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WEIGHTED SOBOLEV TYPE EMBEDDINGS AND COERCIVE QUASILINEAR ELLIPTIC EQUATIONS ON $\mathbb{R}^N$

JIABAO SU AND RUSHUN TIAN

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Abstract. We study weighted Sobolev type embeddings of radially symmetric functions from $W_r^{1,p}(\mathbb{R}^N; V)$ into $L^q(\mathbb{R}^N; Q)$ for $q < p$ with singular potentials. We then investigate the existence of nontrivial radial solutions of quasilinear elliptic equations with singular potentials and sub-p-linear nonlinearity. The model equation is of the form

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + V(|x|)|u|^{p-2}u = Q(|x|)|u|^{q-2}u, & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad |x| \to \infty. \end{cases}$$

1. Introduction

Let $V, Q : (0,\infty) \to (0,\infty)$ be two continuous functions satisfying the following assumptions:

(V) there exist real numbers a and a_0 such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^a} > 0, \quad \liminf_{r\to 0} \frac{V(r)}{r^{a_0}} > 0,$$

(Q) there exist real numbers b and b_0 such that

$$\limsup_{r\to\infty} \frac{Q(r)}{r^b} < \infty, \quad \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty.$$

Let $C_0^\infty(\mathbb{R}^N)$ be the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) \mid u(x) = u(|x|)\}.$$

Denote by $D_r^{1,p}(\mathbb{R}^N)$ the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under the norm

$$\|\nabla u\|_{L^p} = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

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Define for $p \geqslant 1$, $q \geqslant 1$,

$$L^p(\mathbb{R}^N;V) := \{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is Lebesgue measurable}, \int_{\mathbb{R}^N} V(|x|)|u|^p dx < \infty\},$$

$$L^q(\mathbb{R}^N;Q) = \{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is Lebesgue measurable}, \int_{\mathbb{R}^N} Q(|x|)|u|^q dx < \infty\}.$$

Then define

$$W_r^{1,p}(\mathbb{R}^N;V) = D_r^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N;V),$$

which is a Banach space ([1]) equipped with the norm

$$\|u\|_{W_r^{1,p}(\mathbb{R}^N;V)} = \left(\int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p dx \right)^{\frac{1}{p}}.$$

In a recent paper [13], the following embedding theorem was constructed.

Theorem 1.1 ([13]). *Let $1 < p < N$. Assume* (V) *and* (Q)*. Then the embedding*

$$W_r^{1,p}(\mathbb{R}^N;V) \hookrightarrow L^q(\mathbb{R}^N;Q)$$

is continuous for $\underline{q} \leqslant q \leqslant \bar{q}$ when $\bar{q} < \infty$ and for $\underline{q} \leqslant q < \infty$ when $\bar{q} = \infty$, and it is compact for $\underline{q} < q < \bar{q}$, where $\underline{q}$ and $\bar{q}$ are defined as

$$\underline{q} = \begin{cases} \frac{p^2(N-1+b)-ap}{p(N-1)+a(p-1)}, & b \geqslant a > -p, \\ \frac{p(N+b)}{N-p}, & b \geqslant -p \geqslant a, \\ p, & b \leqslant \max\{a, -p\}, \end{cases}$$

$$\bar{q} = \begin{cases} \frac{p(N+b_0)}{N-p}, & b_0 \geqslant -p, a_0 \geqslant -p, \\ \frac{p^2(N-1+b_0)-a_0 p}{p(N-1)+a_0(p-1)}, & -p \geqslant a_0 > \frac{1-N}{p-1}p, b_0 \geqslant a_0, \\ \infty, & a_0 \leqslant -\frac{N-1}{p-1}p, b_0 \geqslant a_0. \end{cases}$$

When $b < \max\{a, -p\}$ and $b_0 > \min\{-p, a_0\}$, the embedding is compact for $q = p$.

