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STAFNEY'S LEMMA HOLDS FOR SEVERAL "CLASSICAL" INTERPOLATION METHODS

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Abstract. Let (B_0, B_1) be a Banach pair. Stafney showed that one can replace the space $\mathcal{F}(B_0, B_1)$ by its dense subspace $\mathcal{G}(B_0, B_1)$ in the definition of the norm in the Calderón complex interpolation method on the strip if the element belongs to the intersection of the spaces B_i. We shall extend this result to a more general setting, which contains well-known interpolation methods: the Calderón complex interpolation method on the annulus, the Lions-Peetre real method (with different choices of norms), and the Peetre "$\pm$" method.

1. Introduction

Stafney showed in his paper [19] (Lemma 2.5, p. 335) that one can replace the space $\mathcal{F}(B_0, B_1)$ by its dense subspace $\mathcal{G}(B_0, B_1)$ in the definition of the norm of an element in the Calderón complex interpolation space $[B_0, B_1]_\theta$ if the element belongs to the intersection of the two Banach spaces. Applications of Stafney's lemma include obtaining properties of the structure space and of the spectrum of complex interpolated commutative Banach algebras ([19]), an apparently simpler proof of one part of Calderón's duality theorem for regular Banach couples ([5]), and alternative constructions of the Calderón complex interpolation method ([15, 12, 11]).

We shall obtain a version of Stafney's lemma in the general setting of pseudolattices, which, for appropriate selections of the parameters, will give us analogues of this lemma for the Calderón complex interpolation method on the annulus, for the "discrete definition" of the Lions-Peetre real method and for the Peetre "$\pm$" method. In the case of the Lions-Peetre method we can work either with the norm defined via the J-functional or with an earlier used variant of that norm introduced in [16].

For mutually closed couples of Banach spaces an estimate similar to Stafney's lemma for the Lions-Peetre method, where the norm estimate given is only to within equivalence of norms, follows from Appendix 3 on pp. 47–8 of [9] and also appears on p. 342 of [8]. See also the "Note added in proof" on p. 49 of [9], which announces (without explicit proof) that the condition of mutual closedness can be removed.

Stafney's lemma cannot be extended to all interpolation methods where there are natural analogues of the space $\mathcal{F}$ and its dense subspace $\mathcal{G}$. For example,

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in [3] an analogue of Calderón's complex interpolation space generated by an n-tuple of Banach spaces, rather than just a couple, is introduced. The example in Appendix 1 on pp. 223–6 of [3] shows that, perhaps surprisingly, for $n \geq 3$, the expected analogue of Stafney's lemma can fail to hold. In fact, in that setting the natural analogues of the two quantities which appear below in the formula (3.1) in Theorem 17 may even fail to be equivalent.

2. Preliminaries and examples

Before stating our main result, we need to provide a number of definitions and examples, most of which are from [7].

Definition 1. Let Ban be the class of all Banach spaces over the complex numbers. A mapping $\mathcal{X} : Ban \to Ban$ will be called a *pseudolattice* if

(i) for each $B \in Ban$ the space $\mathcal{X}(B)$ consists of B-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$,

(ii) whenever A is a closed subspace of B it follows that $\mathcal{X}(A)$ is a closed subspace of $\mathcal{X}(B)$, and

(iii) there exists a positive constant $C = C(\mathcal{X})$ such that, for all $A, B \in Ban$ and all bounded linear operators $T : A \to B$ and every sequence $\{a_n\}_{n\in\mathbb{Z}} \in \mathcal{X}(A)$, the sequence $\{Ta_n\}_{n\in\mathbb{Z}} \in \mathcal{X}(B)$ and satisfies the estimate

$$\left\|\{Ta_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}(B)} \leq C(\mathcal{X}) \|T\|_{A\to B} \left\|\{a_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}(A)} .$$

The following examples will be relevant for our applications.

Example 2. Let X be a Banach lattice of real-valued functions defined on $\mathbb{Z}$. We will use the notation $\mathcal{X} = X$ to mean that, for each $B \in Ban$, $\mathcal{X}(B)$ is the space, usually denoted by $X(B)$, consisting of all B-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ such that $\{\|b_n\|_B\}_{n\in\mathbb{Z}} \in X$. It is normed by $\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{X(B)} = \left\|\{\|b_n\|_B\}_{n\in\mathbb{Z}}\right\|_X$. In particular, we shall be interested in the choices $X = \ell^p$ for $p \in [1, \infty]$ and $X = c_0$.

