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ON AN INCLUSION OF THE ESSENTIAL SPECTRUM OF LAPLACIANS UNDER NON-COMPACT CHANGE OF METRIC

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Abstract. The stability of essential self-adjointness and an inclusion of the essential spectra of Laplacians under the change of a Riemannian metric on a subset K of M are proved. The set K may have infinite volume measured with the new metric, and its completion may contain a singular set such as the fractal set, to which the metric is not extendable.

1. Introduction

Let (M, g) be a connected smooth Riemannian manifold without boundary. The Laplacian Δ of g is called essentially self-adjoint if it has the unique self-adjoint extension $\overline{\Delta}$. In [4], Furutani showed that if Δ with the domain $C_0^\infty(M)$ is essentially self-adjoint and if g is changed on a compact set $K \subset M$ to another smooth metric g' on M, then the Laplacian Δ' of g' with the domain $C_0^\infty(M)$ is essentially self-adjoint; and the essential spectra are stable under this change. In particular, the second result forms a strong contrast to the behavior of the eigenvalues, since eigenvalues change continuously with the perturbation of the metric in a certain way (see e.g. [1]).

Needless to say, there are many important Riemannian manifolds with singularity, by which we mean that g does not extend to the Cauchy boundary (the difference between the completion of M and M), such as algebraic varieties, cone manifolds, edge manifolds, Riemannian orbifolds. In general, the analysis on such a singular space is complicated, and one of the methods to overcome the difficulties is to modify the singularity to a simpler one by the perturbation of the Riemannian metric. The crucial steps in this process is to study the stability of the essential self-adjointness of the Laplacian and to understand the behavior of its spectral structure under the perturbation.

Motivated by these facts, we extend Furutani's theorem to more general K so that K is not compact and its completion $\overline{K}$ includes the singular set. In this

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setting, the natural domain $D(\Delta)$ for the Laplacian is the following:

$$\begin{cases} D(\nabla) = \{u \in C^\infty \cap L^2 : \nabla u \in L^2\}, \\ D(\mathrm{div}) = \{X \in C^\infty \cap L^2 : \mathrm{div} X \in L^2\}, \\ D(\Delta) = \{u \in D(\nabla) : \nabla u \in D(\mathrm{div})\}. \end{cases} \tag{1}$$

(We suppress M and the Riemann measure $d\mu_g$ for the sake of simplicity.) Indeed, if the Cauchy boundary $\partial_C M$ is almost polar, namely,

$$\mathrm{Cap}(\partial_C M) = 0$$

(see Section 2 for the definition and see also e.g. [3]), then M has negligible boundary [9], and by the Gaffney theorem [6], Δ is essentially self-adjoint. Throughout the article, we assume that the Laplacians have the domain defined in (1). The following is our main result:

Theorem 1. *Let g and g' be Riemannian metrics on M such that $g = g'$ outside a subset K of M. If Δ is essentially self-adjoint in L^2 and the Cauchy boundary of K with respect to g' is almost polar, then Δ' is essentially self-adjoint in $L^2(M; d\mu_{g'})$. Additionally, if there is a function χ on M satisfying*

$$\nabla\chi \in L^\infty,\ \Delta\chi \in L^\infty,\ \text{and } \chi|_K = 1, \tag{2}$$

where ∇ is the gradient of g, and the inclusion

$$H^1_0(N;\, d\mu_g) \subset L^2(N;\, d\mu_g) \text{ is compact} \tag{3}$$

for some $N \supset N(supp(\chi); \epsilon)$ with some $\epsilon > 0$, where $N(supp(\chi); \epsilon)$ is the ϵ-neighborhood of the support of χ, then

$$\sigma_{\mathrm{ess}}(\overline{\Delta}) \subset \sigma_{\mathrm{ess}}(\overline{\Delta'}).$$

A special case of Theorem 1 is

Corollary 1 (Furutani's stability result [4])**.** *Let g and g' be Riemannian metrics on M such that $g = g'$ outside a compact subset K of M. If Δ is essentially self-adjoint in L^2, then Δ' is essentially self-adjoint in $L^2(M; d\mu_{g'})$, and*

$$\sigma_{\mathrm{ess}}(\overline{\Delta}) = \sigma_{\mathrm{ess}}(\overline{\Delta'}). \tag{4}$$

A typical example of manifolds which satisfies the condition of Theorem 1 is given as follows:

