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ON SINGLE COMMUTATORS IN II_1–FACTORS

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Abstract. We investigate the question of whether all elements of trace zero in a II_1–factor are single commutators. We show that all nilpotent elements are single commutators, as are all normal elements of trace zero whose spectral distributions are discrete measures. Some other classes of examples are considered.

1. Introduction

In an algebra $\mathfrak{A}$, the *commutator* of $B, C \in \mathfrak{A}$ is $[B, C] = BC - CB$, and we denote by $\mathrm{Comm}(\mathfrak{A}) \subseteq \mathfrak{A}$ the set of all commutators. A *trace* on $\mathfrak{A}$ is by definition a linear functional that vanishes on $\mathrm{Comm}(\mathfrak{A})$. The algebra $M_n(k)$ of $n \times n$ matrices over a field k has a unique trace, up to scalar multiplication (we denote the trace sending the identity element to 1 by tr_n). It is known that every element of $M_n(k)$ that has null trace is necessarily a commutator (see [24] for the case of characteristic zero and [1] for the case of an arbitrary characteristic). For the complex field, $k = \mathbf{C}$, a natural generalization of the algebra $M_n(\mathbf{C})$ is the algebra $B(\mathcal{H})$ of all bounded operators on a separable, possibly infinite dimensional Hilbert space $\mathcal{H}$. Thanks to the ground breaking paper [6] of Brown and Pearcy, $\mathrm{Comm}(B(\mathcal{H}))$ is known: the commutators in $B(\mathcal{H})$ are precisely the operators that are not of the form $\lambda I + K$ for λ a nonzero complex number, I the identity operator and K a compact operator (and an analogous result holds when $\mathcal{H}$ is nonseparable).

Characterizations of $\mathrm{Comm}(B(X))$ for some Banach spaces X are found in [2], [3], [11] and [12].

The von Neumann algebra factors form a natural family of algebras including the matrix algebras $M_n(\mathbf{C})$ and $B(\mathcal{H})$ for infinite dimensional Hilbert spaces $\mathcal{H}$ (these together are the type I factors). The set $\mathrm{Comm}(\mathcal{M})$ was determined by Brown and Pearcy [7] for $\mathcal{M}$ a factor of type III and by Halpern [17] for $\mathcal{M}$ a factor of type II_∞.

The case of type II_1 factors remains open. A type II_1 factor is a von Neumann algebra $\mathcal{M}$ whose center is trivial and that has a trace $\tau : \mathcal{M} \to \mathbf{C}$, which is then unique up to scalar multiplication; by convention, we always take $\tau(1) = 1$. The following question seems natural in light of what is known for matrices:

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Question 1.1. Do we have

$$\mathrm{Comm}(\mathcal{M}) = \ker \tau$$

for any one particular II_1–factor $\mathcal{M}$, or even for all II_1–factors?

Some partial results are known. Fack and de la Harpe [14] showed that every element of $\ker \tau$ is a sum of ten commutators (and with control of the norms of the elements). The number ten was improved to two by Marcoux [19]. Pearcy and Topping, in [22], showed that in the type II_1 factors of Wright (which do not have separable predual), every selfadjoint element of trace zero is a commutator.

In section 2, we employ the construction of Pearcy and Topping for the Wright factors and a result of Hadwin [16] to show that all normal elements of trace zero in the Wright factors are commutators. We then use this same construction to derive that in any II_1–factor, every normal element with trace zero and purely atomic distribution is a single commutator. In section 3, we show that all nilpotent operators in II_1–factors are commutators. Finally, in section 4, we provide classes of examples of elements of II_1–factors that are not normal and not nilpotent but are single commutators, and we ask some specific questions suggested by our examples and results.

2. Some normal operators

The following lemma (but with a constant of 2) was described in Concluding Remark (1) of [22], attributed to unpublished work of John Dyer. That the desired ordering of eigenvalues can be made with bounding constant 4 follows from work of Steinitz [26], the value 2 follows from [15] and the better constant in the version below (which is not actually needed in our application of it) is due to work of Banaszczyk [4], [5].