Example 3. For each $B \in Ban$ let $FC(B)$ be the space of all B-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ such that $b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f\left(e^{it}\right) dt$ for all n and some continuous function $f : \mathbb{T} \to B$. Here $FC(B)$ is normed by $\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{FC(B)} = \sup_{t\in[0,2\pi)} \left\|f\left(e^{it}\right)\right\|_B$. The notation $\mathcal{X} = FC$ will mean that $\mathcal{X}(B) = FC(B)$ for each B.

Example 4. We shall use the notation $\mathcal{X} = UC$, when $\mathcal{X}(B) = UC(B)$ for every $B \in Ban$, where $UC(B)$ denotes the Banach space of all B-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} \lambda_n b_n$ converges for all complex sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ satisfying $|\lambda_n| \leq 1$ for each $n \in \mathbb{Z}$ (i.e. such that the sequence b_n is unconditionally convergent), and $UC(B)$ is normed by $\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{UC(B)} = \sup\left\{\left\|\sum_{n\in F} \lambda_n b_n\right\|_B\right\}$, where the supremum is taken over all finite subsets F of $\mathbb{Z}$ and all sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ which satisfy $|\lambda_n| \leq 1$ for all n (see [18, pp. 174-5]). (Note that, as was pointed out on p. 58 of [13], it suffices to consider sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ with $\lambda_n = \pm 1$, since this yields the same space to within equivalence of norms.)

Analogously, we shall use the notation $\mathcal{X} = WUC$, when $\mathcal{X}(B) = WUC(B)$ for every $B \in Ban$, where $WUC(B)$ denotes the space consisting of all B-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ for which the above norm $\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{UC(B)}$ is finite, but for which the unconditional convergence of the sequence b_n is not required. Such sequences are said to be weakly unconditionally convergent.

Let $\overline{B} = (B_0, B_1)$ be a Banach pair. Let $\mathcal{X}_0$ and $\mathcal{X}_1$ be any two pseudolattices. We consider them as a pair, which we denote by $\mathbf{X} = \{\mathcal{X}_0, \mathcal{X}_1\}$.

In this paper we use the Euler constant e in the definitions of $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ and $\mathbb{A}$. In fact the constant e can be replaced by any other positive constant greater than one in our constructions and the appropriate reformulation of the results in this paper will still hold. For simplicity, we shall use the same notation for spaces constructed using the same method with different choices of the constant.

Definition 5. For each Banach pair $\bar{B}$ and pseudolattice pair $\mathbf{X}$ we define $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ to be the space of all $(B_0 \cap B_1)$-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ for which the sequence $\left\{e^{jn} b_n\right\}_{n\in\mathbb{Z}}$ is in $\mathcal{X}_j\left(B_j\right)$ for $j = 0, 1$. This space is normed by

$$\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} = \max_{j=0,1} \left\|\left\{e^{jn} b_n\right\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}_j\left(B_j\right)} .$$

Definition 6. Let $\mathbb{A}$ denote the annulus $\{z \in \mathbb{C} : 1 \leq |z| \leq e\}$ and let $\mathbb{A}^\circ$ denote its interior. We shall say that the pseudolattice pair $\mathbf{X}$ is *nontrivial* if, for the special one-dimensional Banach pair $\bar{B} = (\mathbb{C}, \mathbb{C})$ and each $s \in \mathbb{A}^\circ$, there exists a sequence $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ such that the limit $\lim_{M,N\to+\infty} \sum_{n=-M}^{N} s^n b_n$ exists and is finite and nonzero.

Definition 7. We shall say that the pseudolattice pair $\mathbf{X}$ is *Laurent compatible* if it is nontrivial and if for every Banach pair $\bar{B}$, every sequence $\{b_n\}_{n\in\mathbb{Z}}$ in $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ and every fixed z in the open annulus $\mathbb{A}^\circ$, the Laurent series $\sum_{n\in\mathbb{Z}} z^n b_n$ converges in $B_0 + B_1$ and

$$\left\|\sum_{n\in\mathbb{Z}} z^n b_n\right\|_{B_0+B_1} \leq C \left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} \tag{2.1}$$

for some constant $C = C\left(z\right)$ independent of the choice of $\{b_n\}_{n\in\mathbb{Z}}$.