Corollary 2 (see Section 3)**.** *Let M be a complete manifold and let $\Sigma \subset M$ be an almost polar compact subset. If Σ is almost polar with respect to a metric g' on $M \setminus \Sigma$ and $g = g'$ outside a compact set $K \subset M$, then the same conclusion in the theorem holds true.*

We may apply Theorem 1 for singular manifolds: we change g to g' on a bounded set $K \supset \partial_C M$ so that g' can be extended to the almost polar Cauchy boundary with respect to g and conclude that Δ is essentially self-adjoint in $L^2(M; d\mu_g)$ and $\sigma_{\mathrm{ess}}(\overline{\Delta}) \subset \sigma_{\mathrm{ess}}(\overline{\Delta'})$.

A sufficient condition for $\partial_C M$ to be almost polar is that it has Minkowski co-dimension greater than 2 [7] (if the metric of g extends to $\partial_C M$ and $\partial_C M$ is a manifold, then it is almost polar if $\partial_C M$ has co-dimension 2).

The idea for proving the inclusion of the essential spectrum in Theorem 1 is to apply Weyl's criteria: a number λ belongs to $\sigma_{ess}(\overline{\Delta})$ if and only if there is a

sequence ϕ_n of "limit-eigenfunctions" of $\overline{\Delta}$ corresponding to λ (see Proposition 2 for details). Indeed, we show that if χ satisfies (2) and (3), then there is a subsequence $\phi_{n(k)}$ such that $(1-\chi)\phi_{n(k)}$ is a limit-eigenfunction of $\overline{\Delta}'$.

Our results differ from Furutani's original results on the following two points. In order to explain those differences, let us employ an example. Let

$$M = K \cup B(1),$$

where $K = \{(x,y,z) \in S^2 : z \geq 0\} \setminus (0,0,1)$ and $B(r) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq r^2,\ z=0\}$. Namely, M is S^2 with flat bottom with deleted north point. (To be more precise, we need to smooth the intersection of K and $B(1)$ so that M is a smooth Riemannian manifold.) Since the Cauchy boundary of M is the north point and it has null capacity, the Laplacian Δ is essentially self-adjoint. We modify M by the stereographic projection so that (M, g') is the 2-dimensional Euclidean space $\mathbb{R}^2$. Since (M, g') is complete, the Cauchy boundary is empty, and the Laplacian Δ' is essentially self-adjoint. Next, we find the function χ which satisfies condition (2) as follows:

$$\chi \in C_0^\infty(M \setminus B(1/3)) \text{ and } \chi = 1 \text{ on } K.$$

By letting $\epsilon = 1/3$ and $N = M \setminus B(1/4)$, condition (3) is satisfied. Indeed, since the north point has null capacity, the spectrum of the Laplacian on M consists only of the eigenvalues with finite multiplicity, whereas Δ' has only essential spectrum. This proves that the inclusion in Theorem 1 holds. This example also shows that the assumptions in Corollary 1 are sharp in the sense that we may not drop the assumption that K must be compact in order to obtain (4), that is, Furutani's stability result. Indeed, if we modify the metric of $\mathbb{R}^2$ to obtain M, then there is no subset N of $\mathbb{R}^2$ which satisfies condition (3).

The second difference is that the essential self-adjointness of Δ does not need to imply that of Δ restricted to $C_0^\infty(M)$; for instance, Δ on M is essentially self-adjoint, but Δ restricted to $C_0^\infty(M)$ has infinitely many self-adjoint extensions. In particular, it is not essentially self-adjoint (see e.g. [2]).

We organize the article in the following manner: in Section 2, we prove Theorem 1, and in Section 3, we present the examples.

2. Proofs

In this section we recall some definitions and prove Theorem 1 and Corollary 1. For the sake of simplicity, we often suppress the symbols M and $d\mu_g$.

