2+k} \exp\left(-\frac{1}{q_2} \sum_{n=n_0+1}^{N+1} (n \log n)^{-\sigma}\right)$$
$$\le c_0 N^k ((\log N)^{\frac{1}{2}-\alpha})^{2k^2+k} \exp\left(-\frac{1}{q_2} \sum_{n=n_0+1}^{N+1} (n \log n)^{-\sigma}\right) \to 0 \quad (N \to \infty).$$

Therefore

$$C_{r,p}\left\{\bigcup_{l=1}^{\infty}\bigcap_{n=l}^{\infty}\Big(LL(t_n^{-1})^{1-\alpha}\sup_{\frac{t_{n+1}}{t_n}\le t<s\le 1}\frac{|w_n(s)-w_n(t)|}{|s-t|^{\alpha}}\ge (\frac{k(\alpha)}{1-I(\varphi)})^{\gamma}(1+2\varepsilon)\Big)\right\}=0,$$

and we can conclude (2.14). □

By Lemma 2.8, we can easily get

Lemma 2.9. *For $\varphi\in K$ with $I(\varphi)<1$, we have*

$$\liminf_{t\to 0} LL(t^{-1})^{1-\alpha}\|\frac{w(t\cdot)}{\sqrt{tLL(t^{-1})}}-\varphi\|_{\alpha}\le (\frac{k(\alpha)}{1-I(\varphi)})^{\gamma},\quad C_{r,p}-q.s.$$

3. The local convergence rate for increments of a Brownian motion in the Hölder norm

In previous sections, we get a quasi sure local functional convergence rate for a Brownian motion in the Hölder norm. In this section, we investigate the local convergence rate for increments of a Brownian motion in the Hölder norm with respect to $C_{r,p}$-capacity. We have the following theorem.

Theorem 3.1. *Let a_u be a non-decreasing function satisfying* (1) $0<a_u\le u<1$ *and* (2) u/a_u *being a non-increasing function. Set*

$$\ell_u=\log\frac{u\log u^{-1}}{a_u}\quad \textit{and}\quad \beta_u=(a_u\ell_u)^{-1/2}.$$

Let $(\mathcal{C}^{\alpha,0},\mathcal{H}^d,\mu)$ be an abstract Wiener space. For $w\in\mathcal{C}^{\alpha,0}$, we denote

$$\Delta(t,u)(s)=w(ut+a_us)-w(ut),\quad s\in[0,1],\ t\in[0,1-\frac{a_u}{u}].$$

We have

$$\liminf_{u\to 0}\ell_u^{1-\alpha}\inf_{t\in[0,1-\frac{a_u}{u}]}\|\beta_u\Delta(t,u)\|_{\alpha}=(k(\alpha))^{\gamma},\quad C_{r,p}-q.s. \tag{3.1}$$

Moreover, if condition (3) $\lim_{u\to 0}\frac{\log(u/a_u)}{LL(u^{-1})}=\infty$ *also holds, then*

$$\lim_{u\to 0}\ell_u^{1-\alpha}\inf_{t\in[0,1-\frac{a_u}{u}]}\|\beta_u\Delta(t,u)\|_{\alpha}=(k(\alpha))^{\gamma},\quad C_{r,p}-q.s. \tag{3.2}$$

Here $k(\alpha)$ is defined as in (2.1).

The proof of Theorem 3.1 is completed by the following lemmas:

Lemma 3.2. *There exists a constant $k(\alpha)>0$, such that for any $f\in K$, $\tau>0, h>0$, we have*

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{1/\gamma}\log C_{r,p}\left(\left\|\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-\frac{f}{\varepsilon^{1/(2\gamma)}}\right\|_{\alpha}\le\varepsilon\tau\right)\\&=\lim_{\varepsilon\to 0}\varepsilon^{1/\gamma}\log \mu\left(\left\|\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-\frac{f}{\varepsilon^{1/(2\gamma)}}\right\|_{\alpha}\le\varepsilon\tau\right)=-\frac{k(\alpha)}{\tau^{1/\gamma}}-I(f).\end{aligned}$$