Lemma 2.1. *Let $A \in M_n(\mathbf{C})$ be a normal element with $\mathrm{tr}_n(A) = 0$. Then there are $B, C \in M_n(\mathbf{C})$ with $A = [B, C]$ and $\|B\|\,\|C\| \leq \frac{\sqrt{5}}{2}\|A\|$.*

Proof. After conjugating with a unitary, we may without loss of generality assume $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and we may choose the diagonal elements to appear in any prescribed order. We have $A = [B, C]$ where

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & & \cdots & 0 & 1 \\ 0 & 0 & \cdots & & 0 \end{pmatrix} \tag{1}$$

and $C = B^* D$, where

$$D = \mathrm{diag}(\lambda_1,\, \lambda_1 + \lambda_2,\, \ldots, \lambda_1 + \cdots + \lambda_{n-1}, 0). \tag{2}$$

By work of Banaszczyk [4], [5], any list $\lambda_1, \ldots, \lambda_n$ of complex numbers whose sum is zero can be reordered so that for all $k \in \{1, \ldots, n-1\}$ we have

$$\left| \sum_{j=1}^{k} \lambda_j \right| \leq \frac{\sqrt{5}}{2} \max_{1 \leq j \leq n} |\lambda_j|. \tag{3}$$

This ensures $\|B\| \leq 1$ and $\|C\| \leq \frac{\sqrt{5}}{2}\|A\|$. $\square$

The II_1–factors of Wright [28] are the quotients of the von Neumann algebra of all bounded sequences in $\prod_{n=1}^{\infty} M_n(\mathbf{C})$ by the ideal I_ω, consisting of all sequences $(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} M_n(\mathbf{C})$ such that $\lim_{n\to\omega} \operatorname{tr}_n(a_n^* a_n) = 0$, where ω is a nontrivial ultrafilter on the natural numbers. The trace of the element of $\mathcal{M}$ associated to a bounded sequence $(b_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} M_n(\mathbf{C})$ is $\lim_{n\to\omega} \operatorname{tr}_n(b_n)$. (See [20] or [18] for ultrapowers of finite von Neumann algebras.) The following result in the case of selfadjoint operators is due to Pearcy and Topping [22].

Theorem 2.2. *If $\mathcal{M}$ is a Wright factor and if $T \in \mathcal{M}$ is normal with $\tau(T) = 0$, then $T \in \operatorname{Comm}(\mathcal{M})$.*

Proof. Let $T \in \mathcal{M}$ be normal and let X and Y be the real and imaginary parts of T, respectively. Let $(S_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} M_n(\mathbf{C})$ be a representative of T, with $\|S_n\| \le \|T\|$ for all n. Let X_n and Y_n be the real and imaginary parts of S_n. Then the mixed $*$-moments of the pair (X_n, Y_n) converge as $n \to \omega$ to the mixed $*$-moments of (X, Y). By standard methods, we can construct some commuting, selfadjoint, traceless $n \times n$ matrices H_n and K_n such that H_n converges in moments to X and K_n converges in moments to Y, as $n \to \infty$. Now using a result of Hadwin (Theorem 2.1 of [16]), we find $n \times n$ unitaries U_n such that

$$\lim_{n\to\omega} \|U_n X_n U_n^* - H_n\|_2 = 0, \qquad \lim_{n\to\omega} \|U_n Y_n U_n^* - K_n\|_2 = 0,$$

where $\|Z\|_2 = \operatorname{tr}_n(Z^* Z)^{1/2}$ is the Euclidean norm resulting from the normalized trace on $M_n(\mathbf{C})$. This shows that T has representative $(T_n)_{n=1}^{\infty}$, where $T_n = U_n^*(H_n + iK_n)U_n$ is normal and, of course, traceless.

By Lemma 2.1, for each n there are $B_n, C_n \in M_n(\mathbf{C})$ with $\|B_n\| = 1$ and $\|C_n\| \le \frac{\sqrt{5}}{2}\|T\|$ such that $T_n = [B_n, C_n]$. Let $B, C \in \mathcal{M}$ be the images (in the quotient $\prod_{n=1}^{\infty} M_n(\mathbf{C})/I_\omega$) of $(B_n)_{n=1}^{\infty}$ and $(C_n)_{n=1}^{\infty}$, respectively. Then $T = [B, C]$. $\square$

The *distribution* of a normal element T in a II_1–factor is the compactly supported Borel probability measure on the complex plane obtained by composing the trace with the projection–valued spectral measure of T.

