

## AN ALMOST SCHUR THEOREM ON 4-DIMENSIONAL MANIFOLDS

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ABSTRACT. In this short paper we prove that the almost Schur theorem, introduced by De Lellis and Topping, is true on 4-dimensional Riemannian manifolds of nonnegative scalar curvature and discuss some related problems on other dimensional manifolds.

### 1. INTRODUCTION

Very recently, De Lellis and Topping proved an interesting result about a generalization of Schur's theorem.

**Theorem 1** (Almost Schur Theorem [1]). *For  $n \geq 3$ , if  $(M^n, g)$  is a closed Riemannian manifold with nonnegative Ricci tensor, then*

$$(1) \quad \int_M \left| \text{Ric} - \frac{\bar{R}}{n}g \right|^2 dv(g) \leq \frac{n^2}{(n-2)^2} \int_M \left| \text{Ric} - \frac{R}{n}g \right|^2 dv(g),$$

where  $\bar{R} = \text{vol}(g)^{-1} \int_M R dv(g)$  is the average of the scalar curvature  $R$  of  $g$ .

It is clear that the Schur theorem follows directly from Theorem 1. The latter can be seen as a quantitative version or a stability result of the Schur Theorem. In [1] they also showed that the constant in inequality (1) is optimal and the nonnegativity of the Ricci tensor cannot be removed in general: When  $n \geq 5$  they gave examples of metrics on  $\mathbb{S}^n$  which make the ratio of the left-hand side of (1) to the right-hand side of (1) arbitrarily large. When  $n = 3$ , they found manifolds which make the ratio arbitrary. At the end they left an open question: *Inequalities of this form may hold for  $n = 3$  and  $n = 4$  with constants depending on the topology of  $M$ .*

In this short paper we will show that Theorem 1 holds under the condition of nonnegativity of the scalar curvature for dimension  $n = 4$ .

**Theorem 2.** *If  $n = 4$  and if  $(M^4, g)$  is a closed Riemannian manifold with nonnegative scalar curvature, then (1) holds. Moreover, equality holds if and only if  $(M^4, g)$  is an Einstein manifold.*

We first observe that inequality (1) is equivalent to

$$(2) \quad \left( \int_M \sigma_1(g) dv(g) \right)^2 \geq \frac{2n}{n-1} \text{vol}(g) \int_M \sigma_2(g) dv(g),$$

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where  $\sigma_k(g)$  is the  $k$ -scalar curvature of metric  $g$ . Its definition will be recalled in Section 2. Then we prove this inequality for  $n = 4$  by using an argument given by Gursky [3].

2. PROOF OF THEOREM 2

Let us first recall the definition of the  $k$ -scalar curvature, which was first introduced by Viaclovsky [4] and has been intensively studied by many mathematicians. Let

$$S_g = \frac{1}{n-2} \left( Ric_g - \frac{R_g}{2(n-1)} \cdot g \right)$$

be the Schouten tensor of  $g$ . For an integer  $k$  with  $1 \leq k \leq n$  let  $\sigma_k$  be the  $k$ -th elementary symmetric function in  $\mathbb{R}^n$ . The  $k$ -scalar curvature is

$$\sigma_k(g) := \sigma_k(\Lambda_g),$$

where  $\Lambda_g$  is the set of eigenvalues of the matrix  $g^{-1} \cdot S_g$ . In particular,  $\sigma_1(g) = \text{tr } S$  and  $\sigma_2 = \frac{1}{2}((\text{tr } S)^2 - |S|^2)$ . It is trivial to see that

$$\begin{aligned} \sigma_1(g) &= \frac{R}{2(n-1)}, \\ \sigma_2(g) &= \frac{1}{2(n-2)^2} \left\{ -|Ric|^2 + \frac{n}{4(n-1)} R^2 \right\}, \\ \left| Ric - \frac{R}{n} g \right|^2 &= |Ric|^2 - \frac{R^2}{n}. \end{aligned}$$

From the above it is easy to have the following observation.

**Observation 1.** Inequality (1) is equivalent to (2).

Hence, instead of proving Theorem 2 we actually prove

**Theorem 3.** *If  $n = 4$  and if  $(M^n, g)$  is a closed Riemannian manifold with non-negative scalar curvature, then (2) holds. Moreover, equality holds if and only if  $(M, g)$  is an Einstein metric.*

The proof of Theorem 3 follows closely a nice argument of Gursky [3].

