PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 3, March 2012, Pages 1069–1074
S 0002-9939(2011)10945-6
Article electronically published on July 6, 2011

ON THE MENGER COVERING PROPERTY AND D-SPACES

DUŠAN REPOVŠ AND LYUBOMYR ZDOMSKYY

(Communicated by Julia Knight)

ABSTRACT. The main results of this paper are:

- It is consistent that every subparacompact space X of size ω_1 is a D-space.
- If there exists a Michael space, then all productively Lindelöf spaces have the Menger property and, therefore, are D-spaces.
- Every locally D-space which admits a σ-locally finite cover by Lindelöf spaces is a D-space.

1. INTRODUCTION

A *neighbourhood assignment* for a topological space X is a function N from X to the topology of X such that $x \in N(x)$ for all x. A topological space X is said to be a *D-space* [6] if for every neighbourhood assignment N for X there exists a closed and discrete subset $A \subset X$ such that $N(A) = \bigcup_{x\in A} N(x) = X$.

It is unknown whether paracompact (even Lindelöf) spaces are D-spaces. Our first result in this note answers [7, Problem 3.8] in the affirmative and may be thought of as a very partial solution to this problem.[1]

Our second result shows that the affirmative answer to [19, Problem 2.6], which asks whether all productively Lindelöf spaces are D-spaces, is consistent. It is worth mentioning that our premises (i.e., the existence of a Michael space) are not known to be inconsistent.

Our third result is a common generalization of two theorems from [10].

Most of our proofs use either the recent important result of Aurichi [2] asserting that every topological space with the Menger property is a D-space or the ideas from its proof. We consider only regular topological spaces. For the definitions of small cardinals $\mathfrak{d}$ and $\mathrm{cov}(\mathcal{M})$ used in this paper we refer the reader to [22].

Received by the editors September 28, 2010 and, in revised form, December 4, 2010 and December 13, 2010.

2010 *Mathematics Subject Classification.* Primary 54D20, 54A35; Secondary 54H05, 03E17.

Key words and phrases. D-space, subparacompactness, Menger property, (productively) Lindelöf space, Michael space.

The first author was supported by SRA grants P1-0292-0101 and J1-2057-0101.

The second author acknowledges the support of FWF grant P19898-N18.

The authors would also like to thank Leandro Aurichi, Franklin Tall, and Hang Zhang for kindly making their recent papers available to us.

[1]While completing this manuscript, we learned that this result was independently obtained by Hang Zhang and Wei-Xue Shi; see [15].

2. Subparacompact spaces of size ω_1

Following [4] we say that a topological space X has the property E^*_ω if for every sequence $\langle u_n : n \in \omega\rangle$ of *countable* open covers of X there exists a sequence $\langle v_n : n \in \omega\rangle$ such that $v_n \in [u_n]^{<\omega}$ and $\bigcup_{n\in\omega} \cup v_n = X$. In the realm of Lindelöf spaces the property E^*_ω is usually called the *Menger property* or $\bigcup_{fin}(\mathcal{O},\mathcal{O})$; see [21] and references therein.

We say that a topological space X has property D_ω if for every neighbourhood assignment N there exists a countable collection $\{A_n : n \in \omega\}$ of closed discrete subsets of X such that $X = \bigcup_{n\in\omega} N(A_n)$. Observe that the property D_ω is inherited by all closed subsets.

The following theorem is the main result of this section.

Theorem 2.1. *Suppose that a topological space X has properties D_ω and E^*_ω. Then X is a D-space.*

The proof of Theorem 2.1 is analogous to the proof of [2, Proposition 2.6]. In particular, it uses the following game of length ω on a topological space X: On the nth move player I chooses a countable open cover $u_n = \{U_{n,k} : k \in \omega\}$ such that $U_{n,k} \subset U_{n,k+1}$ for all $k \in \omega$, and player II responds by choosing a natural number k_n. Player II wins the game if $\bigcup_{n\in\omega} U_{n,k_n} = X$. Otherwise, player I wins. We shall call this game an E^*_ω-game. In the realm of Lindelöf spaces this game is known under the name *Menger game*. It is well known that a Lindelöf space X has the property E^*_ω if and only if the first player has no winning strategy in the E^*_ω-game on X; see [8, 14]. The proof of [14, Theorem 13] also works without any change for non-Lindelöf spaces.