Remark 8. If $\mathbf{X}$ is Laurent compatible, then the $(B_0 + B_1)$-valued function $f\left(z\right) = \sum_{n\in\mathbb{Z}} z^n b_n$ is analytic in $\mathbb{A}^\circ$.

Definition 9. For each Banach pair $\bar{B}$, each Laurent compatible pair $\mathbf{X}$ and each fixed $s \in \mathbb{A}^\circ$ we define the space $\bar{B}_{\mathbf{X},s}$ to consist of all the elements of the form $b = \sum_{n\in\mathbb{Z}} s^n b_n$ where $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}\left(\mathbf{X}, \bar{B}\right)$, with the norm $\|b\|_{\bar{B}_{\mathbf{X},s}} = \inf\left\{\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} : b = \sum_{n\in\mathbb{Z}} s^n b_n\right\}$.

Remark 10. As remarked on p. 251 of [7], in elaboration of a point of view going back to [18], the space $\bar{B}_{\mathbf{X},s}$ coincides with various known interpolation spaces for appropriate choices of $\mathcal{X}_0$, $\mathcal{X}_1$ and s. In each of the three following examples we set $s = e^\theta$ for some $\theta \in (0, 1)$.

(i) If $\mathcal{X}_0 = \mathcal{X}_1 = FC$, the space $\bar{B}_{\mathbf{X},s}$ coincides isometrically with the variant of Calderón's complex interpolation space obtained when an annulus is used instead of a strip in the interpolation method (see Section 4).

(ii) If $\mathcal{X}_0 = \mathcal{X}_1 = \ell^p$, then $\bar{B}_{\mathbf{X},s}$ is the Lions-Peetre real method space $\bar{B}_{\theta,p} = (B_0, B_1)_{\theta,p}$. In fact the norm that we obtain here is exactly the norm introduced in formula (1.3) on p. 17 of [16] for suitable choices of the parameters p_0, p_1, ξ_0 and ξ_1.

(iii) If $\mathcal{X}_0 = \mathcal{X}_1 = UC$, then $\bar{B}_{\mathbf{X},s}$ is the interpolation space $\bar{B}_{\langle\theta\rangle} = \langle B_0, B_1\rangle_\theta$ introduced by Peetre on p. 175–6 of [18], for the function parameter $\rho(t) = t^\theta$, and

if $\mathcal{X}_0 = \mathcal{X}_1 = WUC$, then $\bar{B}_{\mathbf{X},s}$ is the Gustavsson-Peetre variant of $\langle B_0, B_1\rangle_\theta$ which is denoted by $\left\langle \bar{B}, \rho_\theta \right\rangle$ (see p. 45 of [10]).

(The method which yields the spaces $\langle B_0, B_1\rangle_\theta$ is sometimes referred to as the "$\pm$" method, since, as mentioned above, in the definition of unconditional convergence it suffices to consider sequences λ_n whose values are 1 and -1.)

Definition 11. Let $\mathbf{X} = \{\mathcal{X}_0, \mathcal{X}_1\}$ be a pair of pseudolattices. We shall say that $\mathbf{X}$ *admits differentiation* if it is Laurent compatible and, for each complex Banach space B,

(i) for each $r \in (0,1)$, each element $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_0(B)$ satisfies $\lim_{k\to-\infty} r^{-k} \|b_k\|_B = 0$ and each element $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_1(B)$ satisfies $\lim_{k\to\infty} r^k \|b_k\|_B = 0$, and

(ii) for every complex number ρ satisfying $0 < |\rho| < 1$ and for every sequence $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_0(B) \cap \mathcal{X}_1(B)$, the new sequence $\left\{b_n^j\right\}_{n\in\mathbb{Z}}$ is also in $\mathcal{X}_j(B)$ for $j = 0, 1$, where $\left\{b_n^0\right\}_{n\in\mathbb{Z}}$ and $\left\{b_n^1\right\}_{n\in\mathbb{Z}}$ are defined by setting $b_n^0 = \sum_{k<0} \rho^{-k} b_{n+k+1}$ and $b_n^1 = \sum_{k\geq 0} \rho^k b_{n+k+1}$ (where the convergence of these sums in B is guaranteed by condition (i)), and if also

(iii) for $j = 0, 1$ and each ρ as above, the linear map $D_{j,\rho}$ defined on $\mathcal{X}_j(B)$ by setting $D_{j,\rho}\left(\{b_n\}_{n\in\mathbb{Z}}\right) = \left\{b_n^j\right\}_{n\in\mathbb{Z}}$ maps $\mathcal{X}_j(B)$ boundedly into itself.