Theorem 2.3. *If R is the hyperfinite II_1–factor and if μ is a compactly supported Borel probability measure on the complex plane such that $\int z\,\mu(dz) = 0$, then there is a normal element $T \in \operatorname{Comm}(R)$ whose distribution is μ.*

Proof. We will consider a particular instance of the construction from the proof of Theorem 2.2. Let $\mathcal{M}$ be a factor of Wright, with tracial state τ. Let L be the maximum modulus of elements of the support of μ. We may choose complex numbers $(\lambda_j^{(n)})_{j=1}^{n}$ for $n \ge 1$ such that the measures $\frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j^{(n)}}$ converge in weak*–topology to μ and all have support contained inside the disk of radius L centered at the origin and such that $\sum_{j=1}^{n} \lambda_j^{(n)} = 0$ for each n. Let $T_n = \operatorname{diag}(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}) \in M_n(\mathbf{C})$ and let $T \in \mathcal{M}$ be the element associated to the sequence $(T_n)_{n=1}^{\infty}$. Then the distribution of T is μ. By [4], [5], we can order these $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ so that $\left|\sum_{j=1}^{k} \lambda_j^{(n)}\right| \le \frac{\sqrt{5}}{2}\|T\|$ for all $1 \le k \le n$. Then, as in the proof of Lemma 2.1, we have $T_n = [B_n, B_n^* D_n]$ where B_n and D_n are the $n \times n$ matrices B and D of (1) and (2), respectively. If $B, D \in \mathcal{M}$ are the images in the quotient of the sequences $(B_n)_{n=1}^{\infty}$ and $(D_n)_{n=1}^{\infty}$, respectively, then $T = [B, B^* D]$. However, note that $B \in \mathcal{M}$ is a unitary element such that $\tau(B^k) = 0$ for all $k > 0$.

Moreover, the set $\{B^k D B^{-k} \mid k \in \mathbf{Z}\}$ generates a commutative von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ and every element of $\mathcal{A}$ is the image (under the quotient mapping) of a sequence $(A_n)_{n=1}^{\infty}$ where each $A_n \in M_n(\mathbf{C})$ is a diagonal matrix. Thus, the unitary B acts by conjugation on $\mathcal{A}$, and, moreover, we have $\tau(AB^k) = 0$ for all $A \in \mathcal{A}$ and all $k > 0$. Therefore the von Neumann subalgebra generated by $\mathcal{A} \cup \{B\}$ is a case of the group–measure-space construction, $\mathcal{A} \rtimes \mathbf{Z}$, and is a hyperfinite von Neumann algebra by [10] and can, thus, be embedded into the hyperfinite II_1–factor R. □

The above proof actually shows the following.

Corollary 2.4. *Given any compactly supported Borel probability measure μ on the complex plane with $\int z\,\mu(dz) = 0$, there is $f \in L^\infty([0,1])$ and a probability-measure-preserving transformation α of $[0,1]$ such that the distribution of $f - \alpha(f)$ equals μ and the supremum norm of f is no more than $\frac{\sqrt{5}}{2}$ times the maximum modulus of the support of μ.*

Theorem 2.5. *If $\mathcal{M}$ is any II_1–factor and $T \in \mathcal{M}$ is a normal element whose distribution is purely atomic and with trace $\tau(T) = 0$, then $T \in \mathrm{Comm}(\mathcal{M})$.*

Proof. $\mathcal{M}$ contains a (unital) subfactor R isomorphic to the hyperfinite II_1–factor. By Theorem 2.3, there is an element $\widetilde{T} \in \mathrm{Comm}(R)$ whose distribution equals the distribution of T. Since this distribution is purely atomic, there is a unitary $U \in \mathcal{M}$ such that $U\widetilde{T}U^* = T$. Thus, $T \in \mathrm{Comm}(\mathcal{M})$. □

3. Nilpotent operators

The von Neumann algebra $\mathcal{M}$ is embedded in $B(\mathcal{H})$ as a strong–operator–topology closed, selfadjoint subalgebra. If $T \in \mathcal{M}$, we denote the selfadjoint projection onto $\ker(T)$ by $\mathrm{kerproj}(T)$ and the selfadjoint projection onto the closure of the range of T by $\mathrm{ranproj}(T)$. Both of these belong to $\mathcal{M}$, and we have

$$\tau(\mathrm{kerproj}(T)) + \tau(\mathrm{ranproj}(T)) = 1.$$

The following decomposition follows from the usual sort of analysis of subspaces that one does also in the finite dimensional setting.