Proof. Similar to that of Lemma 2.4. □

Lemma 3.3. *Let $k(\alpha)$ be defined as in* (2.1). *We have*

$$\liminf_{u\to 0}\ell_u^{1-\alpha}\inf_{t\in[0,1-\frac{a_u}{u}]}\|\beta_u\Delta(t,u)\|_{\alpha}\ge(k(\alpha))^{\gamma},\quad C_{r,p}-q.s.$$

Proof. Let $l(u) = a_u (\ell_u)^{-\frac{2}{1-2\alpha}}$. For $\theta \in (1,(1-\varepsilon)^{-2})$ we set $u_n = \theta^{-n}$ and $k_n = [\frac{u_n}{l(u_{n+1})}]$ and $t_i = il(u_{n+1}), i = 0,1,2,\cdots,k_n$. Choose $\delta' > 0$ such that $\eta_0 = \frac{1}{(\theta^{1/2}(1-\varepsilon))^{1/\gamma}} - \delta' > 1$. Then

$$\begin{aligned}
&\min_{0<i\le k_n} \|\beta_{u_n}(w(t_i + a_u\cdot) - w(t_i))\|_\alpha \\
\le\ & \max_{0\le i\le k_n} \sup_{0\le S\le l(u_{n+1})} \|\beta_{u_n}(w(S+(t_i+a_u\cdot)) - w(t_i+a_u\cdot))\|_\alpha \\
+\ & \inf_{t\in[0,u_n - a_{u_{n+1}}]} \|\beta_{u_n}(w(t+a_u\cdot) - w(t))\|_\alpha. \qquad (3.3)
\end{aligned}$$

For any $0<\varepsilon<1$, we have

$$\begin{aligned}
& C_{r,p}\left((\ell_{u_n})^{1-\alpha} \min_{0\le i\le k_n} \Big\|\beta_{u_n}(w(t_i+a_u\cdot) - w(t_i))\Big\|_\alpha \le (1-\varepsilon)\,(k(\alpha))^\gamma\right) \\
\le\ & \sum_{0\le i\le k_n} C_{r,p}\left(\Big\|\beta_{u_n}(w(t_i+a_u\cdot) - w(t_i))\Big\|_\alpha \le \frac{(1-\varepsilon)\,(k(\alpha))^\gamma}{\ell_{u_n}^{1-\alpha}}\right) \\
=\ & \sum_{0\le i\le k_n} C_{r,p}\left(\Big\|\frac{1}{\sqrt{\ell_{u_n}}}\frac{(w(t_i+a_u\cdot) - w(t_i))}{\sqrt{a_u}}\Big\|_\alpha \le \left(\frac{a_{u_n}}{a_u}\right)^{1/2} \frac{(k(\alpha))^\gamma\,(1-\varepsilon)}{\ell_{u_n}^{1-\alpha}}\right) \\
\le\ & \sum_{0\le i\le k_n} C_{r,p}\left(\Big\|\frac{1}{\sqrt{\ell_{u_n}}}\frac{(w(t_i+a_u\cdot) - w(t_i))}{\sqrt{a_u}}\Big\|_\alpha \le \theta^{1/2} \frac{(k(\alpha))^\gamma\,(1-\varepsilon)}{\ell_{u_n}^{1-\alpha}}\right).
\end{aligned}$$

By Lemma 3.2, for n large enough, we have

$$\begin{aligned}
& \sum_{0\le i\le k_n} C_{r,p}\left(\Big\|\frac{1}{\sqrt{\ell_{u_n}}}\frac{(w(t_i+a_u\cdot) - w(t_i))}{\sqrt{a_u}}\Big\|_\alpha \le \theta^{1/2} \frac{(k(\alpha))^\gamma\,(1-\varepsilon)}{\ell_{u_n}^{1-\alpha}}\right) \\
\le\ & (1+k_n)\exp\{-\ell_{u_n}\eta_0\} \le \frac{u_n + l(u_{n+1})}{l(u_{n+1})}\left(\frac{a_{u_n}}{u_n \log u_n^{-1}}\right)^{\eta_0}.
\end{aligned}$$