Note that in the above theorem, the embedding from $W_r^{1,p}(\mathbb{R}^N;V)$ into $L^q(\mathbb{R}^N;Q)$ is always with $p \leqslant q$, which can be seen from the definition of $\underline{q}$. This result provides one with a variational basis in finding radial solutions of quasilinear elliptic equations of the form

$$\text{(P)} \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(|x|)|u|^{p-2}u = Q(|x|)|u|^{q-2}u, & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad |x| \to \infty \end{cases}$$

with super-p-linear nonlinearity by variational methods. We refer to [13] for more comments on this subject.

The main purposes of this paper are to establish a compact embedding from $W_r^{1,p}(\mathbb{R}^N;V)$ into $L^q(\mathbb{R}^N;Q)$ with the index q located in a *range* depending on p, N and the real numbers involved in (V) and (Q) so that $1 < q < p$. From this embedding result the functional

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p \, dx - \int_{\mathbb{R}^N} Q(|x|)|u|^q d\, x \tag{1.1}$$

is well-defined and is of C^1 on $W_r^{1,p}(\mathbb{R}^N;V)$. A critical point of Φ is exactly a weak radial solution of (P), that is, a function $u \in W_r^{1,p}(\mathbb{R}^N;V)$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi + V(|x|)|u|^{p-2}u\varphi dx = \int_{\mathbb{R}^N} Q(|x|)|u|^{q-2}u\varphi dx, \tag{1.2}$$

for all $\varphi \in W_r^{1,p}(\mathbb{R}^N; V)$. As applications of the compact embedding results, we investigate via variational arguments ([7, 14, 15]) the existence and multiplicity of nontrivial radial solutions of (P) with sub-p-linear nonlinearity.

Now we state the main results of this paper. Based on the radial lemmas cited in Section 2 we first define the bottom index q_* and the top index q^* for the embedding so that these indices satisfy the prior inequalities $1 \leqslant q_* < p$ and $1 < q^* \leqslant p$.

According to the behaviors of V and Q near infinity, the bottom index q_* can be defined in the following regions of (a, b) and such that $1 \leqslant q_* < p$ (see Figure 1 below):

$A_1^\infty = \{(a,b) \mid b \geqslant \frac{p(N-1)+a(p-1)}{p} - N,\ \ \frac{N-p}{p} - N \leqslant b < -p\}$,

$A_2^\infty = \{(a,b) \mid b < \frac{p(N-1)+a(p-1)}{p^2} - N,\ \ \frac{N-p}{p} - N \leqslant b < -p\}$,

$A_3^\infty = \{(a,b) \mid a \leqslant -p,\ \ \frac{N-p}{p} - N < b < \frac{p(N-1)+a(p-1)}{p} - N\}$,

$A_4^\infty = \{(a,b) \mid b \leqslant \frac{N-p}{p} - N,\ \ \frac{p(N-1)+a(p-1)}{p^2} - N \leqslant b < \frac{p(N-1)+a(p-1)}{p} - N\}$,

$A_5^\infty = \{(a,b) \mid a \geqslant -p,\ \ \frac{p(N-1)+a(p-1)}{p^2} - N \leqslant b < \frac{p(N-1)+a(p-1)}{p} - N\}$.

Define

$$q_* = \begin{cases} \dfrac{p(b+N)}{N-p}, & (a,b) \in A_i^\infty,\ i = 1,2,3, \\[2ex] \dfrac{p^2(b+N)}{p(N-1)+a(p-1)}, & (a,b) \in A_i^\infty,\ i = 4,5. \end{cases}$$

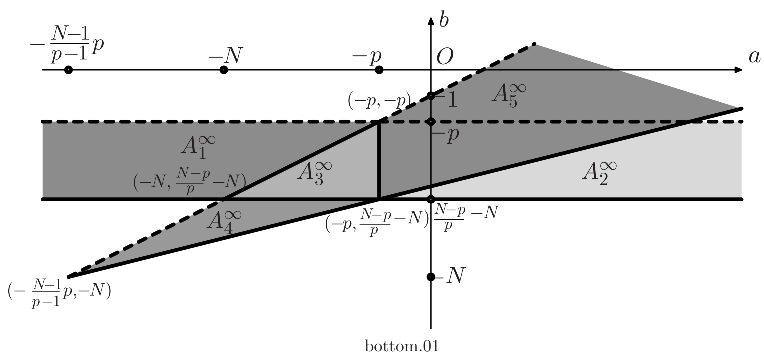

Figure 1. Regions for bottom index

According to the behaviors of V and Q near zero, the top index q^* can be defined in the following regions of (a_0, b_0) and such that $1 < q^* \leqslant p$ (see Figure 2 below).