Lemma 3.1. *Let $\mathcal{M}$ be a II_1–factor and let $T \in \mathcal{M}$ be nilpotent, $T^n = 0$. Then there are integers $n \geq k_1 > k_2 > \ldots > k_m \geq 1$ and for each $j \in \{1, \ldots, m\}$ there are equivalent projections $f_1^{(j)}, \ldots, f_{k_j}^{(j)}$ in $\mathcal{M}$ such that*

(i) *$f^{(j)} := f_1^{(j)} + \cdots + f_{k_j}^{(j)}$ commutes with T,*
(ii) *$f^{(1)} + \cdots + f^{(m)} = 1$,*
(iii) *the $k_j \times k_j$ matrix of $f^{(j)}T$ with respect to these projections $f_1^{(j)}, \ldots, f_{k_j}^{(j)}$ is strictly upper triangular.*

In other words, the lemma says that T lies in a unital $*$–subalgebra of $\mathcal{M}$ that is isomorphic to $M_{k_1}(\mathfrak{A}_1) \oplus \cdots \oplus M_{k_m}(\mathfrak{A}_m)$ for certain compressions $\mathfrak{A}_j$ of $\mathcal{M}$ by projections, and the direct summand component of T in each $M_{k_j}(\mathfrak{A}_j)$ is a strictly upper triangular matrix.

Proof. The proof is by induction on n. The case $n = 1$ is clear, because then $T = 0$. Assume $n \geq 2$. We consider the usual system $p_1, p_2, \ldots, p_n$ of pairwise orthogonal

projections with respect to which T is upper triangular:

$$\begin{aligned} p_1 &= \operatorname{kerproj}(T), \\ p_j &= \operatorname{kerproj}(T^j) - \operatorname{kerproj}(T^{j-1}) \quad (2 \le j \le n). \end{aligned}$$

Then we have

$$\tau(\operatorname{ranproj}(Tp_j)) = \tau(p_j) \qquad (2 \le j \le n), \tag{4}$$

$$\operatorname{ranproj}(Tp_j) \le \operatorname{kerproj}(T^{j-1}) = p_1 + p_2 + \cdots + p_{j-1} \qquad (2 \le j \le n), \tag{5}$$

$$\operatorname{ranproj}(Tp_j) \wedge (p_1 + p_2 + \cdots + p_{j-2}) = 0 \qquad (3 \le j \le n). \tag{6}$$

Indeed, for (4), it will suffice to show $\operatorname{kerproj}(Tp_j) = 1 - p_j$. For this, note that if $p_j\xi = \xi$ and $T\xi = 0$, then $\xi \in \ker T \subseteq \ker T^{j-1}$. Since $p_j \perp \operatorname{kerproj}(T^{j-1})$, this gives $\xi = 0$. The relation (5) is clear. For (6), if $q := \operatorname{ranproj}(Tp_j) \wedge \operatorname{kerproj}(T^{j-2}) \neq 0$, then by standard techniques (see, e.g., Lemma 2.2.1 of [9]), we would have a nonzero projection $r \le p_j$ such that $q = \operatorname{ranproj}(Tr) \le \operatorname{kerproj}(T^{j-2})$. However, this would imply $r \le \operatorname{kerproj}(T^{j-1})$, which contradicts $p_j \perp \operatorname{kerproj}(T^{j-1})$.

Let

$$\begin{aligned} q_n &= p_n\,, \\ q_{n-j} &= \operatorname{ranproj}(T^j q_n) \qquad (1 \le j \le n-1). \end{aligned}$$

Then we have

$$q_k = \operatorname{ranproj}(Tq_{k+1}) \le p_1 + \cdots + p_k \qquad (1 \le k \le n-1), \tag{7}$$

$$q_k \wedge (p_1 + \cdots + p_{k-1}) = 0 \qquad (2 \le k \le n). \tag{8}$$

Now (4) and (7) together imply $\tau(q_k) = \tau(q_{k+1})$, and from (8) we have $\tau(q_1 \vee \cdots \vee q_k) = k\tau(q_1)$. Thus, we have pairwise equivalent and orthogonal projections $f_1, \ldots, f_n$ defined by

$$\begin{aligned} f_n &= q_n\,, \\ f_k &= (q_k \vee \cdots \vee q_n) - (q_{k+1} \vee \cdots \vee q_n) \qquad (1 \le k \le n-1), \end{aligned}$$