Proposition 2.2. *A topological space X has the property E^*_ω if and only if the first player has no winning strategy in the E^*_ω-game.*

A strategy of the first player in the E^*_ω-game may be thought of as a map $\Upsilon : \omega^{<\omega} \to \mathcal{O}(X)$, where $\mathcal{O}(X)$ stands for the collection of all countable open covers of X. The strategy Υ is winning if $X \neq \bigcup_{n\in\omega} U_{z\restriction n, z(n)}$ for all $z \in \omega^\omega$, where $\Upsilon(s) = \{U_{s,k} : k \in \omega\} \in \mathcal{O}(X)$.

We are in a position now to present the proof of Theorem 2.1.

Proof. We shall define a strategy $\Upsilon : X \to \mathcal{O}(X)$ of the player I in the E^*_ω-game on X as follows. Set $F_\emptyset = X$. The property D_ω yields an increasing sequence $\langle A_{\emptyset,k} : k \in \omega\rangle$ of closed discrete subsets of $F_\emptyset$ such that $X = \bigcup_{k\in\omega} N(A_{\emptyset,k})$. Set $\Upsilon(\emptyset) = u_\emptyset = \{N(A_{\emptyset,k}) : k \in \omega\}$.

Suppose that for some $m \in \omega$ and all $s \in \omega^{\leq m}$ we have already defined a closed subset F_s of X, an increasing sequence $\langle A_{s,k} : k \in \omega\rangle$ of closed discrete subsets of F_s, and a countable open cover $\Upsilon(s) = u_s$ of X such that $u_s = \{(X \setminus F_s) \cup N(A_{s,k}) : k \in \omega\}$.

Fix $s \in \omega^{m+1}$. Since X has the property D_ω, so does its closed subspace $F_s := X \setminus \bigcup_{i<m+1} N(A_{s\restriction i, s(i)})$, and hence there exists an increasing sequence $\langle A_{s,k} : k \in \omega\}$ of closed discrete subsets of F_s such that $F_s \subset \bigcup_{k\in\omega} N(A_{s,k})$. Set $\Upsilon(s) = u_s = \{(X \setminus F_s) \cup N(A_{s,k}) : k \in \omega\}$. This completes the definition of Υ.

Since X has the property E^*_ω, Υ is not winning. Thus there exists $z \in \omega^\omega$ such that $X = \bigcup_{n\in\omega} (X \setminus F_{z\restriction n}) \cup N(A_{z\restriction n, z(n)})$. By the inductive construction, $X \setminus F_\emptyset = \emptyset$ and $X \setminus F_{z\restriction n} = \bigcup_{i<n} N(A_{z\restriction i, z(i)})$ for all $n > 0$. It follows from above that $X = \bigcup_{n\in\omega} N(A_{z\restriction n, z(n)})$. In addition, $A_{z\restriction n, z(n)} \subset F_{z\restriction n} = X \setminus \bigcup_{i<n} N(A_{z\restriction i, z(i)})$

for all $n > 0$, which implies that $A := \bigcup_{n\in\omega} A_{z\restriction n,z(n)}$ is a closed discrete subset of X. It suffices to note that $N(A) = X$. □

We recall from [5] that a topological space X is called *subparacompact* if every open cover of X has a σ-locally finite closed refinement.

Lemma 2.3. *Suppose that X is a subparacompact topological space which can be covered by ω_1-many of its Lindelöf subspaces. Then X has the property D_ω.*[2]

In particular, every subparacompact space of size ω_1 has the property D_ω.

Proof. Let $\mathcal{L} = \{L_\xi : \xi < \omega_1\}$ be an increasing cover of X by Lindelöf subspaces, let τ be the topology of X, and let $N : X \to \tau$ be a neighbourhood assignment. Construct by induction a sequence $\langle C_\alpha : \alpha < \omega_1\rangle$ of (possibly empty) countable subsets of X such that

(*i*) $L_0 \subset N(C_0)$;
(*ii*) $C_\alpha \cap N(\bigcup_{\xi<\alpha} C_\xi) = \emptyset$ for all $\alpha < \omega_1$; and
(*iii*) $L_\alpha \setminus N(\bigcup_{\xi<\alpha} C_\xi) \subset N(C_\alpha)$ for all $\alpha < \omega_1$.