Therefore, by Borel-Cantelli's Lemma,

$$\liminf_{n\to\infty} \ell_{u_n}^{1-\alpha} \min_{0\le i\le k_n} \Big\|\beta_{u_n}(w(t_i+a_u\cdot) - w(t_i))\Big\|_\alpha \ge (k(\alpha))^\gamma, \quad C_{r,p} - q.s. \qquad (3.4)$$

On the other hand, for any $\eta > 0$,

$$\begin{aligned}
& C_{r,p}\left\{\ell_{u_n}^{1-\alpha} \sup_{0\le i\le k_n} \sup_{0\le s\le l(u_{n+1})} \beta_{u_n} \Big\|w(s+t_i+a_u\cdot) - w(t_i+a_u\cdot)\Big\|_\alpha \ge \eta\right\} \\
=\ & C_{r,p}\left(\frac{\ell_{u_n}^{1-\alpha}}{\sqrt{a_{u_n}\ell_{u_n}}} \sup_{0\le i\le k_n} \sup_{0\le T\le 1} \Big\|w(Tl(u_{n+1}) + t_i + a_u\cdot) - w(t_i+a_u\cdot)\Big\|_\alpha \ge \eta\right) \\
\le\ & \sum_{i=0}^{k_n} C_{r,p}\left(\frac{\ell_{u_n}^{1-\alpha}}{\sqrt{a_{u_n}\ell_{u_n}}} \sup_{0\le T\le 1} \Big\|w(Tl(u_{n+1}) + t_i + a_u\cdot) - w(t_i+a_u\cdot)\Big\|_\alpha \ge \eta\right) \\
\le\ & \sum_{i=0}^{k_n} \sum_{j=0}^{[\frac{a_u}{l(u_{n+1})}]} C_{r,p}\Big\{\frac{\ell_{u_n}^{1-\alpha}}{\sqrt{a_{u_n}\ell_{u_n}}}\frac{a_u^\alpha}{(l(u_{n+1}))^\alpha}\mathcal{W}(j,n) \ge \eta\Big\},
\end{aligned}$$

where $\mathcal{W}(j,n) = \sup_{0\le T\le 1} \Big\|w(Tl(u_{n+1}) + t_i + jl(u_{n+1}) + l(u_{n+1})\cdot) - w(t_i + jl(u_{n+1}) + l(u_{n+1})\cdot)\Big\|_\alpha$.

We have

$$
\begin{aligned}
&C_{r,p}\Big\{\ell_{u_n}^{1-\alpha}\sup_{0\le i\le k_n}\sup_{0\le s\le l(u_{n+1})}\beta_{u_n}\Big\|w(s+t_i+a_u\cdot)-w(t_i+a_u\cdot)\Big\|_\alpha\ge\eta\Big\}\\
\le\ & \sum_{i=0}^{k_n}\sum_{j=0}^{[\frac{a_u}{l(u_{n+1})}]}C_{r,p}\Big\{\frac{\ell_{u_n}^{1-\alpha}}{\sqrt{a_{u_n}\ell_{u_n}}}\frac{a_u^\alpha}{(l(u_{n+1}))^\alpha}\mathcal{W}(j,n)\ge\eta\Big\}\\
\le\ & \sum_{j=0}^{[\frac{a_u}{l(u_{n+1})}]}\sum_{i=0}^{k_n}C_{r,p}\Big\{\frac{\ell_{u_n}^{\frac12-\alpha}}{\ell_{u_n}}\theta^{\alpha-\frac12}\frac{\ell_{u_n}}{\ell_{u_{n+1}}}\frac{1}{\sqrt{l(u_{n+1})}}\mathcal{W}(j,n)\ge\eta\Big\}\\
\le\ & \sum_{j=0}^{[\frac{a_u}{l(u_{n+1})}]}\sum_{i=0}^{k_n}C_{r,p}\Big\{\frac{1}{\ell_{u_n}^{\frac12+\alpha}}\theta^{\alpha-\frac12}\frac{1}{\sqrt{l(u_{n+1})}}\mathcal{W}(j,n)\ge\eta\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mu\Big\{\frac{\theta^{\alpha-\frac12}}{\ell_{u_n}^{\frac12+\alpha}}\frac{1}{\sqrt{l(u_{n+1})}}\mathcal{W}(j,n)\ge\eta\Big\}\\
=\ &\mu\Big\{\frac{\sqrt{2}\theta^{\alpha-1/2}}{\ell_{u_n}^{\frac12+\alpha}}\sup_{0\le T\le1}\Big\|w(\frac{T}{2}+\frac12\cdot)-w(\frac12\cdot)\Big\|_\alpha\ge\eta\Big\}=\mu\Big\{\frac{\sqrt{2}\theta^{\alpha-1/2}}{\ell_{u_n}^{\frac12+\alpha}}w\in A\Big\},
\end{aligned}
$$

