PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 3, March 2012, Pages 927–930
S 0002-9939(2011)11241-3
Article electronically published on July 13, 2011

STABLE SOLUTIONS OF ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS WITH EUCLIDEAN COVERINGS

ALBERTO FARINA, YANNICK SIRE, AND ENRICO VALDINOCI

(Communicated by Matthew J. Gursky)

Abstract. We investigate the rigidity properties of stable, bounded solutions of semilinear elliptic partial differential equations in Riemannian manifolds that admit a Euclidean universal covering, finding conditions under which the level sets are geodesics or the solution is constant.

Let M be a complete, connected, Riemannian manifold, endowed with a metric g. Let $f \in C^1(\mathbb{R})$. We will consider classical, bounded solutions $u \in C^2(M) \cap L^\infty(M)$ of the following semilinear partial differential equation:

$$\Delta_g u + f(u) = 0. \tag{1}$$

We say that u is stable if, for any $\varphi \in C_0^\infty(M)$, we have that

$$\int_M \left(|\nabla_g \varphi|^2 - f'(u)\varphi^2 \right) \geqslant 0. \tag{2}$$

The notion of stability is a classiclal topic in the calculus of variations, and it may often be related with geometric features (such as monotonicity of the solution) and with quantities of physical relevance (such as energy minimization): we refer to [Dup11] for a thorough discussion on the notion of stability and its importance. Also, as usual, we say that the metric g is flat if its sectional curvature vanishes identically (see, e.g., [GHL90]).

The purpose of this paper is to prove symmetry and rigidity properties for stable, bounded solutions u of (1) under several circumstances. First of all (see Theorem A), we will show that if M admits $\mathbb{R}^2$ with the flat metric as universal covering, then the level sets of u are geodesics.

These results extend some previous works of the authors, such as Theorem 2 of [FSV08b].

The proof is based on some technology developed to study some questions related to a major problem posed by De Giorgi (see [DG79], and also [FV09] for a recent review on this topic).

Though we do not know whether our assumptions in Theorem A are optimal, we recall that, in general, there exist bounded, stable solutions in higher dimensional

Received by the editors December 15, 2010.
2010 *Mathematics Subject Classification.* Primary 35J05, 58J05, 35B53, 35R01.
The third author has been supported by FIRB Analysis and Beyond.

Euclidean spaces whose level sets are not totally geodesic (see [dPKW08]), so our hypotheses cannot be dropped completely.[1]

Next (see Theorem B), we prove that if $f \geqslant 0$ and M has a Euclidean covering space $\mathbb{R}^n$ with $n \leqslant 4$, then u is constant. For results on compact manifolds, see the pioneer work of [Jim84]. For related results on manifolds, see [FSV08b]. See also [DF10] for the Euclidean case up to dimension 4 and [Far07], which studied stable solutions of the Lane-Emden equation, showing that in dimension 11 and higher there are stable, bounded, positive, radial, non-constant solutions even in the Euclidean case (this is somewhat a counterpart of Theorem B, though it leaves open the case of intermediate dimensions).

The proof of Theorem B is based on a parabolicity estimate of [DF10].

We observe (see Corollary C) that the results in Theorems A or B hold if M has dimension $n = 2$ or $n \leqslant 4$, respectively, and it is endowed with the flat metric. In this sense, the scope of this paper is complementary to the one of [FSV08b]: while [FSV08a] aims to classify the stable solutions in a manifold with non-negative Ricci cuvature, here we attempt a similar classifications for a manifold with vanishing sectional curvature.

Following are the results obtained and their proofs.

Theorem A. *Suppose that M has $\mathbb{R}^2$ with the flat metric as universal covering. Let u be a stable, bounded solution of* (1) *on M.*

Then, either u is constant or any connected component of each level set of u is a geodesic.

Proof. If p is the projection from $\mathbb{R}^2$ to M and we set $U(x) := u(p(x))$, we have that

$$U \text{ is a solution of } \Delta U + f(U) \text{ on } \mathbb{R}^2, \tag{3}$$

because p is a local isometry and ∇_g in normal coordinates becomes the Euclidean ∇, since g is pulled back to the Euclidean metric.

Moreover,

$$\begin{gathered} U \text{ is stable; i.e., for any } \varphi \in C_0^\infty(\mathbb{R}^2) \text{ we have that} \\ \int_{\mathbb{R}^2} \Big(|\nabla\varphi|^2 - f'(U)\varphi^2 \Big) \geqslant 0. \end{gathered} \tag{4}$$

Indeed, the stability condition in (2) (resp., in (4)) is equivalent to the existence of a positive solution $v \in C^2(M,(0,+\infty))$ (resp., $V \in C^2(\mathbb{R}^2,(0,+\infty))$) of the linearized equation $\Delta_g v + f'(u)v = 0$ in M (resp., $\Delta V + f'(u)V = 0$ in $\mathbb{R}^2$); see, e.g., [FCS80, FC85]. Hence, the stability of u in (2) gives the existence of v as above. Then, if for any $x \in \mathbb{R}^2$ we define the lifted function $V(x) := v(p(x))$, we have that V is a positive solution of the linearized equation pulled back on the covering, thus proving (4).

Now we apply Theorem 1.1 of [FSV08a] to U. We remark that U satisfies the hypotheses of Theorem 1.1 of [FSV08a] thanks to (3) and (4). As a consequence, U is one-dimensional; i.e., there exist $\omega \in \mathrm{S}^1$ and $h : \mathbb{R} \to \mathbb{R}$ such that

$$U(x) = h(\omega \cdot x) \text{ for any } x \in \mathbb{R}^2. \tag{5}$$

[1]After this paper was completed, we received the very interesting preprint [PW11], where stable solutions were constructed in $\mathbb{R}^8$, which is one dimension less than in [dPKW08].

