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RADIAL SYMMETRY AND DECAY RATES OF POSITIVE SOLUTIONS OF A WOLFF TYPE INTEGRAL SYSTEM

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Abstract. In this paper, we study the properties of the positive solutions of a nonlinear integral system involving Wolff potentials:

$$\begin{cases} u_1 = W_{\beta,\gamma}(f_1(u)) \\ \quad\vdots \\ u_m = W_{\beta,\gamma}(f_m(u)), \end{cases}$$

where $u = (u_1, \dots, u_m)$ and

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty [\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t}$$

with $1 < \gamma < 2$ and $n > \beta\gamma$. First, we estimate the decay rate of the positive solutions at infinity. Based on this, we prove radial symmetry and monotonicity for those solutions by the refined method of moving planes in integral forms, which was established by Chen, Li and Ou. Since the Kelvin transform cannot be used in such a Wolff type system, we have to find a new technique to study the asymptotic estimate, which is essential when we move the planes.

1. Introduction

In this paper, we will study the abstract integral system involving the Wolff potentials

$$\begin{cases} u_1 = W_{\beta,\gamma}(f_1(u)) \\ \quad\vdots \\ u_m = W_{\beta,\gamma}(f_m(u)), \end{cases} \tag{1}$$

where $u_i > 0$ $(i = 1, \dots, m)$, $u = (u_1, \dots, u_m)$, and $f_i(u(x)) = C\prod_{i=1}^m u_i^{e_i}(x)$ with $e_1 + \cdots + e_m = p := \frac{(n+\gamma\beta)(\gamma-1)}{n-\gamma\beta}$. In the following, we will always assume $p > 1$ and $e_i \geq 0$.

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The Wolff potential of a nonnegative function $f \in L^1_{loc}(R^n)$ is defined as (cf. [13])

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty [\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t},$$

where $1 < \gamma < \infty$, $\beta > 0$, $\beta\gamma < n$, and $B_t(x)$ is the ball of radius t centered at x. It is not difficult to see that $W_{1,2}(f)$ is Newton's potential and $W_{\frac{\alpha}{2},2}(f)$ is Reisz's potential.

Recalling the work in [17], [18] and [28], we know that the Wolff potential is helpful for studying the nonlinear PDEs. For example, $W_{1,p}(\omega)$ and $W_{\frac{2k}{k+1},k+1}(\omega)$ can be used to estimate the solutions of the p-Laplace equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \omega$$

and the k-Hessian equation

$$F_k[-u] = \omega, \quad k = 1, 2, \ldots, n,$$

respectively. Here $F_k[u] = S_k(\lambda(D^2u))$, $\lambda(D^2u) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with λ_i being eigenvalues of the Hessian matrix (D^2u), and $S_k(\cdot)$ is the k-th symmetric function:

$$S_k(\lambda) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

Two special cases are $F_1[u] = \Delta u$ and $F_n[u] = \det(D^2u)$.

When $\gamma = 2$ and $\beta = \alpha/2$, (1) becomes

$$\begin{cases} u_i(x) = \displaystyle\int_{R^n} \frac{(f_i(u(y)))dy}{|x-y|^{n-\alpha}}, \quad x \in R^n, \quad m = 1, 2, \ldots, m, \\ \qquad u = (u_1, u_2, \ldots, u_m), \quad 0 < \alpha < n. \end{cases} \tag{2}$$

Here $f_i(u), 1 \le i \le m$, are real-valued functions of homogeneous degree $\frac{n+\alpha}{n-\alpha}$ and are monotone nondecreasing with respect to all the independent variables $u_1, u_2, \ldots, u_m$. Chen and Li proved the radial symmetry of positive solutions (cf. [6]), and the decay rates were obtained in Section 3 of [6] by the Kelvin transform.

When $m = 2$, $f_1(u, v) = u^p$ and $f_2(u, v) = v^q$, (1) becomes

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x), \\ v(x) = W_{\beta,\gamma}(u^p)(x), \end{cases} \tag{3}$$

which was studied by Ma, Chen and Li (cf. [5] and [26]). Here p and q satisfy

$$\frac{\gamma-1}{p+\gamma-1} + \frac{\gamma-1}{q+\gamma-1} = \frac{n-\beta\gamma}{n}.$$

It turns out that the positive solutions of (3) are radially symmetric and decreasing about some point in R^n.