T commutes with $f := f_1 + \cdots + f_n$ and Tf is strictly upper triangular when written as an $n \times n$ matrix with respect to $f_1, \ldots, f_n$. Moreover, we have $(T(1-f))^{n-1} = T^{n-1}(1-f) = 0$ and the induction hypothesis applies to $T(1-f)$. □

Proposition 3.2. *Let $\mathcal{M}$ be a II_1–factor. Then* $\operatorname{Comm}(\mathcal{M})$ *contains all nilpotent elements of $\mathcal{M}$.*

Proof. By Lemma 3.1, we only need to observe that a strictly upper triangular matrix in $M_n(\mathfrak{A})$ is a single commutator, for any algebra $\mathfrak{A}$. But this is easy: if

$$A = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 0 & a_{2,3} & \cdots & a_{2,n} \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & a_{n-1,n} \\ 0 & & \cdots & & 0 \end{pmatrix},$$

then $A = BC - CB$, where B is the matrix in (1),

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & c_{2,2} & \cdots & c_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & c_{n,n} \end{pmatrix}, \tag{9}$$

and where the $c_{i,j}$ are chosen so that

$$\begin{aligned} a_{1,j} &= c_{2,j} \qquad (2 \leq j \leq n), \\ a_{p,j} &= c_{p+1,j} - c_{p,j-1} \qquad (2 \leq p < j \leq n). \end{aligned}$$

□

4. Examples and questions

Example 4.1. A particular case of Theorem 2.5 is that if p is a projection (with irrational trace) in any II_1–factor $\mathcal{M}$, then $p - \tau(p)1 \in \mathrm{Comm}(\mathcal{M})$. We note that a projection with rational trace is contained in some unital matrix subalgebra $M_n(\mathbf{C}) \subseteq \mathcal{M}$; therefore, the case of a projection with rational trace is an immediate application of Shoda's result.

Question 4.2. In light of Theorem 2.5, it is natural to ask: does $\mathrm{Comm}(\mathcal{M})$ contain all normal elements of $\mathcal{M}$ whose trace is zero? (Note that each such element is the limit in norm of a sequence of elements of the sort considered in Theorem 2.5.) It is of particular interest to focus on normal elements that generate maximal selfadjoint abelian subalgebras (masas) in $\mathcal{M}$. Does it make a difference whether the masa is singular or semi-regular? (See [25].)

A particular case:

Question 4.3. If a and b freely generate the group $\mathbf{F}_2$, let λ_a and λ_b be the corresponding unitaries generating the group von Neumann algebra $L(\mathbf{F}_2)$. Do we have $\lambda_a \in \mathrm{Comm}(L(\mathbf{F}_2))$?

Our next examples come from ergodic theory.

Example 4.4. Let α be an ergodic, probability measure–preserving transformation of a standard Borel probability space X that is not weakly mixing. Consider the hyperfinite II_1–factor R realized as the crossed product $R = L^\infty(X) \rtimes_{\tilde{\alpha}} \mathbf{Z}$ where $\tilde{\alpha}$ is the automorphism of $L^\infty(X)$ arising from α by $\tilde{\alpha}(f) = f \circ \alpha$. For $f \in L^\infty([0,1])$, we let $\pi(f)$ denote the corresponding element of R, and we write $U \in R$ for the implementing unitary, so that $U\pi(f)U^* = \pi(\tilde{\alpha}(f))$. By a standard result in ergodic theory (see, for example, Theorem 2.6.1 of [23]), there is an eigenfunction, i.e., $h \in L^\infty(X)\backslash\{0\}$ so that $\tilde{\alpha}(h) = \zeta h$ for some $\zeta \neq 1$; moreover, all eigenfunctions h of an ergodic transformation must have $|h|$ constant. If $g \in L^\infty(X)$, then

$$[U\pi(g), \pi(h)] = U\pi\big(g\big(h - \tilde{\alpha}^{-1}(h)\big)\big).$$

Since $h - \tilde{\alpha}^{-1}(h)$ is invertible, by making appropriate choices of g we get $U\pi(f) = [U\pi(g), \pi(h)] \in \mathrm{Comm}(R)$ for all $f \in L^\infty(X)$.

Question 4.5. If α is a weakly mixing transformation of X (for example, a Bernoulli shift), then, with the notation of Example 4.4, do we have $U\pi(f) \in \mathrm{Comm}(R)$ for all $f \in L^\infty(X)$?