where $A=\{f\in\mathcal{C}^{\alpha,0}:\sup_{0\le t\le1}\|f(\frac12t+\frac12\cdot)-f(\frac12\cdot)\|_\alpha\ge\eta\}$. Since $\inf_{f\in A}I(f)\ge\frac{\eta^2}{32}$, by Theorem 2.1 in[4], for n large enough, we obtain

$$
\begin{aligned}
&C_{r,p}\Big\{\frac{1}{\ell_{u_n}^{\frac12+\alpha}}\theta^{\alpha-\frac12}\frac{1}{\sqrt{l(u_{n+1})}}\mathcal{W}(j,n)>\eta\Big\}\\
&\le\exp\Big(-\frac{\eta^2}{128\theta^{\alpha-1/2}}\ell_{u_n}^{2\alpha}\ell_{u_n}\Big)=\Big(\frac{a_{u_n}}{u_n\log u_n^{-1}}\Big)^{\frac{\eta^2}{128\theta^{\alpha-1/2}}\ell_{u_n}^{2\alpha}}.
\end{aligned}
$$

Taking into account that $\log\frac{u_n\log u_n^{-1}}{a_{u_n}}\to\infty$ as $n\to\infty$, we obtain

$$\sum_n\frac{a_{u_n}}{l(u_{n+1})}\frac{u_n+l(u_{n+1})}{l(u_{n+1})}\Big(\frac{a_{u_n}}{u_n\log u_n^{-1}}\Big)^{\frac{\eta^2}{128\theta^{\alpha-1/2}}\ell_{u_n}^{2\alpha}}<\infty.$$

By Borel-Cantelli's Lemma,

(3.5)

$$\limsup_{n\to\infty}\ell_{u_n}^{1-\alpha}\sup_{0\le i\le k_n}\sup_{0\le s\le l(u_{n+1})}\beta_{u_n}\Big\|w(s+t_i+a_u\cdot)-w(t_i+a_u\cdot)\Big\|_\alpha=0,\quad C_{r,p}-q.s.$$

By (3.3), (3.4) and (3.5), we obtain

$$\liminf_{n\to\infty}\ell_{u_n}^{1-\alpha}\inf_{t\in[0,u_n-a_{u_{n+1}}]}\Big\|\beta_{u_n}(w(t+a_u\cdot)-w(t))\Big\|_\alpha\ge(k(\alpha))^\gamma,\quad C_{r,p}-q.s.\tag{3.6}$$