$A_1^0 = \{(a_0,b_0) \mid b_0 > \frac{p(N-1)+a_0(p-1)}{p} - N,\ \ \frac{N-p}{p} - N < b_0 \leqslant -p\}$,

$A_2^0 = \{(a_0,b_0) \mid a_0 \geqslant -p,\ \ \frac{N-p}{p} - N < b_0 \leqslant -p\}$,

$A_3^0 = \{(a_0,b_0) \mid a_0 < -p,\ \ \frac{N-p}{p} - N < b_0 \leqslant \frac{p(N-1)+a_0(p-1)}{p} - N\}$,

$A_4^0 = \{(a_0,b_0) \mid b_0 < \frac{N-p}{p} - N,\ \ \frac{p(N-1)+a_0(p-1)}{p^2} - N < b_0 \leqslant \frac{p(N-1)+a_0(p-1)}{p} - N\}$,

$A_5^0 = \{(a_0,b_0) \mid b_0 \geqslant -p,\ \ \frac{p(N-1)+a_0(p-1)}{p^2} - N < b_0 \leqslant \frac{p(N-1)+a_0(p-1)}{p} - N\}$.

Define

$$q^* = \begin{cases} \dfrac{p(b_0+N)}{N-p}, & (a_0,b_0) \in A_i^0,\ i = 1,2, \\[2ex] \dfrac{p^2(b_0+N)}{p(N-1)+a_0(p-1)}, & (a_0,b_0) \in A_i^0,\ i = 3,4,5. \end{cases}$$

In the region $A_6^0 = \{(a_0, b_0) \mid \frac{p(N-1)+a_0(p-1)}{p} - N < b_0 \leqslant \frac{p(N-1)+a_0(p-1)}{p^2} - N\}$, define

$$q_{**} = \frac{p^2(b_0 + N)}{p(N-1) + a_0(p-1)}.$$

One sees that $1 \leqslant q_{**} < p$.

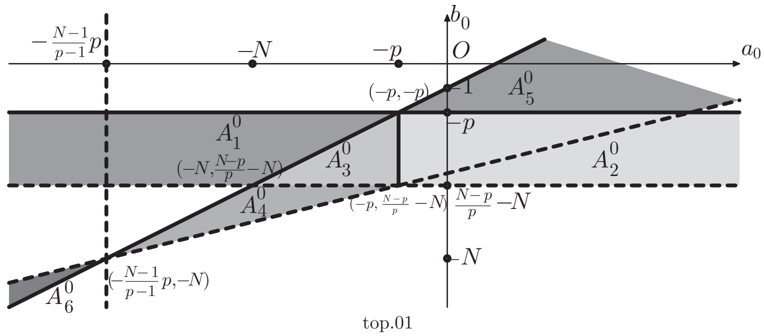

FIGURE 2. Regions for top index

The embedding result reads as follows.

Theorem 1.2. *Let $1 < p < N$. Assume* (V) *and* (Q) *with q_* and q^* or q_{**} being defined. Then the embedding*

$$W_r^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N; Q)$$

is continuous and compact for $q_* < q < q^*$ or for $\max\{q_*, q_{**}\} < q < p$.