Remark 12. The pair $\mathbf{X} = \{\mathcal{X}_0, \mathcal{X}_1\}$ admits differentiation whenever $\mathcal{X}_0$ and $\mathcal{X}_1$ are each chosen to be any of ℓ^p ($p \in [1,\infty]$), c_0, FC, UC or WUC (see p. 256 of [7]).

The property of admitting differentiation has the following consequence (which also explains the choice of terminology for this property).

Lemma 13. *Let $\mathbf{X}$ be a pair of pseudolattices which admits differentiation and let $\bar{B}$ be a Banach pair. Let the sequence $\{f_n\}_{n\in\mathbb{Z}}$ be an element of $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ and let $f : \mathbb{A}^\circ \to B_0 + B_1$ be the analytic function defined by $f(z) = \sum_{n\in\mathbb{Z}} z^n f_n$. Suppose that $f(s) = 0$ for some point $s \in \mathbb{A}^\circ$ and let $g : \mathbb{A}^\circ \to B_0 + B_1$ be the analytic function obtained by setting $g(s) = f'(s)$ and $g(z) = \frac{1}{z-s} f(z)$ for all $z \in \mathbb{A}^\circ \setminus \{s\}$. Let $\{g_n\}_{n\in\mathbb{Z}}$ be the sequence of coefficients in the Laurent expansion $g(z) = \sum_{n\in\mathbb{Z}} z^n g_n$ of g in $\mathbb{A}^\circ$. Then $\{g_n\}_{n\in\mathbb{Z}}$ is also an element of $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$.*

For the proof we refer the reader to pp. 258-9 of [7].

We conclude this section with two more definitions of notions which will appear explicitly in our main theorem.

Definition 14. For each Banach pair $\bar{B}$ we define $\mathcal{J}_0\left(\bar{B}\right)$ to be the space of all $(B_0 \cap B_1)$-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ with finite support.

Remark 15. For every Banach pair $\bar{B}$, we obviously have that

$$\mathcal{J}_0\left(\bar{B}\right) \subset \mathcal{J}\left(\{\mathcal{X}_0, \mathcal{X}_1\}, \bar{B}\right)$$

whenever $\mathcal{X}_0$ and $\mathcal{X}_1$ are chosen to be any of the pseudolattices FC, UC, WUC, ℓ^p for $p \in [1,\infty]$ or c_0. Furthermore, one can verify that $\mathcal{J}_0\left(\bar{B}\right)$ is dense in $\mathcal{J}\left(\{FC, FC\}, \bar{B}\right)$ and also dense in $\mathcal{J}\left(\{\mathcal{X}_0, \mathcal{X}_1\}, \bar{B}\right)$ for $\mathcal{X}_i \in \{UC, c_0, \ell^p\}$, $1 \leq p < \infty$, $i = 0, 1$. But, in general, $\mathcal{J}_0\left(\bar{B}\right)$ is not dense in $\mathcal{J}\left(\{WUC, WUC\}, \bar{B}\right)$ and, except for trivial Banach spaces B_0, B_1, is never dense in $\mathcal{J}\left(\{\ell^\infty, \ell^\infty\}, \bar{B}\right)$.

Definition 16. Let S denote the right-shift operator on two-sided sequences defined by $S\left(\{b_n\}_{n\in\mathbb{Z}}\right) = \{b_{n-1}\}_{n\in\mathbb{Z}}$.

3. The main theorem

We can now state and prove our main theorem. As might be expected, there are some analogies between its proof and Stafney's arguments on p. 335 of [19].

Theorem 17. *Let* $\mathbf{X}$ *be a pair of pseudolattices which admits differentiation and let* $\bar{B}$ *be a Banach pair. Suppose that*

(i) $\mathcal{J}_0\left(\bar{B}\right) \subset \mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ *and* $\mathcal{J}_0\left(\bar{B}\right)$ *is dense in* $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ *and that*

(ii) the right-shift operator S *maps* $\mathcal{X}_j\left(B_j\right)$ *boundedly into itself for* $j = 0, 1$.