Let $\phi_{t,u}(s)=\beta_u(w(t+a_us)-w(t)),s\in[0,1],t\in[0,u-a_u]$, for $u\in(u_{n+1},u_n]$. Then

$$
\begin{aligned}
\inf_{t\in[0,u-a_u]}\Big\|\phi_{t,u}(\cdot)\Big\|_\alpha&\ge\inf_{t\in[0,u_n-a_{u_{n+1}}]}\frac{\beta_u}{\beta_{u_n}}\Big\|\phi_{t,u_n}(\frac{a_u}{a_{u_n}}\cdot)\Big\|_\alpha\\
&\ge\inf_{t\in[0,u_n-a_{u_{n+1}}]}\beta_{u_n}\Big\|w(t+a_u\cdot)-w(t)\Big\|_\alpha.
\end{aligned}
$$

Thus

$$\ell_u^{1-\alpha} \inf_{t\in[0,u-a_u]} \Big\|\phi_{t,u}(\cdot)\Big\|_\alpha \geq \ell_{u_n}^{1-\alpha} \inf_{t\in[0,u_n-a_{u_{n+1}}]} \beta_{u_n}\Big\|w(t+a_u\cdot)-w(t)\Big\|_\alpha. \tag{3.7}$$

We finally obtain

$$\liminf_{u\to 0} \ell_u^{1-\alpha} \inf_{t\in[0,u-a_u]} \Big\|\phi_{t,u}(\cdot)\Big\|_\alpha \geq (k(\alpha))^\gamma, \quad C_{r,p}-q.s.$$

□

Lemma 3.4. *Letting* $k(\alpha)$ *be as in* (2.1), *we have* $\liminf_{u\to 0} \ell_u^{1-\alpha} \inf_{t\in[0,1-\frac{a_u}{u}]} \|\beta_u\Delta(t,u)\|_\alpha \leq (k(\alpha))^\gamma, \quad C_{r,p}-q.s.$

Proof. Step 1: Let $\lim_{u\to 0}\frac{\log(\frac{u}{a_u})}{\log\log u^{-1}}<\infty$. Set $\rho=\lim_{u\to 0}\frac{a_u}{u}$.

Case 1. Letting $\rho<1$, we can define u_n as follows: $u_{n+1}=u_n-a_{u_n}, u_1=\frac{1}{2}$. Then $\{\|\beta_{u_n}\Delta(1-\frac{a_{u_n}}{u_n},u_n)-f\|_\alpha>\varepsilon\}(n\geq 1)$ are independent. We can prove that $u_n\to 0$ as $n\to\infty$. Taking $k=[r]+1$, by Lemma 2.3, we have

$$\begin{aligned}
&C_{r,p}\left(\bigcap_{n=m_0}^{l}\Big(\ell_{u_n}^{1-\alpha}\Big\|\beta_{u_n}\Delta(1-\frac{a_{u_n}}{u_n},u_n)\Big\|_\alpha \geq (k(\alpha))^\gamma(1+2\varepsilon)\Big)\right)^{1/p}\\
&=C_{r,p}\left(\bigcap_{n=m_0}^{l}\left(\left\|\frac{1}{\sqrt{\ell_{u_n}}}\frac{\Delta(1-\frac{a_{u_n}}{u_n},u_n)}{\sqrt{a_{u_n}}}\right\|_\alpha \geq \frac{(k(\alpha))^\gamma(1+2\varepsilon)}{\ell_{u_n}^{1-\alpha}}\right)\right)^{1/p}\\
&\leq cl^k\left(\frac{k(\alpha)^\gamma\varepsilon}{\ell_{u_l}^{1-\alpha}}\right)^{-2k^2-k}\mu\left(\bigcap_{n=m_0}^{l}\left(\left\|\frac{\Delta(1-\frac{a_{u_n}}{u_n},u_n)}{\sqrt{\ell_{u_n}}\sqrt{a_{u_n}}}\right\|_\alpha \geq \frac{k(\alpha)^\gamma(1+\varepsilon)}{\ell_{u_n}^{1-\alpha}}\right)\right)^{\frac{1}{q_2}}\\
&\leq cl^k\left(\frac{k(\alpha)^\gamma\varepsilon}{\ell_{u_l}^{1-\alpha}}\right)^{-2k^2-k}\prod_{n=m_0}^{l}\left(1-\mu\left(\left\|\frac{\Delta(1-\frac{a_{u_n}}{u_n},u_n)}{\sqrt{\ell_{u_n}}\sqrt{a_{u_n}}}\right\|_\alpha < \frac{k(\alpha)^\gamma(1+\varepsilon)}{\ell_{u_n}^{1-\alpha}}\right)\right)^{\frac{1}{q_2}};
\end{aligned}$$