We denote by $(\cdot,\cdot)$ and $(\cdot,\cdot)_1$ the inner product in L^2 and the Sobolev space H^1 of order $(1,2)$, respectively. H_0^1 is the completion of the set C_0^∞ of smooth functions with compact support with respect to the norm $\|\cdot\|_1 = \sqrt{(\cdot,\cdot)_1}$. Let $\mathcal{O}$ be the family of all open subsets of $\overline{M}$. For $A \in \mathcal{O}$ we define $\mathcal{L}_A = \{u \in H^1 : u \geq 1\ \mu_g\text{-a.e. on } M \cap A\}$,

$$\mathrm{Cap}(A) = \begin{cases} \inf_{u \in \mathcal{L}_A} \|u\|_1, & \mathcal{L}_A \neq \phi, \\ \infty, & \mathcal{L}_A = \phi, \end{cases}$$

and

$$\mathrm{Cap}(\partial_C M) = \inf_{A \in \mathcal{O},\ \partial_C M \subset A} \mathrm{Cap}(A).$$

We will use

Proposition 1 (Lemma 2.1.1 of [3]). *If $\mathcal{L}_A \neq \phi$ for $A \in \mathcal{O}$, there exists a unique element $e_A \in \mathcal{L}_A$ called the* equilibrium potential *of A such that the following hold:*
(*i*) $\|e_A\|_1^2 = Cap(A)$.
(*ii*) $0 \leq e_A \leq 1$ μ_g*-a.e. and* $e_A = 1$ μ_g*-a.e. on* $A \cap M$.
(*iii*) *If* $A, B \in \mathcal{O}$, $A \subset B$, *then* $e_A \leq e_B$ μ_g*-a.e.*

We prove Theorem 1. We start from:

Proof of the essential self-adjointness of Δ'. For arbitrary $u \in H^1(M; d\mu_{g'})$, we have to find $\hat{u}_n \in H^1(M; d\mu_{g'})$ which converges to u in $H^1(M; d\mu_{g'})$. Indeed, this implies that (M, g') has negligible boundary, and hence Δ' is essentially self-adjoint by Gaffney's theorem [6].

Since $L^\infty(M) \cap H^1(M; d\mu_{g'})$ is dense in $H^1(M; d\mu_{g'})$, we may assume that $u \in L^\infty(M)$ without loss of generality. Let

$$\psi := (1 - r)_+, \tag{5}$$

where r is the distance from K. The function $\psi \in L^\infty(M)$ enjoys the properties

$$\psi|_K = 1 \quad \text{and} \quad \|\nabla\psi\|_{L^\infty} \leq 1.$$

Since

$$|(1 - \psi)u(x)| \leq (1 + \|\psi\|_{L^\infty})|u(x)|$$

and

$$|\nabla((1 - \psi)u(x))| \leq (1 + \|\psi\|_{L^\infty})|\nabla u(x)| + |u(x)|,$$

for almost every $x \in M$, it follows that $(1 - \psi)u \in H^1$. Recalling that the essential self-adjointness of Δ implies $H_0^1 = H^1$ [9], we find $v_n \in C_0^\infty(M \setminus K)$ such that

$$v_n \to (1 - \psi)u \text{ as } n \to \infty$$

in $H^1(M \setminus K)$. Because the Cauchy boundary $\partial_C M$ of M associated to g' is almost polar, there is a sequence of the equilibrium potentials e_n of $O_n \supset \partial_C M$ such that $\bigcap_{n>1} O_n = \partial_C M$ and

$$\|e_n\|_{H^1(M; d\mu_{g'})} \to 0 \text{ as } n \to \infty.$$

Then $u_n := (1 - e_n)\psi u \in H_0^1(M; d\mu_{g'})$ satisfies

$$u_n \to \psi u \text{ as } n \to \infty$$

in $H^1(M; d\mu_{g'})$, and we get $\hat{u}_n = u_n + v_n \in H_0^1(M; d\mu_{g'})$ such that

$$\hat{u}_n \to (1 - \psi)u + \psi u = u \text{ as } n \to \infty$$

in $H^1(M; d\mu_{g'})$. □

Next, we prove the inclusion of the essential spectrum and complete the proof of Theorem 1. We use the following characterization:

Proposition 2 (Weyl's criterion). *A number λ belongs to the essential spectrum of Δ if and only if there is a sequence of orthonormal vectors $\{\phi_n\}$ of L^2 such that*

$$\|(\Delta - \lambda)\phi_n\|_{L^2} \to 0 \text{ as } n \to \infty.$$

Proof of the inclusion of the essential spectrum. We assume (2) and (3) to prove the inclusion of the essential spectrum. Hereafter, we denote $\Delta = \overline{\Delta}$ and $\Delta' = \overline{\Delta'}$ because of their essential self-adjointness. Let $\lambda \in \sigma_{\mathrm{ess}}(\Delta)$ and $\phi_n \in D(\Delta)$ such that

$$(\phi_i, \phi_j) = \delta_{ij},$$
$$\|(\Delta - \lambda)\phi_n\| \to 0 \text{ as } n \to \infty,$$

where $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. Let χ be the function satisfying (2) and let ϕ be the function defined as

$$\phi = (1 - \hat{r}/\epsilon)_+,$$

where $\hat{r}$ is the distance from the support of χ. Clearly, we have

$$\sup_{n>0} \|\phi\phi_n\| < \infty. \tag{6}$$

Moreover, taking into account that $\phi_n \in D(\Delta)$ implies

$$\|\nabla\phi_n\|^2 = -(\phi_n, \Delta\phi_n),$$

it follows that

$$\begin{aligned}\limsup_{n\to\infty} \|\nabla(\phi\phi_n)\| &\le \|\nabla\phi\|_{L^\infty} + \limsup_{n\to\infty} \|\nabla\phi_n\| \\ &\le \|\nabla\phi\|_{L^\infty} + \limsup_{n\to\infty} (\|\Delta\phi_n\|\|\phi_n\|)^{1/2} < \infty.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \|\phi\phi_n\|_1 < \infty. \tag{7}$$

Now, specifying $\epsilon > 0$ as in the statement, by (7) and the fact that the embedding $H_0^1(N) \subset L^2(N)$ is compact, there exists a subsequence $\phi_{n(k)}$ of ϕ_n and $\phi' \in L^2$ such that

$$\phi\phi_{n(k)} \to \phi' \text{ strongly in } L^2 \text{ as } k \to \infty.$$

However, if $f \in L^2$, then $f\phi \in L^2$ and

$$(f, \phi\phi_n) = (f\phi, \phi_n) \to 0 \text{ as } n \to \infty;$$

hence, $\phi\phi_{n(k)} \to 0$ weakly in L^2 as $k \to \infty$. Because of the uniqueness of the weak-limits, it follows that $\phi' = 0$, and we may assume

$$\|\phi\phi_n\| \to 0 \text{ as } n \to \infty \tag{8}$$

without loss of generality. Since $\phi = 1$ on $\mathrm{supp}(\chi)$,

$$\begin{aligned}&\|(\Delta - \lambda)(\chi\phi_n)\| \\ &\le \|(\Delta\chi)\phi\phi_n\| + 2\|(\nabla\chi, \nabla\phi_n)\| + \|\chi(\Delta - \lambda)\phi_n\| \\ &\le \|\Delta\chi\|_{L^\infty}\|\phi\phi_n\| + 2\|\nabla\chi\|_{L^\infty}\|\nabla\phi_n\|_{L^2(\mathrm{supp}(\chi))} + \|\chi\|_{L^\infty}\|(\Delta - \lambda)\phi_n\|.\end{aligned}$$

The first and third terms in the last line tend to 0 as $n \to \infty$ because of (8) and (6). The second term can be estimated as

$$\|\nabla\chi\|_{L^\infty}\|\nabla(\phi\phi_n)\| \le \|\nabla\chi\|_{L^\infty}\sqrt{\|\Delta\phi_n\|\|\phi\phi_n\|} \to 0 \text{ as } n \to \infty.$$

Thus, since $1 - \chi = 0$ and $g = g'$ on $M \setminus K$,

$$\|(\Delta' - \lambda)[(1-\chi)\phi_n]\|_{L^2(M;d\mu_{g'})} = \|(\Delta - \lambda)[(1-\chi)\phi_n]\| \to 0 \text{ as } n \to \infty.$$

On the other hand,

$$\|(1-\chi)\phi_n\|_{L^2(M;d\mu_{g'})} \geq |\|\phi_n\| - \|\chi\phi_n\|| \geq |1 - \|\phi\phi_n\|| \to 1 \text{ as } n \to \infty,$$

and we conclude that $\lambda \in \sigma_{\mathrm{ess}}(\Delta')$ by Weyl's criterion. □

Finally, we assume that K is compact to prove the essential self-adjointness of the Laplacian and the stability of the essential spectrum, namely, Corollary 1.

Proof of Corollary 1. We will show that the Laplacian Δ' is essentially self-adjoint and that there exist the function χ and the subset N of M which satisfy conditions (2) and (3) for each metric g and g'. This will imply

$$\sigma_{\mathrm{ess}}(\overline{\Delta}) = \sigma_{\mathrm{ess}}(\overline{\Delta'})$$

by Theorem 1.