Then, for each $x \in B_0 \cap B_1$ *and* $s \in \mathbb{A}^\circ$,

$$\|x\|_{\bar{B}_{\mathbf{X},s}} = \inf\left\{\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} : \sum_{n\in\mathbb{Z}} s^n b_n = x,\ \{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}_0\left(\bar{B}\right)\right\}. \tag{3.1}$$

Proof of the theorem. Let x be in $B_0 \cap B_1$, s in $\mathbb{A}^\circ$ and let ε be an arbitrary positive number. The sequence $\{b_n\}_{n\in\mathbb{Z}}$ defined by setting $b_0 = x$ and $b_n = 0$ for $n \neq 0$ is in $\mathcal{J}_0\left(\bar{B}\right)$ and satisfies $\sum_{n\in\mathbb{Z}} s^n b_n = x$. It is clear from the definition of the norm $\|\cdot\|_{\bar{B}_{\mathbf{X},s}}$ that, for some $\{c_n\}_{n\in\mathbb{Z}}$ in the subspace $\mathcal{N}_s\left(\mathbf{X}, \bar{B}\right)$ of $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$, consisting of all sequences $\{d_n\}_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} s^n d_n = 0$, $\left\|\{b_n\}_{n\in\mathbb{Z}} - \{c_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} < \|x\|_{\bar{B}_{\mathbf{X},s}} + \varepsilon/2$. We need the following proposition:

Proposition. $\mathcal{J}_0\left(\bar{B}\right) \cap \mathcal{N}_s\left(\mathbf{X}, \bar{B}\right)$ *is dense in* $\mathcal{N}_s\left(\mathbf{X}, \bar{B}\right)$ *with respect to the norm of* $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ *restricted to* $\mathcal{N}_s\left(\mathbf{X}, \bar{B}\right)$.

We will first give a proof of the proposition and then continue with the proof of the theorem. Let $\{c_n\}_{n\in\mathbb{Z}}$ be in $\mathcal{N}_s\left(\mathbf{X}, \bar{B}\right)$. Set $f(z) = \sum_{n\in\mathbb{Z}} z^n c_n$. Then, by Lemma 13, the function $g : \mathbb{A}^\circ \to B_0 + B_1$ defined by setting $g(s) = f'(s)$ and $g(z) = \frac{1}{z-s} f(z)$ for all $z \in \mathbb{A}^\circ \setminus \{s\}$ has a Laurent expansion $g(z) = \sum_{n\in\mathbb{Z}} z^n g_n$ with $\{g_n\}_{n\in\mathbb{Z}} \in \mathcal{J}\left(\mathbf{X}, \bar{B}\right)$. Set $C = \max_{j=0,1} \|S\|_{\mathcal{X}_j(B_j)\to\mathcal{X}_j(B_j)}$. Since $\mathcal{J}_0\left(\bar{B}\right)$ is dense in $\mathcal{J}\left(\mathbf{X}, \bar{B}\right)$, we can find some $\{h_n\}_{n\in\mathbb{Z}} \in \mathcal{J}_0\left(\bar{B}\right)$ such that

$$\left\|\{h_n\}_{n\in\mathbb{Z}} - \{g_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} < \frac{\varepsilon}{e\left(1+C\right)}. \tag{3.2}$$

For every analytic function $f : \mathbb{A}^\circ \to B_0 + B_1$ with a Laurent expansion $f(z) = \sum_{n\in\mathbb{Z}} z^n b_n$ with $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ we shall define

$$\|f\|_{\mathcal{J},\bar{B}} := \left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)}.$$

(This is well-defined due to the uniqueness of the Laurent expansion in the annulus.) Set $h(z) = \sum_{n\in\mathbb{Z}} z^n h_n$.

Note that for every element $\{k_n\}_{n\in\mathbb{Z}} \in \mathcal{J}\left(\mathbf{X}, \bar{B}\right)$ if $k(z) = \sum_{n\in\mathbb{Z}} z^n k_n$ and if $r(z) = z - s$, then $(rk)(z) = \sum_{n\in\mathbb{Z}} z^n \left(k_{n-1} - s k_n\right)$ and thus

$$\begin{aligned} \|rk\|_{\mathcal{J},\bar{B}} &= \left\|\{k_{n-1} - s k_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} \\ &\leq \left\|\{k_{n-1}\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} + |s| \left\|\{k_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)}. \end{aligned}$$