Set $C = \bigcup_{\alpha<\omega_1} C_\alpha$. The subparacompactness of X yields a closed cover $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ of X which refines $\mathcal{U} = \{N(x) : x \in C\}$ and such that each $\mathcal{F}_n$ is locally finite. Since every element of $\mathcal{U}$ contains at most countably many elements of C, so do elements of $\mathcal{F}$. Therefore for every $F \in \mathcal{F}_n$ such that $C \cap F \neq \emptyset$ we can write this intersection in the form $\{x_{n,F,m} : m \in \omega\}$. Now it is easy to see that $A_{n,m} := \{x_{n,F,m} : F \in \mathcal{F}_n, C \cap F \neq \emptyset\}$ is a closed discrete subset of X and $\bigcup_{n,m\in\omega} A_{n,m} = C$. □

Remark 2.4. What we have actually used in the proof of Lemma 2.3 is the following weakening of subparacompactness: every open cover $\mathcal{U}$ which is closed under unions of its countable subsets admits a σ-locally finite closed refinement. We do not know whether this property is strictly weaker than subparacompactness.

Corollary 2.5. *Let X be a countably tight paracompact topological space of density ω_1. Then X has the property D_ω.*

Proof. Let $\{x_\alpha : \alpha < \omega_1\}$ be a dense subspace of X. Since X has countable tightness, $X = \bigcup_{\alpha<\omega_1} \overline{\{x_\xi : \xi < \alpha\}}$. It suffices to note that the closure of any countable subspace of a paracompact space is Lindelöf. □

It is well known [9, Theorem 4.4] (and it easily follows from corresponding definitions) that any Lindelöf space of size $< \mathfrak{d}$ has the Menger property. The same argument shows that every topological space of size $< \mathfrak{d}$ has the property E^*_ω. Combining this with Theorem 2.1 and Lemma 2.3, we get the following corollary, which implies the first of the results mentioned in our abstract.

Corollary 2.6. *Suppose that X is a subparacompact topological space of size $|X| < \mathfrak{d}$ which can be covered by ω_1-many of its Lindelöf subspaces. Then X is a D-space.*

3. Concerning the existence of a Michael space

A topological space X is said to be *productively Lindelöf* if $X \times Y$ is Lindelöf for all Lindelöf spaces Y. It was asked in [19] whether productively Lindelöf spaces are D-spaces. The positive answer to the above question has been proved consistent,

[2]By the methods of [15] the submetalindelöfness is sufficient here.

and in a stream of recent papers (see the list of references in [19]) several sufficient set-theoretical conditions were established. The following statement gives a uniform proof for some of these results. In particular, it implies [16, Theorems 5 and 7] and [1, Corollary 4.5] and answers [17, Question 15] in the affirmative.

A Lindelöf space Y is called a *Michael* space if $\omega^\omega \times Y$ is not Lindelöf.

Proposition 3.1. *If there exists a Michael space, then every productively Lindelöf space has the Menger property.*

We refer the reader to [11], where the existence of a Michael space was reformulated in a combinatorial language and a number of set-theoretical conditions guaranteeing the existence of Michael spaces were established.

In the proof of Proposition 3.1 we shall use set-valued maps; see [13]. By a *set-valued map* Φ from a set X into a set Y we understand a map from X into $\mathcal{P}(Y)$ and write $\Phi : X \Rightarrow Y$ (here $\mathcal{P}(Y)$ denotes the set of all subsets of Y). For a subset A of X we set $\Phi(A) = \bigcup_{x\in A} \Phi(x) \subset Y$. A set-valued map Φ from a topological spaces X to a topological space Y is said to be

- *compact-valued* if $\Phi(x)$ is compact for every $x \in X$;
- *upper semicontinuous* if for every open subset V of Y the set $\Phi^{-1}_{\subset}(V) = \{x \in X : \Phi(x) \subset V\}$ is open in X.

The proof of the following claim is straightforward.