Letting $\gamma = 2$ and $\beta = \alpha/2$ in (3), we obtain a Hardy-Littlewood-Sobolev (HLS) type integral system

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \frac{v^q(y)dy}{|x-y|^{n-\alpha}}, \\ v(x) = \displaystyle\int_{R^n} \frac{u^p(y)dy}{|x-y|^{n-\alpha}}. \end{cases} \tag{4}$$

Chen, Li and Ou used an integral form of the method of moving planes to prove the radial symmetry of the positive solutions of (4) (cf. [7]). Moreover, if $p = q$, then $u = v$, and hence (4) reduces to a single equation (cf. [8]):

$$u(x) = \int_{R^n} \frac{u^{\frac{n+\alpha}{n-\alpha}}(y)dy}{|x-y|^{n-\alpha}}.$$

The radial symmetry of the positive solutions was obtained in [8] and [24].

For a general weighted HLS type integral system (WHLS)

$$\begin{cases} u(x) = \dfrac{1}{|x|^\alpha} \displaystyle\int_{R^n} \frac{v^q(y)dy}{|y|^\beta |x-y|^{n-\alpha}}, \\ v(x) = \dfrac{1}{|x|^\beta} \displaystyle\int_{R^n} \frac{u^p(y)dy}{|y|^\alpha |x-y|^{n-\alpha}}, \end{cases} \tag{5}$$

the radial symmetry of the positive solutions is proved by Jin and Li (cf. [4]). Afterwards, they also derived the integrability intervals of those solutions in [14]. Based on this result, Li, Lim and Lei, Ma obtained the asymptotic estimates when $|x| \to 0$ and $|x| \to \infty$ (cf. [22], [20]).

In this paper, we will also prove the radial symmetry and estimate the decay rates of positive solutions u_i of (1) if $u_i \in L^{p+\gamma-1}(R^n)$. We state the main result of this paper, which will be proved in Section 3.

Theorem 1.1. *Assuming $u_i \in L^{p+\gamma-1}(R^n)$ solves (1), $i = 1, 2, \ldots, m$, and assuming f_i is simple and monotone nondecreasing with respect to the variables $u_1, u_2, \ldots, u_m$, f is simple means that:*

$$(f_{i_1}(u), \ldots, f_{i_k}(u)) \neq (f_{i_1}(v), \ldots, f_{i_k}(v))$$

whenever

$$(u_{i_1}, \ldots, u_{i_k}) = (v_{i_1}, \ldots, v_{i_k})$$

and

$$u_{i_{k+1}} > v_{i_{k+1}}, \ldots, u_{i_m} > v_{i_m}.$$

Then $u_i(x)$ is radially symmetric and decreasing about some point $x_0 \in R^n$.

In order to use the moving plane method to prove the radial symmetry, we need to find out the asymptotic behavior of u_i, which is proved in the next theorem. Notice that the f_i are homogeneous and the integrability of the u_i is equivalent to the integrability of $\overline{u}(x) = u_1 + \cdots + u_m$. In fact, $|f_i(u)| \leq C \cdot |\overline{u}|^p$. Then there exists a function $R(x)$ satisfying

$$0 < R(x) \leq C, \tag{6}$$

such that

$$\overline{u}(x) = R(x) W_{\beta,\gamma}(\overline{u}^p). \tag{7}$$

Clearly, $\overline{u} \in L^r(R^n)$ is equivalent to $u_i \in L^r(R^n)$. The regularity results for (3) are derived in [26]. Letting $p = q$ in Theorem 2.1 of [26], we can also see that

$$\overline{u} \in L^r(R^n), \quad \text{for} \quad \frac{1}{r} \in (0, \frac{n-\gamma\beta}{n(\gamma-1)}). \tag{8}$$

Moreover, according to Theorem 3.1 in [26], we also have

$$\overline{u} \in L^\infty(R^n). \tag{9}$$

Theorem 1.2. *Assuming $u_i \in L^{p+\gamma-1}(R^n)$ solves (1), $i = 1, 2, \ldots, m$, if f_i is monotone nondecreasing with respect to the variables $u_1, u_2, \ldots, u_m$, then there exist positive constants c and C, such that for sufficiently large $|x|$,*

$$c|x|^{-\frac{n-\beta\gamma}{\gamma-1}} \le \bar{u}(x) \le C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

Remark. For (2), the decay rate as in Theorem 1.2 holds naturally by the Kelvin transform. However, for (1), the Kelvin transform is not valid. In addition, the proof of Theorem 1.2 is different from the argument in [20], which estimates the decay rates there by using the radial symmetry of positive solutions. Now, we will apply the conclusion of Theorem 1.2 to establish the radial symmetry. We have to find another method to obtain the decay rate.