moreover, by a small deviation, for n large enough, we have

$$\mu\left(\left\|\frac{1}{\sqrt{\ell_{u_n}}}\frac{\Delta(1-\frac{a_{u_n}}{u_n},u_n)}{\sqrt{a_{u_n}}}\right\|_\alpha < \frac{(k(\alpha))^\gamma(1+\varepsilon)}{\ell_{u_n}^{1-\alpha}}\right)\geq\left(\frac{a_{u_n}}{u_n\log u_n^{-1}}\right)^{\sigma_1},$$

where $\sigma_1=\frac{1}{(1+\varepsilon)^{1/\gamma}}+\sigma_0<1$, for some $\sigma_0>0$. Thus

$$\begin{aligned}
&C_{r,p}\left(\bigcap_{n=m_0}^{l}\Big(\ell_{u_n}^{1-\alpha}\|\beta_{u_n}\Delta(1-\frac{a_{u_n}}{u_n},u_n)\|_\alpha \geq (k(\alpha))^\gamma(1+2\varepsilon)\Big)\right)^{1/p}\\
\leq\quad & cl^k\left(\frac{\ell_{u_l}^{1-\alpha}}{\varepsilon(k(\alpha))^\gamma}\right)^{2k^2+k}\exp\left(-\frac{1}{q_2}\sum_{n=m_0}^{l}\ell_{u_l}^{-\sigma_1}\right).
\end{aligned}$$

Note that for l large enough, there exists $C>0$ such that

$$\sum_{n=m_0}^{l}\ell_{u_l}^{-\sigma_1}>C(\log u_l^{-1})^{\varepsilon'},$$

where $\varepsilon'=1-\sigma_1>0$.

Since $\lim_{u\to 0}\frac{\log(ua_u^{-1})}{\log\log u^{-1}}<\infty$, for some $0<M<\infty$, we have $u/a_u\le(\log u^{-1})^M$. Taking $\theta>2/\varepsilon', u_0=e^{-(\log l_0)^\theta}$, for l_0 large enough, we can prove that

$$\log u_l^{-1}\ge(\log l)^\theta,\ \text{ as } l\ge l_0;$$

thus we get

$$(\log u_l^{-1})^{\varepsilon'}>(\log l)^2.$$

We can also prove that when l is large enough, there exists a constant $c_1>0$ such that $c_1 l\ge\log u_l^{-1}$. Thus we have

$$\begin{aligned}&C_{r,p}\left(\bigcap_{n=m_0}^{l}\left(\ell_{u_n}^{1-\alpha}\Big\|\beta_{u_n}\Delta(1-\frac{a_{u_n}}{u_n},u_n)\Big\|_\alpha\ge(k(\alpha))^\gamma(1+2\varepsilon)\right)\right)^{1/p}\\ \le\ & cl^kLL(u_l^{-1})^{(1-\alpha)(4k^2+2k)}\exp\Big(-\frac{C}{q_2}(\log u_l^{-1})^{\varepsilon'}\Big)\\ \le\ & \frac{c_2l^k(\log l)^{(1-\alpha)(4k^2+2k)}}{\exp(\frac{C}{q_2}(\log l)^2)}\to 0,\quad l\to\infty,\end{aligned}$$

where $c_2=c_2(k,p,q_1,k(\alpha),\alpha)$. We get

$$C_{r,p}\left\{\bigcup_{l=1}^{\infty}\bigcap_{n=l}^{\infty}\Big(\ell_{u_n}^{1-\alpha}\Big\|\beta_{u_n}\Delta(1-\frac{a_{u_n}}{u_n},u_n)\Big\|_\alpha\ge(k(\alpha))^\gamma(1+2\varepsilon)\Big)\right\}=0.$$

Consequently,

$$\liminf_{u\to 0}\ell_u^{1-\alpha}\|\beta_u\Delta(1-\frac{a_u}{u},u)\|_\alpha\le(k(\alpha))^\gamma,\quad C_{r,p}-q.s.$$

Case 2. Let $\rho=1$; then $a_u=u$. See Theorem 2.1.