The idea for defining embedding indices is quite clear although it seems to be complicated apparently. To understand the embeddings, we look at a special case $a = a_0 = b = b_0 := \alpha$. In this case Theorem 1.2 reads as

$$W_r^{1,p}(\mathbb{R}^N, |x|^\alpha) \hookrightarrow L^q(\mathbb{R}^N, |x|^\alpha)$$

is continuous and compact for $q_* < q < q^*$, where

$$q_* = \frac{p(\alpha + N)}{N - p},\ q^* = \frac{p^2(\alpha + N)}{p(N-1) + \alpha(p-1)},\ \frac{N-p}{p} - N < \alpha < -p.$$

As an application of Theorem 1.2, we obtain the following existence results for (P).

Theorem 1.3. *Let $1 < p < N$. Assume* (V) *and* (Q) *with the corresponding q_* and q^* (resp., q_* and q_{**}) defined such that $q_* < q < q^*$ (resp., $p > q > \max\{q_*, q_{**}\}$). Then* (P) *has a ground state solution $u \in W_r^{1,p}(\mathbb{R}^N; V)$; namely, u satisfies*

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)\ u^p\ dx}{\left(\displaystyle\int_{\mathbb{R}^N} Q(|x|)\ u^q\ dx\right)^{\frac{p}{q}}} = \inf_{\substack{v \in W_r^{1,p}(\mathbb{R}^N;V)\\ v \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v|^p + V(|x|)|v|^p\ dx}{\left(\displaystyle\int_{\mathbb{R}^N} Q(|x|)\ |v|^q\ dx\right)^{\frac{p}{q}}}.$$

Moreover, (P) *has infinitely many radial solutions (u_k) in $W_r^{1,p}(\mathbb{R}^N; V)$ such that $\Phi(u_k) < 0$ and $\Phi(u_k) \to 0$ as $k \to \infty$.*

We give some comments. Nonlinear elliptic equations on the whole space with potentials have been extensively studied in the past decades with different aims. Most known works are concerned with the case $p = 2$ and such that the nonlinearity has a subcritical growth and satisfies a global Ambrosetti-Rabinowitz type condition ([2, 7]) from which the nonlinearity has a superlinear growth. The main reason may be that the embeddings used there are similar to the form $H^1 \hookrightarrow L^q$ with $q > 2$. We mention the works [3, 4], where some embeddings $H^1(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N)(2 \leqslant q < 2^* := \frac{2N}{N-2}, N \geqslant 3)$ with various potentials V were built. We refer to [12, 13] and the references therein for more comments and comparisons and some historical results. Due to the reason for the embedding, there are few works concerning elliptic problems on the whole space $\mathbb{R}^N$ with sublinear nonlinearity. We are only aware of the work [5] which studied Schrödinger equations with *concave* and convex nonlinearities. Note that in [5], the continuous embedding $H^1(\mathbb{R}^N; V) \hookrightarrow L^1(\mathbb{R}^N)$ was used where the potential V was not radial and satisfied $V \geqslant 1$ and $\int_{\mathbb{R}^N} V^{-1} dx < \infty$.

The classical Sobolev embedding theorem says that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is not compact for any q, but restricted to the radial case, $H^1_r(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ are compact for all $q \in (2, 2^*)$(see [10]) and it is not true for $q = 2$ and $q = 2^* = \frac{2N}{N-2}(N \geqslant 3)$. When radial potentials are involved, the compactness of the embeddings may be valid for a wider range of q. In [12, 13], the authors developed techniques and ideas in establishing weighted Sobolev type embeddings (one is Theorem 1.1) for singular radial potentials V and Q. Note that Theorem 1.1 includes some cases in which the embedding $W^{1,p}_r(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N; Q)$ is compact for all $p \leqslant q < \infty$. In the present paper, we further explore the effects of the potentials and extend the techniques and ideas developed in [12, 13] to establish the embedding $W^{1,p}_r(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N; Q)$ with $1 < q < p$. Theorem 1.2 is new and complements Theorem 1.1. The result for the case $p = 2$ complements some embedding results in [12, 8, 9]. It provides a basic variational tool to study the quasilinear elliptic equations on the whole space $\mathbb{R}^N$ with the nonlinearity having a global sub-p-linear growth. This implies that the functional Φ is coercive. For simplicity, Theorem 1.3 considers a special case of pure "power" nonlinearity $|u|^{q-2}u$. In fact we can consider the case with general sub-p-linear nonlinearities. See the comments in Section 3 where Theorem 1.3 is proved. In Section 2 we prove Theorem 1.2.