2. Asymptotic estimate

In this section, we prove Theorem 1.2.

Theorem 2.1. *Let $u \in L^{p+\gamma-1}(R^n)$ and u be a positive solution of (1). Then there exists a positive constant c, such that for large $|x|$,*

$$u_i(x) \ge \frac{c}{|x|^{\frac{n-\beta\gamma}{\gamma-1}}}, \quad i = 1, 2, \ldots, m. \tag{10}$$

Proof. Clearly, $\int_{B_1(0)} f_i(u(y))dy \ge c > 0$, $i = 1, 2, \ldots, m$. Therefore, it follows that

$$\begin{aligned} u_i(x) &= \int_0^\infty [\frac{\int_{B_t(x)} f_i(u(y))dy}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\ge \int_{|x|+1}^\infty [\frac{\int_{B_1(0)} f_i(u(y))dy}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\ge c \int_{|x|+1}^\infty t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{dt}{t} \\ &\ge c|x|^{-\frac{n-\beta\gamma}{\gamma-1}}. \end{aligned}$$

Theorem 2.1 is proved. □

Proposition 2.2. *We claim that for $i = 1, 2, \ldots, m$,*

$$\lim_{|x|\to\infty} u_i(x) = 0. \tag{11}$$

Proof. Take $x_0 \in R^n$. By (9), $\|\bar{u}\|_\infty < \infty$. Thus, $\forall \varepsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$R(x) \int_0^\delta [\frac{\int_{B_t(x_0)} \bar{u}^p(z)dz}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \le C\|\bar{u}\|_\infty^{\frac{p}{\gamma-1}} \int_0^\delta t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} < \varepsilon.$$

As $|x - x_0| < \delta$,

$$\begin{aligned} &\int_\delta^\infty [\frac{\int_{B_t(x_0)} f_i(u(z))dz}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\le \int_\delta^\infty [\frac{\int_{B_{t+\delta}(x)} f_i(u(z))dz}{(t+\delta)^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} (\frac{t+\delta}{t})^{\frac{n-\beta\gamma}{\gamma-1}+1} \frac{d(t+\delta)}{t+\delta} \\ &\le C \int_0^\infty [\frac{\int_{B_t(x)} f_i(u(z))dz}{t^{n-\beta\gamma}}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &\le Cu_i(x), \quad i = 1, 2, \ldots, m. \end{aligned}$$

Combining these estimates, we get

$$\bar{u}(x_0) < \varepsilon + C\bar{u}(x), \quad \text{for} \quad |x - x_0| < \delta.$$

Noting $\lim_{|x_0|\to\infty} \int_{B_\delta(x_0)} \bar{u}^{p+\gamma-1}(x)dx = 0$ since $\bar{u} \in L^{p+\gamma-1}(R^n)$, we have

$$\begin{aligned} \bar{u}^{p+\gamma-1}(x_0) &= |B_\delta(x_0)|^{-1} \int_{B_\delta(x_0)} \bar{u}^{p+\gamma-1}(x_0)dx \\ &\le C\varepsilon^{p+\gamma-1} + C|B_\delta(x_0)|^{-1} \int_{B_\delta(x_0)} \bar{u}^{p+\gamma-1}(x)dx \to 0 \end{aligned} \tag{12}$$

when $|x_0| \to \infty$. This result means that (11) holds. □

Theorem 2.3. *Let $u_i \in L^{p+\gamma-1}(R^n)$ and u be a positive solution of* (1). *Assume $\gamma \in (\frac{3n}{2n+\beta}, 2]$. Then there exists a positive constant C, such that for large $|x|$,*

$$\bar{u}(x) \le \frac{C}{|x|^{\frac{n-\beta\gamma}{\gamma-1}}}. \tag{13}$$

Proof.