Recall that if K is compact, then its Cauchy boundary is empty so that the Laplacian Δ' is essentially self-adjoint.

Since K is compact and g is smooth, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the metric g and its higher order (up to 2nd) derivatives are bounded on $N = N(K; 2\epsilon) = \{x \in M : d(x, K) < 2\epsilon\}$. Let

$$\hat{\chi}(x) = (1 \wedge (2 - 3\tilde{r}(x)/\epsilon))_+,$$

where $\tilde{r}$ is the distance from K. The function $\hat{\chi}$ is 1 on $N(K;\, \epsilon/3)$ and has support in $N(K;\, 2\epsilon/3)$, and it satisfies $\|\nabla\hat{\chi}\|_{L^\infty} \leq 3/\epsilon$. However, since $\hat{\chi}$ does not need to be in the Sobolev space H^2 of order $(2,2)$, we apply the Friedrichs mollifier j with radius $\delta > 0$ for $\hat{\chi}$ to find the smooth function $\chi = j * \hat{\chi}$. If $\delta < \epsilon/3$, then χ satisfies

$$\begin{cases} \chi(x) = 1 \text{ for } x \in K, \\ \mathrm{supp}(\chi) \subset N(K;\, 2\epsilon/3), \\ \|\nabla\chi\|_{L^\infty} \leq 3/\epsilon, \\ \Delta\chi \in L^\infty, \end{cases}$$

namely, condition (2). On the other hand, since K is compact, N_ϵ and N are relatively compact in M with sufficiently small $\epsilon > 0$. Hence, N has finite volume and finite diameter, and the Poincaré inequality holds on N. It follows that the embedding $H^1_0(N;\, d\mu_g) \subset L^2(N;\, d\mu_g)$ is compact, that is, condition (3). We obtain the inclusion: $\sigma_{\mathrm{ess}}(\overline{\Delta}) \subset \sigma_{\mathrm{ess}}(\overline{\Delta'})$.

This argumentation holds true if we replace g by g', and we arrive at the conclusion. □

3. Examples

In this section, we present examples of manifolds for which Theorem 1 can be applied.

Example 1 (see [8])**.** Let (M, g) be an m-dimensional complete Riemannian manifold and let $\Sigma \subset M$ be an n-dimensional compact manifold with $m \geq n+2$. Assume that M has the product structure $M^{m-n} \times M^n$ near Σ and that g can be diagonalized. Choose local coordinates in a neighborhood K of Σ so that

$$g = g_1 \oplus g_2$$

in K, where g_1 is a metric on M^{m-n} and g_2 is a metric on M^n. Let g' be another smooth metric on $M \setminus \Sigma$ so that

$$g' = \begin{cases} f^2 g_1 \oplus g_2, & \text{on } K, \\ g, & \text{on } M \setminus K. \end{cases}$$

If $m = 2$, assume that $f \in L^{2+\epsilon}(K; d\mu_g)$ for some $\epsilon \in (0, \infty)$.

If $m = 3$, assume that $\inf(f) > 0$ and $f \in L^{(m(m-2)/2)+\epsilon}(K; d\mu_g)$ for some $\epsilon \in (0, \infty)$.

Then the manifold $M \setminus \Sigma$ with metrics g and g' satisfies the assumption of Theorem 1. In particular, if M is compact, Δ' on $(M \setminus \Sigma, g')$ has discrete spectrum, which satisfies the Weyl asymptotic formula [8].

In the next example, the manifold has fractal singularity.

Example 2. Let (M, g) be a complete Riemannian manifold with dimension greater than 2. Let $\Sigma \subset M$ be the Cantor set, r the distance in M from Σ, and $B(R)$ the R-neighborhood of Σ. Set

$$g' = f^2 g,$$

where

$$f(x) = \begin{cases} r^\epsilon, & x \in B = B(1), \\ 1, & x \in M \setminus B. \end{cases}$$