Step 2: Let $\lim_{u\to 0}\frac{\log\frac{u}{a_u}}{\log\log u^{-1}}=\infty$. The conclusion follows from Lemma 3.6. □

We have proven (3.1). We are now in a position to prove (3.2).

Lemma 3.5. *Let $k(\alpha)$ be as in (2.1). There exists a decreasing sequence $u_n\to 0$ $(n\to\infty)$ such that*

$$\limsup_{n\to\infty}\ell_{u_n}^{1-\alpha}\inf_{t\in[0,u_{n+1}-a_{u_n}]}\|\beta_{u_n}(w(t+a_us)-w(t))\|_\alpha\le(k(\alpha))^\gamma,\quad C_{r,p}-q.s.$$

Proof. Since $\lim_{u\to 0}\frac{\log\frac{u}{a_u}}{\log\log u^{-1}}=\infty$, there exists a subsequence $\{u_n;n\ge 1\}$ such that $\frac{u_n}{a_{u_n}}=n^{d'},d'>1$. Letting $t_i=ia_{u_n},i=0,1,2,\cdots,k_n=[\frac{u_{n+1}}{a_{u_n}}]-1$ and $h(n)=\frac{\log\frac{u_n}{a_{u_n}}}{\log\log u_n^{-1}}=\frac{\log n^{d'}}{\log\log u_n^{-1}}$, we have $u_n^{-1}=\exp(n^{\frac{d'}{h(n)}})$ and $h(n)\to\infty$ as $n\to\infty$. Moreover, for any $\kappa>0$,

$$\frac{n^\kappa}{\log u_n^{-1}}\to\infty,\quad n\to\infty,$$

and

$$1\le\frac{u_n}{u_{n+1}}=\exp\{(n+1)^{\frac{d'}{h(n+1)}}-n^{\frac{d'}{h(n)}}\}\le\exp\{n^{\frac{d'}{h(n)}-1}\}\to 1,\quad n\to\infty.$$

Choose $\delta_0 > 0$ such that $\delta_1 = \frac{1}{(1+\varepsilon)^{1/\gamma}} + \delta_0 < 1$. Take $k = [r] + 1$. By Lemma 2.3 and a small deviation, for n large enough, we have

$$\begin{aligned}
& C_{r,p}\left(\ell_{u_n}^{1-\alpha} \inf_{t\in[0,u_{n+1}-a_{u_n}]} \|\beta_{u_n}(w(t+a_u s)-w(t))\|_\alpha \geq k(\alpha)^\gamma(1+2\varepsilon)\right)^{1/p} \\
\leq\ & C_{r,p}\left(\min_{0\leq i\leq k_n} \|\beta_{u_n}(w(t_i+a_u s)-w(t_i))\|_\alpha \geq \frac{k(\alpha)^\gamma(1+2\varepsilon)}{\ell_{u_n}^{1-\alpha}}\right)^{1/p} \\
\leq\ & (1+k_n)^k \left(\frac{\ell_{u_n}^{1-\alpha}}{k(\alpha)^\gamma\varepsilon}\right)^{2k^2+k} \cdot \mu\left(\left\|\frac{w(t_i+a_u s)-w(t_i)}{\sqrt{a_{u_n}}}\right\|_\alpha \geq \frac{k(\alpha)^\gamma(1+\varepsilon)}{\ell_{u_n}^{\frac{1}{2}-\alpha}}\right)^{\frac{1+k_n}{q_2}} \\
\leq\ & (1+k_n)^k \left(\frac{\ell_{u_n}^{1-\alpha}}{k(\alpha)^\gamma\varepsilon}\right)^{2k^2+k} \left(1-\mu\left(\|w(\cdot)\|_\alpha < \frac{k(\alpha)^\gamma(1+\varepsilon)}{\ell_{u_n}^{1/2-\alpha}}\right)\right)^{\frac{1+k_n}{q_2}} \\
\leq\ & (1+k_n)^k \left(\frac{\ell_{u_n}^{1-\alpha}}{k(\alpha)^\gamma\varepsilon}\right)^{2k^2+k} \left(1-\left(\frac{a_{u_n}}{u_n\log u_n^{-1}}\right)^{\delta_1}\right)^{\frac{1+k_n}{q_2}} \\
\leq\ & cn^{kd'}(\log n)^{(2k^2+k)(1-\alpha)} \exp\left\{-\frac{1}{q_2}\left(\frac{a_{u_n}}{u_n\log u_n^{-1}}\right)^{\delta_1}\left[\frac{u_{n+1}}{a_{u_n}}\right]\right\},
\end{aligned}$$