Step 1. $\forall \rho > 0$, take a cutoff function $\psi_\rho(x)$ as follows: $\psi(x) \in C_0^\infty(B_2 \setminus B_1)$ satisfying

$$\begin{array}{ll} 0 \le \psi(x) \le 1, & \text{for} \quad 1 \le |x| \le 2; \\ \psi(x) = 1, & \text{for} \quad \frac{5}{4} \le |x| \le \frac{7}{4}. \end{array}$$

Define $\psi_\rho(x) = \psi(\frac{x}{\rho})$ and set $h(x) = \bar{u}(x)|x|^{n/p}\psi_\rho(x)$. Then either we can find $C > 0$ such that for any x,

$$h(x) \le C, \tag{14}$$

where C is independent of ρ, or there exists an increasing sequence $\{\rho_j\}_{j=1}^\infty$ satisfying $\lim_{j\to\infty} \rho_j = \infty$, with some $x_j \in B_{2\rho_j} \setminus B_{\rho_j}$,

$$\lim_{j\to\infty} h(x_j) = \infty. \tag{15}$$

Step 2. If (14) is true, then for large $|x|$,

$$\bar{u}(x) \le C|x|^{-n/p}. \tag{16}$$

Then by (6) and (16), we have

$$R(x) \int_0^{\frac{|x|}{2}} \Big(\frac{\int_{B_t(x)} \bar{u}^p(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \le \frac{C}{|x|^{\frac{n}{\gamma-1}}} \int_0^{\frac{|x|}{2}} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \le \frac{C}{|x|^{\frac{n-\beta\gamma}{\gamma-1}}}. \tag{17}$$

On the other hand, (8) shows that $u \in L^p(R^n)$. Write $C_p = \|u\|_p^p$. Then

$$R(x) \int_{|x|/2}^\infty \Big(\frac{\int_{B_t(x)} \bar{u}^p(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t} \le CC_p^{\frac{1}{\gamma-1}} \int_{|x|/2}^\infty t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{dt}{t} \le C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

Combining this result with (17), and using (7), we obtain

$$\bar{u}(x) = R(x)W_{1,\gamma}(\bar{u}^p)(x) \le C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

This is (13).

Step 3. Let x_ρ be the maximum point of $h(x)$ in $B_{2\rho}\setminus B_\rho$. We will prove (13) under the assumption of (15). First, by (15) we can find $\delta>0$ which is independent of ρ, such that (cf. [19])

$$\psi_\rho(x_\rho)>\delta. \tag{18}$$

Since ψ is smooth, (18) shows that there exists a suitable small constant $\sigma>0$, such that $\psi_\rho(y)>\delta/2$ for $|y-x_\rho|<\sigma|x_\rho|$. Hence, from $h(y)\le h(x_\rho)$, we deduce that

$$\bar u(y)\le C\frac{\bar u(x_\rho)}{\psi_\rho(y)}\le C(\delta)\bar u(x_\rho),\quad \text{as}\quad |y-x_\rho|<\sigma|x_\rho|. \tag{19}$$

Equations (7) and (6) imply that

$$\begin{aligned}\bar u(x_\rho)\le C[&\int_0^{\sigma|x_\rho|}\Big(\frac{\int_{B_t(x_\rho)}\bar u^p(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}\\ &+\int_{\sigma|x_\rho|}^{\infty}\Big(\frac{\int_{B_t(x_\rho)}\bar u^p(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}]:=C(I_1+I_2).\end{aligned} \tag{20}$$

Clearly,

$$I_2\le C_p^{\frac{1}{\gamma-1}}\int_{\sigma|x_\rho|}^{\infty}t^{-\frac{n-\beta\gamma}{\gamma-1}}\frac{dt}{t}\le C|x_\rho|^{-\frac{n-\beta\gamma}{\gamma-1}}. \tag{21}$$

Using (19), we obtain that, for $r\in(0,\sigma|x_\rho|)$,

$$\begin{aligned}I_1\le C\bar u(x_\rho)[&\int_0^{r}\Big(\frac{\int_{B_t(x_\rho)}\bar u^{p-\gamma+1}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}\\ &+\int_{r}^{\sigma|x_\rho|}\Big(\frac{\int_{B_t(x_\rho)}\bar u^{p-\gamma+1}(y)dy}{t^{n-\beta\gamma}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}]:=C\bar u(x_\rho)(J_1+J_2).\end{aligned} \tag{22}$$