It is shown in [9] that Σ is the almost polar Cauchy boundary of $(M \setminus \Sigma; g')$ if

$$\epsilon > \frac{\ln 2 - \ln 3}{2 \ln 3 - \ln 2}.$$

The compact inclusion

$$H_0^1(B \setminus \Sigma) \subset L^2(B \setminus \Sigma)$$

can be seen as follows. By definition, $H_0^1(B \setminus \Sigma) \subset H_0^1(B)$, and the inclusion $H_0^1(B) \subset L^2(B) = L^2(B \setminus \Sigma)$ is compact; thus, it suffices to show

$$H_0^1(B \setminus \Sigma) \supset H_0^1(B).$$

Let $u \in H_0^1(B)$. Since $L^\infty \cap H_0^1(B) \subset H_0^1(B)$ is dense, we may assume that $u \in L^\infty$ without loss of generality. Let e_n be the equilibrium potential as in the proof of Theorem 1. Then $u_n = u(1 - e_n) \in H_0^1(B \setminus \Sigma)$ and

$$u_n \to u \text{ in } H_0^1(B; d\mu_g),$$

and hence $u \in H_0^1(B \setminus \Sigma)$. The function χ can be found as the relative equilibrium potential of $B(1)$ and $B(2)$ applied to the Friedrichs mollifier. Therefore, $M \setminus \Sigma$ together with g and g' satisfies the condition of Theorem 1. Let us point out the following:

- $(M \setminus \Sigma, g')$ is $C^{1,1}$ and is not smooth, but Theorem 1 can be applied to this setting.
- We can show the compact embedding $H_0^1(B \setminus \Sigma; d\mu_{g'}) \subset L^2(B; \setminus \Sigma; d\mu_{g'})$ only for $\epsilon \geq 0$.

In the next example, K has infinite volume with g'.

Example 3. Let M be a 2-dimensional complete Riemannian manifold. Delete a point $p \in M$ and set

$$g' = f^2 g,$$

where

$$f(x) = \begin{cases} r^{-\epsilon}, & x \in B = B(1), \\ 1, & x \in M \setminus B, \end{cases}$$

and r is the distance from p. For any $\epsilon \geq 1$, $(M \setminus \{p\}, g')$ is complete and $\mu_{g'}(B \setminus \{p\}) = \infty$.

More generally, if M is a complete manifold and $\Sigma \subset M$ is a compact set, then there is a smooth Riemannian metric g' on $M \setminus \Sigma$ and a compact set $K \subset M$ such that $g = g'$ on $M \setminus K$, $(M \setminus \Sigma; g')$ is complete and there exist a function χ and a subset N of $M \setminus \Sigma$ satisfying conditions (2) and (3), respectively.

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References

[1] Bando, S.; and Urakawa, H., Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds. Tohoku Math. J. (2) **35** (1983), no. 2, 155-172. MR699924 (84h:58146)

[2] Colin de Verdière, Yves, Pseudo-laplaciens. I. Ann. Inst. Fourier (Grenoble) **32** (1982), no. 3, xiii, 275-286. MR688031 (84k:58221)

[3] Fukushima, M; Oshima, Y.; Takeda, M., Dirichlet forms and symmetric Markov processes. de Gruyter Studies in Mathematics, 19. Walter de Gruyter & Co., Berlin, 1994. MR1303354 (96f:60126)

[4] Furutani, K., On a stability of essential spectra of Laplace operators on non-compact Riemannian manifolds. Proc. Japan Acad. **56**, Ser. A (1980), 425-428. MR603058 (82c:58055)

[5] Gaffney, M.P., A special Stokes's theorem for complete Riemannian manifolds. Ann. of Math. (2) **60** (1954), 140-145. MR0062490 (15:986d)

[6] Gaffney, M.P., Hilbert space methods in the theory of harmonic integrals. Trans. Amer. Math. Soc. **78** (1955), 426-444. MR0068888 (16:957a)

[7] Masamune, J., Essential self-adjointness of Laplacians on Riemannian manifolds with fractal boundary. Comm. Partial Differential Equations **24** (1999), no. 3-4, 749-757. MR1683058 (2000m:58035)

[8] Masamune, J.; Rossman, W., Discrete spectrum and Weyl's asymptotic formula for incomplete manifolds. Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999), 219-229, Adv. Stud. Pure Math., 34, Math. Soc. Japan, Tokyo, 2002. MR1925741 (2004b:58052)

[9] Masamune, J., Analysis of the Laplacian of an incomplete manifold with almost polar boundary. Rend. Mat. Appl. (7) **25** (2005), no. 1, 109-126. MR2142127 (2006a:58040)

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