Claim 3.2.
(1) Suppose that X, Y are topological spaces, X is Lindelöf, and $\Phi : X \Rightarrow Y$ is a compact-valued upper semicontinuous map such that $Y = \Phi(X)$. Then Y is Lindelöf.
(2) If $\Phi_0 : X_0 \Rightarrow Y_0$ and $\Phi_0 : X_1 \Rightarrow Y_1$ are compact-valued upper semicontinuous, then so is the map $\Phi_0 \times \Phi_1 : X_0 \times X_1 \Rightarrow Y_0 \times Y_1$ assigning to each $(x_0, x_1) \in X_0 \times X_1$ the product $\Phi_0(x_0) \times \Phi_1(x_1)$.

Proof of Proposition 3.1. Suppose, contrary to our claim, that X is a productively Lindelöf space which does not have the Menger property and Y is a Michael space. It suffices to show that $X \times Y$ is not Lindelöf.

Indeed, by [23, Theorem 8] there exists a compact-valued upper semicontinuous map $\Phi : X \to \omega^\omega$ such that $\Phi(X) = \omega^\omega$. By Claim 3.2(2) the product $\omega^\omega \times Y$ is the image of $X \times Y$ under a compact-valued upper semicontinuous map. By the definition of a Michael space, $\omega^\omega \times Y$ is not Lindelöf. By applying Claim 3.2(1) we can conclude that $X \times Y$ is not Lindelöf either. □

By a result of Tall [16] the existence of a Michael space implies that all productively Lindelöf analytic metrizable spaces are σ-compact. Combining recent results obtained in [1] and [12], we can consistently extend this result to all Σ^1_2 definable subsets of 2^ω.

Theorem 3.3. *Suppose that* $\mathrm{cov}(\mathcal{M}) > \omega_1$ *and that there exists a Michael space. Then every productively Lindelöf* Σ^1_2 *definable subset of* 2^ω *is* σ*-compact.*

Proof. Let X be a productively Lindelöf Σ^1_2 definable subset of 2^ω.

If X cannot be written as a union of ω_1-many of its compact subspaces, then it contains a closed copy of ω^ω [12], and hence the existence of the Michael space implies that X is not productively Lindelöf, a contradiction.

Thus X can be written as a union of ω_1-many of its compact subspaces, and therefore it is σ-compact by [1, Corollary 4.15]. □

We do not know whether the assumption $\operatorname{cov}(\mathcal{M}) > \omega_1$ can be dropped from Theorem 3.3.

Question 3.4. Suppose that there exists a Michael space. Is every coanalytic productively Lindelöf space σ-compact?

By [18, Proposition 31] the affirmative answer to the question above follows from the Axiom of Projective Determinacy.

4. Locally finite unions

Theorem 4.1. *Suppose that X is a locally D-space which admits a σ-locally finite cover by Lindelöf spaces. Then X is a D-space.*

Proof. Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ be a cover of X by Lindelöf subspaces such that $\mathcal{F}_n$ is locally finite. Fix $F \in \mathcal{F}_n$. For every $x \in F$ there exists an open neighbourhood U_x of x such that $\bar{U}_x$ is a D-space. Let C_F be a countable subset of F such that $F \subset \bigcup_{x\in C_F} U_x$. Then $\mathcal{Z}_F = \{\overline{F \cap U_x} : x \in C_F\}$ is a countable cover of F consisting of closed D-subspaces of X such that $F \cap Z$ is dense in Z for all $Z \in \mathcal{Z}_F$. It follows from the above that X admits a σ-locally finite cover consisting of closed D-subspaces. Since a union of a locally finite family of closed D-subspaces is easily seen to be a closed D-subspace, X is a union of an increasing sequence of its closed D-subspaces. Therefore it is a D-space by results of [3]. □

Corollary 4.2. *If a topological space X admits a σ-locally finite locally countable cover by topological spaces with the Menger property, then it is a D-space. In particular, a locally Lindelöf space admitting a σ-locally finite cover by topological spaces with the Menger property is a D-space.*

Proof. The second part is a direct consequence of the first one since every σ-locally countable family of subspaces of a locally Lindelöf space is locally countable.

To prove the first assertion, note that by local countability every point $x \in X$ has a closed neighbourhood which is a countable union of its subspaces with the Menger property, and hence it has the Menger property itself. Therefore X is a locally D-space. It now suffices to apply Theorem 4.1. □

It is known that every Lindelöf $\mathcal{C}$-scattered space is $\mathcal{C}$-like and that $\mathcal{C}$-like spaces have the Menger property; see [20, p. 247] and references therein. Thus Corollary 4.2 implies Theorems 2.2 and 3.1 from [10].