By (11), for any $\varepsilon\in(0,1)$, we have

$$J_1\le\varepsilon^{\frac{p-\gamma+1}{\gamma-1}}\int_0^r t^{\frac{\beta\gamma}{\gamma-1}}\frac{dt}{t}\le\frac{1}{4C^2}$$

as long as ρ is sufficiently large. On the other hand, by Hölder's inequality and (9),

$$\int_{B_t(x_\rho)}\bar u^{p-\gamma+1}(y)dy\le\|\bar u\|_s^{p-\gamma+1}|B_t(x_\rho)|^{1-\frac{p-\gamma+1}{s}}\le Ct^{n-n(p-\gamma+1)/s},$$

and hence

$$J_2\le C\int_r^{\sigma|x_\rho|}t^{\frac{\beta\gamma-n(p-\gamma+1)/s}{\gamma-1}}\frac{dt}{t}\le\frac{1}{4C^2}$$

if ρ is sufficiently large and r is chosen suitably large. Substituting the estimates of J_1 and J_2 into (22), we obtain

$$I_1\le\frac{1}{2C}\bar u(x_\rho)$$

when ρ is sufficiently large. Inserting this result and (21) into (20) yields

$$\bar u(x_\rho)\le C|x_\rho|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

By (19), we obtain that as $|x-x_\rho|<\sigma|x_\rho|$,

$$\bar u(x)\le C\bar u(x_\rho)\le C|x_\rho|^{-\frac{n-\beta\gamma}{\gamma-1}}\le C|x|^{-\frac{n-\beta\gamma}{\gamma-1}}.$$

Since ρ is arbitrary, the result above still holds for all x as long as $|x|$ is large. Then (13) is verified, and the proof of Theorem 2.3 is complete. $\square$

3. Radial symmetry

In this section, we prove Theorem 1.1.

Choose a special direction, for example x_1. Write $x = (x_1, x')$ and set $H_\lambda = \{x \in R^n : x_1 < \lambda\}$. Let $x_\lambda = (2\lambda - x_1, x')$ and $u^\lambda(x) = u(x^\lambda)$.

Step 1. Moving plane from negative infinity.

Set

$$E_t := \{t| \int_{B_t(x)} f_i(u^\lambda)\, dy \le \int_{B_t(x)} f_i(u)\, dy\}.$$

By the mean value theorem,

$$u_i(x) - u_i^\lambda(x) \le \frac{1}{\gamma - 1} \int_{E_t} \frac{\int_{B_t(x)} f_i(u) - f_i(u^\lambda)\, dy}{t^{n-\gamma\beta}} \xi^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t}.$$

For $x \in E_t$, we have

$$\frac{\int_{B_t(x)} f_i(u^\lambda)\, dy}{t^{n-\gamma\beta}} \le \xi \le \frac{\int_{B_t(x)} f_i(u)\, dy}{t^{n-\gamma\beta}}.$$

Thus, we have the following basic estimate:

$$\begin{aligned} & u_i(x) - u_i^\lambda(x) \\ & \le \frac{1}{\gamma - 1} \int_{E_t} \frac{\int_{B_t(x)} f_i(u) - f_i(u^\lambda)\, dy}{t^{n-\gamma\beta}} \Big(\frac{\int_{B_t(x)} f_i(u)\, dy}{t^{n-\gamma\beta}}\Big)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t} \\ & \le \frac{1}{\gamma - 1} \int_0^\infty \frac{\int_{B_t(x)\cap D_i^\lambda} f_i(u) - f_i(u^\lambda)\, dy}{t^{n-\gamma\beta}} \Big(\frac{\int_{B_t(x)} f_i(u)\, dy}{t^{n-\gamma\beta}}\Big)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t}. \end{aligned}$$

Define

$$\begin{aligned} B_i^\lambda &= \{x \in H_\lambda : u_i(x^\lambda) < u_i(x)\}, \\ D_i^\lambda &= \{x \in H_\lambda : f_i(u(x^\lambda)) < f_i(u(x))\}. \end{aligned}$$

We can restrict the integration region to $B_t(x) \setminus B_t(x^\lambda)$, which lies in H_λ, and then restrict to $(B_t(x) \setminus B_t(x^\lambda)) \cap D_i^\lambda \subseteq B_t(x) \cap D_i^\lambda$.