We claim that

$$\text{(6)} \qquad \begin{array}{l} \text{either } h \text{ is constant or} \\ \text{the level sets of } h \text{ are made by points that do not accumulate.} \end{array}$$

To establish this we suppose that there exist points t_k, each different from the others, that accumulate to some t_o, for which $h(t_k) = 0$. By Rolle's Theorem, there exists τ_k on the segment joining t_k to t_{k+1} such that $h'(\tau_k) = 0$. Of course, τ_k accumulates to t_o as well. We have

$$h(t_o) = \lim_{k \to +\infty} h(t_k) = 0,$$
$$h'(t_o) = \lim_{k \to +\infty} \frac{h(t_k) - h(t_o)}{t_k - t_o} = 0,$$

and

$$h''(t_o) = \lim_{k \to +\infty} \frac{h'(\tau_k) - h'(t_o)}{\tau_k - t_o} = 0.$$

In particular, $f(h(t_o)) = -h''(t_o) = 0$. Thus, if we set $g(t) = (g_1(t), g_2(t)) := (h(t), h'(t))$ and $\psi(g_1, g_2) = (g_2, -f(g_1))$, we have that $g'(t) = \psi(g(t))$. Also, the constant function $g_o(t) := (h(t_o), 0)$ satisfies $g_o'(t) = \psi(g_o(t))$. Since $g(t_o) = g_o(t_o)$, we have that $g(t) = g_o(t)$ for any $t \in \mathbb{R}$, due to the Uniqueness Theorem for ODEs; hence h is constant. This proves (6).

Then, by (5) and (6) we have that either U is constant or that all the level sets of U are straight lines with direction perpendicular to ω. In the latter case, the level sets of u are the image by the projection p of such straight lines, and therefore they are geodesics, because all the Cristoffel symbols vanish when they are written in local coordinates, since the pullback of g is Euclidean. □

Theorem B. *Suppose that $f(r) \geqslant 0$ for any $r \in \mathbb{R}$ and that M has a Euclidean covering space $\mathbb{R}^n$ with $n \leqslant 4$.*

Let u be a stable, bounded solution of (1) *on M. Then u is constant.*

Proof. By arguing as in (3) and (4), we see that, if p is the projection from $\mathbb{R}^n$ to M and $U(x) := u(p(x))$, then U is a stable solution of $\Delta U + f(U)$ on $\mathbb{R}^n$. Accordingly, by Theorem 1.1 of [DF10], we have that U is constant. □

Corollary C. *Suppose that the metric g of M is flat. Let u be a stable, bounded solution of* (1) *on M.*

Then,

- *if* $\dim M = 2$*, then either u is constant or any connected component of each level set of u is a geodesic;*
- *if* $\dim M \leqslant 4$*, and $f(r) \geqslant 0$ for any $r \in \mathbb{R}$, then u is constant.*

Proof. By Theorem 3.82 on page 135 of [GHL90], we have that M has a Euclidean covering space $\mathbb{R}^n$.

So the results follow from Theorems A and B. □

References

[DF10] L. Dupaigne and A. Farina, *Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 4, 855–882. MR2654082

[DG79] Ennio De Giorgi, *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978) (Bologna), Pitagora, 1979, pp. 131–188. MR533166 (80k:49010)

[dPKW08] Manuel del Pino, Michał Kowalczyk, and Juncheng Wei, *A counterexample to a conjecture by De Giorgi in large dimensions*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 23-24, 1261–1266. MR2473304 (2009k:35084)

[Dup11] Louis Dupaigne, *Stable solutions of elliptic partial differential equations*, Taylor & Francis, 2011. MR2779463

[Far07] Alberto Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of $\mathbb{R}^N$*, J. Math. Pures Appl. (9) **87** (2007), no. 5, 537–561. MR2322150 (2008c:35070)

[FC85] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math. **82** (1985), no. 1, 121–132. MR808112 (87b:53090)

[FCS80] Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211. MR562550 (81i:53044)

[FSV08a] Alberto Farina, Berardino Sciunzi, and Enrico Valdinoci, *Bernstein and De Giorgi type problems: new results via a geometric approach*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 4, 741–791. MR2483642 (2009j:58020)

[FSV08b] Alberto Farina, Yannick Sire, and Enrico Valdinoci, *Stable solutions of elliptic equations on Riemannian manifolds*, preprint (2008), `www.mat.uniroma2.it/~valdinoc/manifold.pdf`.

[FV09] Alberto Farina and Enrico Valdinoci, *The state of the art for a conjecture of De Giorgi and related problems*, Recent progress on reaction-diffusion systems and viscosity solutions, World Sci. Publ., Hackensack, NJ, 2009, pp. 74–96. MR2528756

[GHL90] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, second ed., Universitext, Springer-Verlag, Berlin, 1990. MR1083149 (91j:53001)

[Jim84] Shuichi Jimbo, *On a semilinear diffusion equation on a Riemannian manifold and its stable equilibrium solutions*, Proc. Japan Acad. Ser. A Math. Sci. **60** (1984), no. 10, 349–352. MR778527 (86g:58144)

[PW11] Frank Pacard and Juncheng Wei, *Stable solutions of the Allen-Cahn equation in dimension 8 and minimal cones*, preprint (2011), `http://arxiv.org/abs/1102.3446`.

LAMFA, CNRS UMR 6140, Université de Picardie Jules Verne, Amiens, France
E-mail address: `alberto.farina@u-picardie.fr`

LATP, Université Aix-Marseille 3, Marseille, France
E-mail address: `sire@cmi.univ-mrs.fr`

Dipartimento di Matematica, Università di Roma Tor Vergata, Rome, Italy
E-mail address: `enricovaldinoci@gmail.com`