Assumption (ii) of the theorem yields that

$$\begin{aligned} \left\|\{k_{n-1}\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} &= \max\left\{\left\|\{k_{n-1}\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}_0(B_0)}, \left\|\{e^n k_{n-1}\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}_1(B_1)}\right\} \\ &\leq C \max\left\{\left\|\{k_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}_0(B_0)}, e\left\|\{e^n k_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}_1(B_1)}\right\} \\ &\leq eC \left\|\{k_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\overline{B}\right)} \end{aligned}$$

and thus $\|rk\|_{\mathcal{J},\bar{B}} \leq e\,(1+C)\,\|k\|_{\mathcal{J},\bar{B}}$. The preceding calculation, along with equation (3.2), shows, in particular, that

$$\begin{aligned}\left\|\{h_{n-1}-sh_n\}_{n\in\mathbb{Z}}-\{c_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)} &= \|rh-f\|_{\mathcal{J},\bar{B}}\\ &= \|rh-rg\|_{\mathcal{J},\bar{B}}\\ &\leq e\,(1+C)\,\|h-g\|_{\mathcal{J},\bar{B}}<\varepsilon\,.\end{aligned}$$

Since $\{h_{n-1}-sh_n\}_{n\in\mathbb{Z}}$ is in $\mathcal{J}_0\left(\bar{B}\right)\cap\mathcal{N}_s\left(\mathbf{X},\bar{B}\right)$, the proposition follows.

Continuing with the proof of the theorem, we choose an element $\{u_n\}_{n\in\mathbb{Z}}$ in $\mathcal{J}_0\left(\bar{B}\right)\cap\mathcal{N}_s\left(\mathbf{X},\bar{B}\right)$ such that $\left\|\{c_n\}_{n\in\mathbb{Z}}-\{u_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)}<\varepsilon/2$.

We have that $\{b_n-u_n\}_{n\in\mathbb{Z}}\in\mathcal{J}_0\left(\bar{B}\right)$ and $\sum_{n\in\mathbb{Z}}s^n\left(b_n-u_n\right)=x$. Furthermore,

$$\begin{aligned}&\left\|\{b_n-u_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)}\\ \leq\ &\left\|\{b_n\}_{n\in\mathbb{Z}}-\{c_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)}+\left\|\{c_n\}_{n\in\mathbb{Z}}-\{u_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\mathbf{X},\bar{B}\right)}\\ <\ &\|x\|_{\bar{B}_{\mathbf{X},s}}+\varepsilon\,.\end{aligned}$$

So the proof of the theorem is complete. □

Remark 18. By Remarks 10, 12, and 15 we can obtain an appropriate formulation of Theorem 17 for the "annulus" variant of Calderón's complex interpolation method space, for the Lions-Peetre real method space $(B_0,B_1)_{\theta,p}$ for $1\leq p<\infty$ (and in fact also when it is equipped with the norm defined via the J-functional if we make slight modifications to our proof) and for the Peetre interpolation space $\bar{B}_{\langle\theta\rangle}=\langle B_0,B_1\rangle_\theta$ for $\theta\in(0,1)$. (Of course condition (ii) of Theorem 17 obviously holds in these cases and in fact S is even an isometry.)

Our theorem in the case of the "annulus" variant of Calderón's complex interpolation method space can also be obtained by an alternative argument similar to Stafney's proof on p. 335 of [19] if one replaces Calderón's space $\mathcal{G}\left(B_0,B_1\right)$ by the space of all Laurent polynomials with coefficients in $B_0\cap B_1$ and if $\mathcal{F}\left(B_0,B_1\right)$ is replaced by its counterpart for the annulus (see also Section 4 and the paragraph which precedes Definition 4.1 on p. 80 of [6]).

Remark 19. We can also obtain a version of our theorem for a discrete version of the (generalised) $\mathcal{J}$-method which is discussed (for example) on p. 381 of [1] and apparently originated in the work of Peetre in [17].

Remark 20. Janson showed that if we equip $B_0\cap B_1$ with the norms of $\left\langle\bar{B},\rho_\theta\right\rangle$ and $\bar{B}_{\langle\theta\rangle}$, we obtain two normed spaces with equivalent norms (see pp. 59-60 of [13]), and thus a weaker version of (3.1), i.e. that the left and right sides are equivalent, can also be obtained for $\mathcal{X}_0=\mathcal{X}_1=WUC$, even though, as pointed out in Remark 15, condition (i) fails to hold in this case.

4. Additional remarks regarding complex interpolation

We begin by explicitly recalling the definition of complex interpolation spaces on the annulus.