Now, investigate $f_i(u) - f_i(u^\lambda)$. Let

$$w^{\lambda,+}(y) = (u(y) - u^\lambda(y))^+, \quad \overline{u}(y) = u_1(y) + \cdots + u_m(y).$$

For any fixed $y \in D_i^\lambda$, there are two possibilities.

Case 1. $\frac{1}{2}u_l(y) \ge u_l^\lambda(y)$ for some l.

So $u_l(y) \le 2(u_l(y) - u_l^\lambda(y))$. Moreover, for $|y|$ large, $\overline{u}(y) \le C u_l(y)$, since they have the same asymptotic behavior. Thus for large $|y|$ we have:

$$\begin{aligned} & f_i(u(y)) - f_i(u^\lambda(y)) \le f_i(u(y)) \le C|\overline{u}(y)|^p \\ & \le C|\overline{u}(y)|^{p-1}|\overline{u}(y)| \le C|\overline{u}(y)|^{p-1} u_l(y) \le C|\overline{u}(y)|^{p-1}|w^{\lambda,+}(y)|. \end{aligned}$$

Case 2. $u_l(y) \le u_l^\lambda(y)$ for $l \in I$. $u_l(y) \ge u_l^\lambda(y) \ge \frac{1}{2}u_l(y)$ for $l \in J$. Here both I and J are subsets of $\{1, \ldots, m\}$.

Since $y \in D_i^\lambda$ means J is nonempty, without loss of generality, we assume $J = \{1, \dots, k\}$ for fixed y. Thus,

$$\begin{aligned}
&f_i(u(y)) - f_i(u^\lambda(y)) \\
&\quad \le [f_i(u_1, \dots, u_m) - f_i(u_1^\lambda, u_2, \dots, u_m)] \\
&\qquad + \cdots + [f_i(u_1^\lambda, \dots, u_{k-1}^\lambda, u_k, \dots, u_m) - f_i(u_1^\lambda, \dots, u_k^\lambda, u_{k+1}, \dots, u_m)] \\
&\qquad + [f_i(u_1^\lambda, \dots, u_k^\lambda, u_{k+1}, \dots, u_m) - f_i(u_1^\lambda, \dots, u_m^\lambda)] \\
&\quad \le [f_i(u_1, \dots, u_m) - f_i(u_1^\lambda, u_2, \dots, u_m)] \\
&\qquad + \cdots + [f_i(u_1^\lambda, \dots, u_{k-1}^\lambda, u_k, \dots, u_m) - f_i(u_1^\lambda, \dots, u_k^\lambda, u_{k+1}, \dots, u_m)] \\
&\quad \le \frac{\partial f_i(\xi_1, u_2, \dots, u_m)}{\partial u_1} w_1^{\lambda,+}(y) \\
&\qquad + \cdots + \frac{\partial f_i(u_1^\lambda, \dots, u_{k-1}^\lambda, \xi_k, u_{k+1}, \dots, u_m)}{\partial u_k} w_k^{\lambda,+}(y).
\end{aligned}$$

Noticing that for $l \in J$, $u_l(y) \ge \xi_l(y) \ge \frac{1}{2} u_l(y)$, we can see that

$$\frac{\partial f_i}{\partial u_l}(u_1^\lambda, \dots, \xi_l, \dots, u_m) \le C \frac{f_i(u_1^\lambda, \dots, \xi_l, \dots, u_m)}{\xi_l}.$$

Therefore,

$$f_i(u(y)) - f_i(u^\lambda(y)) \le C|\overline{u}(y)|^{p-1}|w^{\lambda,+}(y)|.$$