2. Proof of Theorem 1.2

In this section we give the proofs of the embedding Theorem 1.2. We use some notions. In the following, B_r denotes the open ball in $\mathbb{R}^N$ centered at 0 with radius r and $B_R \backslash B_r$ denotes the annulus with interior radius r and exterior radius R. For any set $A \subset \mathbb{R}^N$, A^c denotes the complement of A. We use C to denote different positive constants independent of functions in $W^{1,p}_r(\mathbb{R}^N; V)$. We recall some radial lemmas built in [13], where one can refer to the proofs of these lemmas and related comments.

Lemma 2.1 ([13]). *Let $1 < p < N$. There exists $\widehat{C} = \widehat{C}(N, p) > 0$ such that for any $u \in D^{1,p}_r(\mathbb{R}^N)$, the following inequality holds:*

$$|u(x)| \leqslant \widehat{C} |x|^{-\frac{N-p}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

Lemma 2.2 ([13])**.** *Assume* (V) *with* $a > -\frac{N-1}{p-1}p$. *Then there exists* $C > 0$ *such that for all* $u \in W_r^{1,p}(\mathbb{R}^N; V)$,

$$|u(x)| \leqslant C|x|^{-\frac{p(N-1)+a(p-1)}{p^2}} \ \|u\|_{W_r^{1,p}(\mathbb{R}^N;V)}, \quad |x| \gg 1.$$

Lemma 2.3 ([13])**.** *Assume* (V). *Then there exists* $r_0 > 0$ *and* $\widetilde{C}_0 > 0$ *such that for all* $u \in W_r^{1,p}(\mathbb{R}^N; V) \cap D_0^{1,p}(B_{r_0}(0))$,

$$|u(x)| \leqslant \widetilde{C}_0 \ |x|^{-\frac{p(N-1)+a_0(p-1)}{p^2}} \ \|u\|_{W_r^{1,p}(\mathbb{R}^N;V)}, \quad 0 < |x| \leqslant r_0.$$

Lemma 2.4 ([13])**.** *Let* $1 < p < N$, $1 \leqslant q \leqslant \infty$. *Then for any* $0 < r < R < \infty$ *with* $R \gg 1$, *the following embedding is compact:*

$$W_r^{1,p}(B_R \backslash B_r; V) \hookrightarrow L^q(B_R \backslash B_r; Q).$$

Now we give the proofs.

Proof of Theorem 1.2. We will show the continuity of the embedding by proving

$$S_r(V, Q) := \inf_{u \in W_r^{1,p}(\mathbb{R}^N;V)} \frac{\displaystyle\int_{\mathbb{R}^N} (|\nabla u|^p + V(|x|)\ |u|^p)\ dx}{\left(\displaystyle\int_{\mathbb{R}^N} Q(|x|)\ |u|^q\ dx\right)^{\frac{p}{q}}} > 0. \tag{2.1}$$

Assume that

$$S_r(V, Q) = 0.$$

Then there exists a sequence $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N; V)$ such that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^p + V(|x|)|u_n|^p)\ dx = o(1) \quad \text{as} \quad n \to \infty, \tag{2.2}$$

$$\int_{\mathbb{R}^N} Q(|x|)|u_n|^q dx \equiv 1, \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

We will obtain a contradiction with (2.3) by proving that (2.2) implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} Q(|x|)|u_n|^q dx = 0. \tag{2.4}$$

By (V) and (Q), there exist constants $R_0 > r_0 > 0$ such that for some $C_0 > 0$,

$$V(|x|) \geqslant C_0|x|^a, \quad Q(|x|) \leqslant C_0|x|^b, \quad \text{for} \quad |x| \geqslant R_0;$$
$$V(|x|) \geqslant C_0|x|^{a_0}, \quad Q(|x|) \leqslant C_0|x|^{b_0}, \text{ for } 0 < |x| \leqslant r_0.$$