Let $\bar{B}=(B_0,B_1)$ be a Banach pair. Let $\mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)$ be the space of all continuous functions $f:\mathbb{A}\to B_0+B_1$ such that f is analytic in $\mathbb{A}^\circ$ and for $j=0,1$ the

restriction of f to the circle $e^j\mathbb{T}$ is a continuous map of $e^j\mathbb{T}$ into B_j. We norm $\mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)$ by $\|f\|_{\mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)} = \sup\left\{\left\|f\left(e^{j+it}\right)\right\|_{B_j} : t \in [0,2\pi),\, j=0,1\right\}$. For each $\theta \in (0,1)$, let $\left[\bar{B}\right]_{\theta,\mathbb{A}}$ denote the space of all elements in B_0+B_1 of the form $b = f\left(e^{\theta}\right)$ for some $f \in \mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)$. It is normed by

$$\|b\|_{\left[\bar{B}\right]_{\theta,\mathbb{A}}} = \inf\left\{\|f\|_{\mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)} : f \in \mathcal{F}_{\mathbb{A}}\left(\bar{B}\right),\, b = f\left(e^{\theta}\right)\right\}.$$

This variant of Calderón's complex interpolation space, which was apparently first considered in [18], coincides with $[B_0, B_1]_\theta$ to within equivalence of norms, as was shown on pp. 1007-9 of [4].

Here we give a detailed proof that

$$\bar{B}_{\{FC,FC\},e^{\theta}} = \left[\bar{B}\right]_{\theta,\mathbb{A}} \text{ with equality of norms, for each } \theta \in (0,1)$$

as was stated in [7]. Some parts of the proof can also be found on pp. 78-9 of [6]. Related ideas appear already in [2].

Let $b \in \left[\bar{B}\right]_{\theta,\mathbb{A}}$ and let ε be an arbitrary positive number. We can find some $f \in \mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)$ which satisfies $b = f\left(e^{\theta}\right)$ and $\|f\|_{\mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)} < \|b\|_{\left[\bar{B}\right]_{\theta,\mathbb{A}}} + \varepsilon$. As shown on pp. 78–9 of [6], $f(z) = \sum_{n\in\mathbb{Z}} z^n \hat{f}(n)$ for every $z \in \mathbb{A}^\circ$, where

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-nit} f\left(e^{it}\right) dt = \frac{1}{2\pi}\int_0^{2\pi} e^{-n(1+it)} f\left(e^{1+it}\right) dt,$$

from which we obtain that $b = \sum_{n\in\mathbb{Z}} e^{\theta n}\widehat{f}(n)$, and thus $b \in \bar{B}_{\{FC,FC\},e^{\theta}}$ and $\left[\bar{B}\right]_{\theta,\mathbb{A}} \subset \bar{B}_{\{FC,FC\},e^{\theta}}$. Furthermore

$$\begin{aligned}
\|b\|_{\left[\bar{B}\right]_{\theta,\mathbb{A}}} + \varepsilon &> \sup\left\{\left\|f\left(e^{j+it}\right)\right\|_{B_j} : t \in [0,2\pi), j=0,1\right\} \\
&= \max_{j=0,1}\left\|\left\{e^{jn}\hat{f}(n)\right\}_{n\in\mathbb{Z}}\right\|_{FC(B_j)} \\
&= \left\|\left\{\hat{f}(n)\right\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\{FC,FC\},\bar{B}\right)} \\
&\geq \|b\|_{\bar{B}_{\{FC,FC\},e^{\theta}}}.
\end{aligned}$$

It follows that $\|b\|_{\left[\bar{B}\right]_{\theta,\mathbb{A}}} \geq \|b\|_{\bar{B}_{\{FC,FC\},e^{\theta}}}$ for all $b \in \left[\bar{B}\right]_{\theta,\mathbb{A}}$.

Now, for the reverse inclusion and norm inequality, let $b \in \bar{B}_{\{FC,FC\},e^{\theta}}$ and let ε be an arbitrary positive number. We can find some $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}\left(\{FC,FC\},\bar{B}\right)$ such that

$$b = \sum_{n\in\mathbb{Z}} e^{\theta n} b_n \tag{4.1}$$

and $\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\{FC,FC\},\bar{B}\right)} < \|b\|_{\bar{B}_{\{FC,FC\},e^{\theta}}} + \varepsilon$. By the definition of FC, we can find for $j=0,1$ continuous functions $f_j : \mathbb{T} \to B_j$ such that

$$e^{jn} b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f_j\left(e^{it}\right) dt \quad \text{for all } n.$$

We shall define a sequence of functions $g_N : \mathbb{A} \to B_0 \cap B_1$ by

$$g_N(z) = \sum_{n=-N}^{N} z^n \left(1 - \frac{|n|}{N+1}\right) b_n \,,$$

and we shall first show that $g_N(z)$ converges in $B_0 + B_1$ for each $z \in \mathbb{A}$.