where $c > 0$ is a constant. Choosing a suitable d', we have

$$\sum_{n=1}^\infty c^p n^{pkd'}(\log n)^{p(2k^2+k)(1-\alpha)} \exp\left\{-\frac{1}{q_2}\left(\frac{a_{u_n}}{u_n\log u_n^{-1}}\right)^{\delta_1}\left[\frac{u_{n+1}}{a_{u_n}}\right]\right\} < \infty.$$

By Borel-Cantelli's Lemma,

$$\begin{aligned}
& \limsup_{n\to\infty}\left(\log\frac{u_n\log u_n^{-1}}{a_{u_n}}\right)^{1-\alpha} \inf_{t\in[0,u_{n+1}-a_{u_n}]} \|\beta_{u_n}(w(t+a_u s)-w(t))\|_\alpha \\
\leq\ & k(\alpha)^\gamma(1+\varepsilon), \quad C_{r,p}-q.s.
\end{aligned}$$

□

Lemma 3.6. *Letting $k(\alpha)$ be as in (1), we have*

(3.8)

$$\limsup_{u\to 0}\left(\log\frac{u\log u^{-1}}{a_u}\right)^{1-\alpha} \inf_{t\in[0,u-a_u]} \|\beta_u(w(t+a_u s)-w(t))\|_\alpha \leq k(\alpha)^\gamma, \quad C_{r,p}-q.s.$$

Proof. Let $\psi_{t,u}(s) = \beta_u(w(t+a_u s)-w(t)),\ s\in[0,1], t\in[0,u-a_u]$. Let u_n be defined as in Lemma 3.5. Then $\psi_{t,u}(s) = \frac{\beta_u}{\beta_{u_n}}\psi_{t,u_n}\left(\frac{a_u}{a_{u_n}}s\right)$. We have for $u \in (u_{n+1}, u_n]$,

$$\begin{aligned}
\inf_{t\in[0,u-a_u]} \|\beta_u(w(t+a_u s)-w(t))\|_\alpha &= \inf_{t\in[0,u-a_u]} \left\|\frac{\beta_u}{\beta_{u_n}}\psi_{t,u_n}\left(\frac{a_u}{a_{u_n}}\cdot\right)\right\|_\alpha \\
&\leq \inf_{t\in[0,u_{n+1}-a_{u_n}]} \frac{\beta_u}{\beta_{u_n}}\left\|\psi_{t,u_n}\left(\frac{a_u}{a_{u_n}}\cdot\right)\right\|_\alpha.
\end{aligned}$$

Taking into account $\frac{\beta_{u_n}}{\beta_{u_{n+1}}} \to 1$ as $n\to\infty$, by Lemma 3.5, we obtain (3.8). □

The conclusion (3.2) follows from Lemmas 3.3 and 3.6.

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