References

1. Alas, O.; Aurichi, L.F.; Junqueira, L.R.; Tall, F.D., *Non-productively Lindelöf spaces and small cardinals,* Houston J. Math., to appear.
2. Aurichi, L.F., *D-spaces, topological games, and selection principles,* Topology Proc. **36** (2010), 107–122. MR2591778
3. Borges, C.R.; Wehrly, A.C., *A study of D-spaces,* Topology Proc. **16** (1991), 7–15. MR1206448 (94a:54059)
4. Bukovský, L.; Haleš, J., *On Hurewicz properties,* Topology Appl. **132** (2003), 71–79. MR1990080 (2004f:03085)
5. Burke, D.K., *Covering properties*, in: *Handbook of Set-Theoretic Topology* (K. Kunen, J.E. Vaughan, eds.), North Holland, Amsterdam, 1984, 347–422. MR776628 (86e:54030)
6. Van Douwen, E.K.; Pfeffer, W.F., *Some properties of the Sorgenfrey line and related spaces,* Pacific J. Math. **81** (1979), 371–377. MR547605 (80h:54027)
7. Gruenhage, G., *A survey on D spaces*, Contemp. Math., Vol. 533, Amer. Math. Soc., Providence, RI, 2011.

8. Hurewicz, W., *Über die Verallgemeinerung des Borellschen Theorems*, Math. Z. **24** (1925), 401–421.
9. Just, W.; Miller, A.W.; Scheepers, M.; Szeptycki, P.J., *The combinatorics of open covers. II,* Topology Appl. **73** (1996), 241–266. MR1419798 (98g:03115a)
10. Martínez, J.C.; Soukup, L., *The D-property in unions of scattered spaces,* Topology Appl. **156** (2009), 3086–3090. MR2556068 (2010k:54027)
11. Moore, J.T., *Some of the combinatorics related to Michael's problem,* Proc. Amer. Math. Soc. **127** (1999), 2459–2467. MR1486743 (99j:54008)
12. Repický, M., *Another proof of Hurewicz theorem,* Tatra Mt. Math. Publ., to appear.
13. Repovš, D.; Semenov, P., *Continuous selections of multivalued mappings.* Mathematics and its Applications, 455. Kluwer Academic Publishers, Dordrecht, 1998. MR1659914 (2000a:54002)
14. Scheepers, M., *Combinatorics of open covers. I. Ramsey theory,* Topology Appl. **69** (1996), 31–62. MR1378387 (97h:90123)
15. Shi, W.; Zhang, H., *A note on D-spaces*, preprint, 2010.
16. Tall, F.D., *Productively Lindelöf spaces may all be D,* Canad. Math. Bulletin, to appear.
17. Tall, F.D., *Lindelöf spaces which are Menger, Hurewicz, Alster, productive, or D,* Topology Appl., to appear.
18. Tall, F.D., *A note on productively Lindelöf spaces,* preprint, 2010.
19. Tall, F.D., *Set-theoretic problems concerning Lindelöf spaces*, preprint, 2010.
20. Telgársky, R., *Topological games: On the 50th anniversary of the Banach-Mazur game,* Rocky Mountain J. Math. **17** (1987), 227–276. MR892457 (88d:54046)
21. Tsaban, B., *Selection principles and special sets of reals,* in: *Open problems in topology. II* (edited By Elliott Pearl), Elsevier Sci. Publ., 2007, pp. 91–108.
22. Vaughan, J., *Small uncountable cardinals and topology*, in: *Open problems in topology* (J. van Mill, G.M. Reed, eds.), Elsevier Sci. Publ., 1990, pp. 195-218. MR1078647
23. Zdomskyy, L., *A semifilter approach to selection principles,* Comment. Math. Univ. Carolin. **46** (2005), 525–539. MR2174530 (2006g:54028)

Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, P.O. Box 2964, Ljubljana, Slovenia 1001

E-mail address: `dusan.repovs@guest.arnes.si`

URL: `http://www.fmf.uni-lj.si/~repovs/index.htm`

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Strasse 25, A-1090 Wien, Austria

E-mail address: `lzdomsky@gmail.com`

URL: `http://www.logic.univie.ac.at/~lzdomsky/`