If $z \in \mathbb{A}^\circ$, then, by Remark 8, the sequence of Laurent polynomials $S_N(z) := \sum_{n=-N}^{N} z^n b_n$ converges in B_0+B_1, and thus, since in fact $g_N(z) = \frac{1}{N+1}\sum_{n=0}^{N} S_N(z)$, it follows, by standard arguments, that $g_N(z)$ also converges in B_0+B_1 to the same limit, $\sum_{n\in\mathbb{Z}} z^n b_n$.

If we apply the lemma on pp. 10–11 of [14] with the Fejér summability kernel $K_n(t) = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right) e^{ijt}$ and with $\varphi(\tau) = f_j\left(e^{i(t-\tau)}\right)$ and $B = B_j$ for $j = 0, 1$, then we obtain that

$$\lim_{N\to\infty} \left\| g_N\left(e^{j+it}\right) - f_j\left(e^{it}\right)\right\|_{B_j} = 0 \tag{4.2}$$

for each $t \in [0, 2\pi)$ and thus $g_N(z)$ also converges for $z \in \mathbb{A}\backslash\mathbb{A}^\circ$. Moreover, since

$$\lim_{\tau\to 0} \sup\left\{ \left\| f_j\left(e^{i(t-\tau)}\right) - f_j\left(e^{it}\right)\right\|_{B_j} : t \in [0, 2\pi),\ j = 0, 1\right\} = 0\,,$$

we can in fact obtain, by making a slight modification to the proof of the lemma in [14] for our particular case, that

$$\lim_{N\to\infty} \sup\left\{ \left\| g_N\left(e^{j+it}\right) - f_j\left(e^{it}\right)\right\|_{B_j} : t \in [0, 2\pi),\ j = 0, 1\right\} = 0\,. \tag{4.3}$$

We shall denote the pointwise limit of g_N in $B_0 + B_1$ by g. By equation (4.2), we obtain that $g\left(e^{j+it}\right) = f_j\left(e^{it}\right)$ for each $t \in [0, 2\pi)$ and $j = 0, 1$, and thus the restriction of g to the circle $e^j\mathbb{T}$ is a continuous map of $e^j\mathbb{T}$ into B_j. Since $g(z) = \sum_{n\in\mathbb{Z}} z^n b_n$ for every $z \in \mathbb{A}^\circ$, by Remarks 8 and 12 and equation (4.1), g is an analytic function on $\mathbb{A}^\circ$ which satisfies

$$g\left(e^\theta\right) = b\,. \tag{4.4}$$

By equation (4.3) and the maximum principle, the sequence of continuous functions g_N converges in $B_0 + B_1$ uniformly on $\mathbb{A}$ and consequently its limit is also a continuous (B_0+B_1)-valued function. Thus, $g \in \mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)$ and so, by equation (4.4), $b \in \left[\bar{B}\right]_{\theta,\mathbb{A}}$ and $\bar{B}_{\{FC,FC\},e^\theta} \subset \left[\bar{B}\right]_{\theta,\mathbb{A}}$. Furthermore, the preceding calculations show that

$$\begin{aligned}
\left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{J}\left(\{FC,FC\},\bar{B}\right)} &= \max_{j=0,1} \sup_{t\in[0,2\pi)} \left\| f_j\left(e^{it}\right)\right\|_{B_j} \\
&= \max_{j=0,1} \sup_{t\in[0,2\pi)} \left\| g\left(e^{j+it}\right)\right\|_{B_j} \\
&= \|g\|_{\mathcal{F}_{\mathbb{A}}\left(\bar{B}\right)} \geq \|b\|_{\left[\bar{B}\right]_{\theta,\mathbb{A}}}\,.
\end{aligned}$$

Therefore, $\|b\|_{\left[\bar{B}\right]_{\theta,\mathbb{A}}} \leq \|b\|_{\bar{B}_{\{FC,FC\},e^\theta}}$ for all $b \in \bar{B}_{\{FC,FC\},e^\theta}$. This completes the proof.

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