Combining these results into the previous estimate, we have

$$\begin{aligned}
&u_i(x) - u_i^\lambda(x) \\
&\quad \le \frac{1}{\gamma - 1} \int_0^\infty \frac{\int_{D_i^\lambda \bigcap B_t(x)} f_i(u) - f_i(u^\lambda)\, dy}{t^{n-\gamma\beta}} \Big(\frac{\int_{B_t(x)} f_i(u)\, dy}{t^{n-\gamma\beta}}\Big)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t} \\
&\quad \le C \int_0^\infty \frac{\int_{D_i^\lambda \bigcap B_t(x)} \overline{u}(y)^{p-1} w^{\lambda,+}(y)\, dy}{t^{n-\gamma\beta}} \Big(\frac{\int_{B_t(x)} f_i(u)\, dy}{t^{n-\gamma\beta}}\Big)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t} \\
&\quad = C \int_0^\infty \frac{\int_{B_t(x)} \chi_{D_i^\lambda}(y) \overline{u}(y)^{p-1} w^{\lambda,+}(y)\, dy}{t^{n-\gamma\beta}} \Big(\frac{\int_{B_t(x)} f_i(u)\, dy}{t^{n-\gamma\beta}}\Big)^{\frac{2-\gamma}{\gamma-1}} \frac{dt}{t} \\
&\quad \le C(u_i)^{2-\gamma} \Big(\int_0^\infty \Big(\frac{\int_{B_t(x)} \chi_{D_i^\lambda}(y) \overline{u}(y)^{p-1} w^{\lambda,+}(y)\, dy}{t^{n-\gamma\beta}}\Big)^{\frac{1}{\gamma-1}} \frac{dt}{t}\Big)^{\gamma-1} \\
&\quad := C(u_i)^{2-\gamma} \cdot A^{\gamma-1}.
\end{aligned}$$

By the Wolff inequality and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
\|A\|_r^{\gamma-1} &\le C\|I_{\gamma\beta}(\overline{u}^{p-1} w^{\lambda,+} \chi_{D_i^\lambda})\|_{\frac{r}{\gamma-1}} \\
&\le C\|\overline{u}^{p-1} w^{\lambda,+} \chi_{D_i^\lambda}\|_{\frac{nr}{n(\gamma-1)+\gamma\beta r}} \\
&\le \|\overline{u}\|_{p+\gamma-1, D_i^\lambda}^{p-1} \|w^{\lambda,+}\|_{s, D_i^\lambda},
\end{aligned}$$

where $\frac{\gamma-1}{r} - \frac{1}{s} = \frac{p-1}{p+\gamma-1} - \frac{\gamma\beta}{n}$. We choose $r = p + \gamma - 1$. Then all the requirements of the above inequalities are met, and moreover $s = p + \gamma - 1$. Thus,

$$\begin{aligned}
\|u_i - u_i^\lambda\|_{p+\gamma-1, D_i^\lambda} &\le C\|u_i\|_{p+\gamma-1}^{2-\gamma} \|A\|_{p+\gamma-1}^{\gamma-1} \\
&\le C\|u_i\|_{p+\gamma-1}^{2-\gamma} \|\overline{u}\|_{p+\gamma-1, D_i^\lambda}^{p-1} \|w^{\lambda,+}\|_{p+\gamma-1, D_i^\lambda}.
\end{aligned}$$

Let $B^\lambda = \bigcup B_i^\lambda$ and $D^\lambda = \bigcup D_i^\lambda$. Then $D^\lambda \subset B^\lambda$. Therefore, we obtain the basic estimate

$$\|w^{\lambda,+}\|_{p+\gamma-1,H_\lambda} \le C\|\overline{u}\|_{p+\gamma-1}^{2-\gamma}\|\overline{u}\|_{p+\gamma-1,B^\lambda}^{p-1}\|w^{\lambda,+}\|_{p+\gamma-1,H_\lambda}. \tag{23}$$

Now for λ near negative infinity, $\|\overline{u}\|_{p+\gamma-1,B^\lambda}$ is so small that

$$\|\overline{u}\|_{p+\gamma-1}^{2-\gamma}\|\overline{u}\|_{p+\gamma-1,B^\lambda}^{p-1} \le \frac{1}{2C}.$$

Inserting this result into (23) yields

$$\|w^{\lambda,+}\|_{p+\gamma-1,H_\lambda} = 0.$$

In other words, $u_i(y) \le u_i^\lambda(y)$ and $f_i(u(y)) \le f_i(u^\lambda(y))$ for almost all $y \in H_\lambda$.

Step 2. First, we introduce a simple lemma.

Lemma 3.1. *Let $\Omega_t(x) = B_t(x) \setminus B_t(x^\lambda) \subset H_\lambda$. Then we have*

$$\begin{aligned} u_i(x) - u_i^\lambda(x) = \quad & \int_0^\infty \Big(\frac{\int_{\Omega_t(x)} f_i(u(y))\,dy}{t^{n-\gamma\beta}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t} \\ & - \int_0^\infty \Big(\frac{\int_{\Omega_t(x)} f_i(u^\lambda(y))\,dy}{t^{n-\gamma\beta}}\Big)^{\frac{1}{\gamma-1}}\frac{dt}{t}. \end{aligned}$$

Define $\lambda_0 = \sup\{\lambda \in R | \mu(B^s) = 0, \text{for all } s \le \lambda\}$.