**Lemma 1.** *For any  $n \geq 3$  and any closed Riemannian manifold  $(M^n, g)$ , there exists a conformal metric  $g_1 \in [g]$  satisfying*

$$(3) \quad \frac{2n}{n-1} \frac{\int_M \sigma_2(g_1) dv(g_1)}{(\text{vol}(g_1))^{\frac{n-4}{n}}} \leq Y_1([g])^2,$$

where  $Y_1([g])$  is the first Yamabe invariant defined by

$$(4) \quad Y_1([g]) := \inf_{g \in [g]} \frac{\int_M \sigma_1(g) dv(g)}{(\text{vol}(g))^{\frac{n-2}{n}}}$$

and  $[g]$  is the conformal class of the metric  $g$ .

Here our definition of the Yamabe constant is different from the standard one by a multiple factor  $\frac{1}{2(n-1)}$ .

*Proof of Lemma 1.* The proof follows closely an argument given by Gursky in [3]. Let  $g_1$  be a solution of the Yamabe problem. Thus the scalar curvature, and hence  $\sigma_1(g)$ , is constant. We have a simple fact: for any  $n \times n$  symmetric matrix  $A$  such that

$$(\sigma_1(A))^2 \geq \frac{2n}{n-1} \sigma_2(A),$$

equality holds if and only if the matrix is a multiple of the identity matrix. Now the following calculations lead to

$$(5) \quad \frac{2n}{n-1} \text{vol}(g_1) \int_M \sigma_2(g_1) dv(g_1) \leq \text{vol}(g_1) \int_M (\sigma_1(g_1))^2 dv(g_1) = \left( \int_M \sigma_1(g_1) dv(g_1) \right)^2.$$

Here we have used the fact that  $\sigma_1(g_1)$  is a constant. Therefore,

$$\frac{2n}{n-1} \frac{\int_M \sigma_2(g_1) dv(g_1)}{(\text{vol}(g_1))^{\frac{n-4}{n}}} \leq \left( \frac{\int_M \sigma_1(g_1) dv(g_1)}{\text{vol}(g_1)^{\frac{n-2}{n}}} \right)^2 = Y_1([g])^2,$$

since  $g_1$  is a Yamabe solution.  $\square$

*Proof of Theorem 3.* In the case of dimension  $n = 4$ , it is well known that  $\int_M \sigma_2(g) dv(g)$  is constant in any given conformal class. Hence by Lemma 1 we have

$$\begin{aligned} \frac{2n}{n-1} \int_M \sigma_2(g) dv(g) &= \frac{2n}{n-1} \int_M \sigma_2(g_1) dv(g_1) \leq Y_1([g])^2 \\ &\leq \left( \frac{\int_M \sigma_1(g) dv(g)}{\text{vol}(g)^{\frac{1}{2}}} \right)^2. \end{aligned}$$

In the last inequality we have used the condition  $\sigma_1(g) \geq 0$ , which implies that  $Y_1([g]) \geq 0$ . The equality holds if and only if the Schouten tensor  $S_g$  is proportional to the metric  $g$ ; i.e.,  $g$  is an Einstein metric.  $\square$

We conjecture that Theorem 2 is true for  $n = 3$ . To attack this conjecture one needs to study a corresponding Yamabe-type problem. The methods developed, especially in [2], for a  $\sigma_k$ -Yamabe problem would be helpful to study this problem.

**Note added in proof.**

1. The conjecture proposed at the end of the paper was proved in [6].
2. The rigidity of Theorem 1, i.e., equality in (1) implies that the metric is Einstein, was proved in [5], among other generalizations of Theorems 1 and 2. It was also proved independently in the last version of [1].

REFERENCES

- [1] C. De Lellis and P. Topping, Almost Schur Theorem, to appear in Calc. Var. PDE, arXiv 1003.3527.
- [2] Y. Ge, C.-S. Lin and G. Wang, On  $\sigma_2$ -scalar curvature, J. Diff. Geo., **84** (2010), 45–86. MR2629509
- [3] M. Gursky, The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE, Comm. Math. Phys., **207** (1999), 131–143. MR1724863 (2000k:58029)

- [4] J. Viaclovsky, Conformal geometry, contact geometry, and the calculus of variations, *Duke Math. J.*, **101** (2000), 283–316. MR1738176 (2001b:53038)
- [5] Y. Ge, G. Wang and C. Xia, On problems related to an inequality of DeLellis and Topping, preprint, 2011.
- [6] Y. Ge and G. Wang, A new conformal invariant on 3-dimensional manifolds and its applications, arXiv 1103.3838.

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