Example 4.6. Assume that $\tilde{\alpha}$ from Example 4.4 has infinitely many distinct eigenvalues. This is the case for every compact ergodic action α (for example, an irrational rotation of the circle or the odometer action) but can also hold for a noncompact action (for example, a skew rotation of the torus). For every finite set $F \subset \mathbf{Z} \setminus \{0\}$, there is an eigenvalue ζ such that $\zeta^k \neq 1$, for any $k \in F$. Let h be an eigenfunction of $\tilde{\alpha}$ corresponding to this eigenvalue ζ; clearly, $|h|$ is a constant. Then, for $g_k \in L^\infty(X)$,

$$\left[\sum_{k\in F} U^k\pi(g_k), \pi(h)\right] = \sum_{k\in F} \left[U^k\pi(g_k), \pi(h)\right] = \sum_{k\in F} U^k\pi\big(g_k\big(h - \tilde{\alpha}^{-k}(h)\big)\big).$$

Thus, for any $f_k \in L^\infty(X)$, by choosing $g_k = f_k\big(h - \tilde{\alpha}^{-k}(h)\big)^{-1}$, we obtain

$$\sum_{k\in F} U^k\pi(f_k) \in \mathrm{Comm}(R).$$

Question 4.7. It is natural to ask Question 1.1 in the particular case of quasinilpotent elements T of $\mathcal{M}$: must they lie in $\mathrm{Comm}(\mathcal{M})$? From Proposition 4 of [21], it follows that every quasinilpotent operator T in a II_1–factor has trace zero. (Alternatively, use L. Brown's analogue [8] of Lidskii's theorem in II_1–factors and the fact that the Brown measure of T must be concentrated at 0.)

Question 4.8. Consider the quasinilpotent DT–operator T (see [13]), which is a generator of the free group factor $L(\mathbf{F}_2)$. Do we have $T \in \mathrm{Comm}(L(\mathbf{F}_2))$?

Example 4.9. Consider G. Tucci's quasinilpotent operator

$$A = \sum_{n=1}^{\infty} a_n V_n \in R$$

from [27], where $a = (a_n)_{n=1}^\infty \in \ell^1_+$, the set of summable sequences of nonnegative numbers. Here $R = \overline{\bigotimes_1^\infty M_2(\mathbf{C})}$ is the hyperfinite II_1–factor and

$$V_n = I^{\otimes n-1} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \otimes I \otimes I \otimes \cdots . \tag{10}$$

Tucci showed in Remark 3.7 (p. 2978) of [27] that A is a single commutator whenever $a = (b_n c_n)_{n=1}^\infty$ for some $b = (b_n)_{n=1}^\infty \in \ell^1$ and $c = (c_n)_{n=1}^\infty \in \ell^1$, by writing $A = [B, C]$, where

$$B = \sum_{n=1}^{\infty} b_n V_n V_n^*, \tag{11}$$

$$C = \sum_{n=1}^{\infty} c_n V_n\,. \tag{12}$$

Note that, for $a \in \ell^1_+$, there exist b and c in ℓ^1 such that $a = (b_n c_n)_{n=1}^\infty$ if and only if $\sum_{n=1}^\infty a_n^{1/2} < \infty$, i.e., if and only if $a \in \ell^{1/2}_+$.

The rest of the paper is concerned with some further results and remarks about Tucci's operators.

We might try to extend the formula $A = [B, C]$ for B and C as in (11) and (12), respectively, to other sequences $a \in \ell^1_+$, i.e., for b and c not necessarily in ℓ^1, and where the convergence in (11) and (12) might be in some weaker topology.

We first turn our attention to (12). Denoting the usual embedding $R \hookrightarrow L^2(R,\tau)$ by $X \mapsto \widehat{X}$, from (10) we see that the vectors $\widehat{V}_n$ are orthogonal and all have $L^2(R,\tau)$-norm equal to $1/\sqrt{2}$; therefore, the series (12) converges in $L^2(R,\tau)$ as soon as $c \in \ell^2$, and we have

$$\widehat{C} = \sum_{n=1}^{\infty} c_n \widehat{V}_n. \tag{13}$$

We easily see (below) that only for $c \in \ell^1$ there is a bounded operator $C \in R$ such that $\widehat{C}$ is given by (13).