Claim. If u_i is not identically equal to $u_i^{\lambda_0}$, then $u_i(x) < u_i^{\lambda_0}(x), x \in H_{\lambda_0}$.

By Lemma 3.1, without loss of generality, we can assume there is some k such that $f_i(u(x)) < f_i(u^{\lambda_0}(x))$ on a set of positive measure for $i \in \{1,\dots,k\}$ and $f_i(u(x)) = f_i(u^{\lambda_0}(x))$ a.e. for $i \in \{k+1,\dots,m\}$. Using Lemma 3.1 again, we have

$$u_i(x) < u_i^{\lambda_0}(x), \quad \text{as} \quad x \in H_{\lambda_0}, \quad \text{for} \quad i \in \{1,\dots,k\}$$

and

$$u_i(x) = u_i^{\lambda_0}(x) \quad \text{a.e.} \quad \text{for} \quad i \in \{k+1,\dots,m\}.$$

If $k = m$, the claim is proved. If $k < m$, $\{u_1,\dots,u_m\} \ne \{u_1^{\lambda_0},\dots,u_m^{\lambda_0}\}$ on a positive measure set and $u \le u^{\lambda_0}$. Since the system is simple, we must have $\{f_{k+1}(u),\dots,f_m(u)\} \ne \{f_{k+1}(u^{\lambda_0}),\dots,f_m(u^{\lambda_0})\}$ on a positive measure set; that is, $f_i(u(x)) \le f_i(u^{\lambda_0}(x))$ on a positive measure set. This contradicts our assumption.

By this claim, we can move the plane further to some $\lambda \ge \lambda_0$, which contradicts our assumption. First, we assume that the $L^{p+\gamma-1}$ norm of $\overline{u}$ outside some fixed bounded region B is small. Since

$$w^\lambda(x) = u(x) - u^\lambda(x) \to w^{\lambda_0}$$

in the norm of $L^{p+\gamma-1}(B)$, $w^\lambda(x)$ converges to w^{λ_0} in measure (here we always restrict our attention in B). Let

$$A_\delta = \{x \in B \cap H_{\lambda_0} | 0 > w^{\lambda_0}(x) > -\delta\}.$$

By virtue of $w^{\lambda_0} < 0$ on $B \cap H_{\lambda_0}$, for any given $\varepsilon > 0$, $\mu(A_\delta) < \varepsilon$ for some δ small enough. When λ is close enough to λ_0, $B \cap (H_\lambda - H_{\lambda_0})$ has small measure. Since w^λ converges to w^{λ^0} in measure, we have $w^\lambda(x) < 0$ on $(B \cap H_{\lambda_0} - A_\delta)$ except for a small measure subset, say E. Thus, combining all the above estimates, we get

$$\{x : w^\lambda(x) > 0\} \subseteq B^C \cup (B \cap (H_\lambda - H_{\lambda_0})) \cup E \cup A_\delta.$$

Therefore, we can still deduce that $\|\overline{u}\|_{p+\gamma-1,B^\lambda}$ is sufficiently small. Theorem 1.2, together with (9), implies that for any $\delta > 0$, there exist two positive constants c, C such that

$$c \leq u_i(y) \leq C, \quad \forall y \in R^n \setminus B_\delta(x).$$

There are two cases. If $\lambda_0 < 0$, then we can still apply the basic estimate (23) to derive $u(x) \leq u_\lambda(x)$ when λ is close to λ_0. This leads to a contradiction, which implies that $u(x) = u^{\lambda_0}(x)$ almost everywhere. If $\lambda_0 > 0$, the moving plane can stop at $x_1 = 0$. Then we repeat the above argument from $x_1 = \infty$. Now, we also have two cases. One is that the plane stops before the origin. The argument is the same as in the first case. The other case is that the plane stops at the origin, which still shows that $u(x) = u_0^\lambda(x)$. Finally, since the direction x_1 is chosen arbitrarily, we know that $u_1(x), u_2(x), \ldots, u_m(x)$ are radially symmetric and decreasing about the same point x_0. This completes our proof.

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