Proposition 4.10. *Let $c \in \ell^2$. Suppose there is a bounded operator $C \in R$ such that $\widehat{C}$ is given by* (13)*. Then $c \in \ell^1$.*

Proof. For any sequence $(\zeta_n)_{n=1}^{\infty}$ of complex numbers of modulus 1, there is an automorphism of R sending V_n to $\zeta_n V_n$ for all n. Thus, without loss of generality we may assume $c_n \geq 0$ for all n.

Letting $E_n : R \to M_2(\mathbf{C})^{\otimes n} \otimes I \otimes I \otimes \cdots \cong M_{2^n}(\mathbf{C})$ be the conditional expectation onto the tensor product of the first n copies of the 2×2 matrices (see Example 4.9), we must have $C_n := E_n(C) = \sum_{k=1}^{n} c_k V_k \in M_{2^n}(\mathbf{C})$. Let $x = 2^{-n/2}(1,1,\ldots,1)^t$ be the normalization of the column vector of length 2^n with all entries equal to 1. Taking the usual inner product in $\mathbf{C}^{2^n}$, we see $\langle V_k x, x\rangle = 1/2$ for all $k \in \{1,\ldots,n\}$. Thus,

$$\frac{1}{2}\sum_{k=1}^{n} c_k = \big|\, \langle C_n x, x\rangle \,\big| \leq \|C_n\| \leq \|C\|.$$

This shows $c \in \ell^1$. □

Let us now investigate the series (11) for some sequence $b = (b_n)_{n=1}^{\infty}$ of complex numbers. We claim that this series gives rise (in a weak sense explained below) to a bounded operator if and only if $b \in \ell^1$. Indeed, for K a finite subset of $\mathbf{N}$, we have

$$\left\| \sum_{n\in K} b_n V_n V_n^* \right\|_{L^2(R,\tau)}^2 = \frac{1}{4}\sum_{n\in K} |b_n|^2 + \frac{1}{4}\left| \sum_{n\in K} b_n \right|^2.$$

Now suppose $K_1 \subseteq K_2 \subseteq \cdots$ are finite sets whose union is all of $\mathbf{N}$. Then $\sum_{n\in K_p} b_n V_n V_n^*$ converges in $L^2(R,\tau)$ as $p \to \infty$ if and only if $b \in \ell^2$ and $y := \lim_{p\to\infty} \sum_{n\in K_p} b_n$ exists. Then the limit in $L^2(R,\tau)$ is

$$\widehat{B} = \sum_{n=1}^{\infty} b_n \left(V_n V_n^* - \frac{1}{2} \right)\widehat{\ } + \frac{y}{2}\hat{1}. \tag{14}$$

If there is a bounded operator B such that $\widehat{B}$ is given by (14), then for every finite $F \subseteq \mathbf{N}$, the conditional expectation $E_F(B)$ of B onto the (finite dimensional) subalgebra of R generated by $\{V_n V_n^* \mid n \in F\}$ will be $\sum_{n\in F} b_n (V_n V_n^* - \frac{1}{2}) + \frac{y}{2}$. Taking the projection $P = \prod_{n\in F} V_n V_n^*$, we have $E_F(B)P = \frac{1}{2}(y + \sum_{n\in F} b_n)P$, so

$$\left| \frac{1}{2}\left(y + \sum_{n\in F} b_n \right) \right| \leq \|E_F(B)\| \leq \|B\|.$$

As F was arbitrary, this implies $b \in \ell^1$.

Suppose $b_n c_n = \frac{1}{n^r}$ and $b = (b_n)_1^\infty \in \ell^1$. Letting $(b_n^*)_1^\infty$ denote the nonincreasing rearrangement of $(|b_n|)_1^\infty$, we have $b_n^* = o(\frac{1}{n})$ and standard arguments show $c_n^* \geq \frac{K}{n^{r-1}}$ for some constant K. Thus, by Proposition 4.10, Tucci's formula for writing $A = [B, C]$ does not work if $a_n = \frac{1}{n^r}$ for $1 < r \leq 2$, while of course for $r > 2$ it works just fine.

Question 4.11. Fix $1 < r \leq 2$, and let

$$A = \sum_{n=1}^{\infty} \frac{1}{n^r} V_n \in R$$

be Tucci's quasinilpotent operator in the hyperfinite II_1–factor. Do we have $A \in \mathrm{Comm}(R)$?

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