We will separate $\mathbb{R}^N$ into the following three areas and estimate the integrals:

$$\int_{B_r} Q(|x|)|u_n|^q dx, \quad \int_{B_R\backslash B_r} Q(|x|)|u_n|^q dx, \quad \int_{B_R^c} Q(|x|)|u_n|^q dx$$

for $R \geqslant R_0$ and $r \leqslant r_0$. It follows from Lemma 2.4 that for any fixed $R > r > 0$, (2.2) implies that

$$\int_{B_R\backslash B_r} Q(|x|)|u_n|^q dx \to 0 \quad \text{as} \quad n \to \infty. \tag{2.5}$$

Hence we need to estimate the integrals on $0 < |x| \leqslant r$ and $|x| \geqslant R$.

We estimate $\int_{B_R^c} Q(|x|)|u_n|^q dx$ in different regions A_i^∞ of (a,b) given in the introduction. For $(a,b)\in A_i^\infty (i=1,2,3)$, $q>q_*=\frac{p(b+N+1)}{N-p}$ implies that

$$\alpha_i^\infty := b+N-\frac{N-p}{p}q<0, \quad i=1,2,3.$$

By Lemma 2.1, we have

$$\begin{aligned}\int_{B_R^c} Q(|x|)|u_n|^q dx &\leqslant C_0\int_{B_R^c}|x|^b|u_n|^q dx\\ &\leqslant C\int_{B_R^c}|x|^{b-\frac{N-p}{p}q}\|\nabla u_n\|_{L^p}^q dx\\ &= C\|\nabla u_n\|_{L^p}^q\int_{B_R^c}|x|^{b-\frac{N-p}{p}q}dx\\ &\leqslant CR^{b+N-\frac{N-p}{p}q}\|\nabla u_n\|_{L^p}^q\\ &\leqslant CR^{b+N-\frac{N-p}{p}q}\|u_n\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}.\end{aligned}\tag{2.6}$$

For $(a,b)\in A_i^\infty (i=4,5)$, $q>q_*=\frac{p^2(b+N)}{p(N-1)+a(p-1)}$ implies that

$$\alpha_i^\infty := b+N-\frac{p(N-1)+a(p-1)}{p^2}q<0, \quad i=4,5.$$

By Lemma 2.2, we have

$$\begin{aligned}\int_{B_R^c} Q(|x|)|u_n|^q dx &\leqslant C_0\int_{B_R^c}|x|^b|u_n|^q dx\\ &\leqslant C\|u_n\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}\int_{B_R^c}|x|^{b-\frac{p(N-1)+a(p-1)}{p^2}q}dx\\ &= C\ R^{b+N-\frac{p(N-1)+a(p-1)}{p^2}q}\|u_n\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}.\end{aligned}\tag{2.7}$$

From (2.6) and (2.7) we see that for any fixed $R\geqslant R_0$, (2.2) implies that

$$\int_{B_R^c} Q(|x|)|u_n|^q dx\to 0 \quad \text{as} \quad n\to\infty. \tag{2.8}$$

Next we estimate $\int_{B_r} Q(|x|)|u_n|^q dx$ in different regions A_i^0 of (a_0,b_0) given in Section 1. For $(a_0,b_0)\in A_i^0 (i=1,2)$, $q<q^*=\frac{p(b_0+N)}{N-p}$ implies that

$$\beta_i^0 := b_0+N-\frac{N-p}{p}q>0, \quad i=1,2,3.$$

By Lemma 2.1, we have that

$$\begin{aligned}\int_{B_r} Q(|x|)|u_n|^q dx &\leqslant C_0 \textstyle\int_{B_r} |x|^{b_0}|u_n|^q dx \\ &\leqslant C\int_{B_r} |x|^{b_0-\frac{N-p}{p}q}\|\nabla u_n\|_{L^p}^q dx \\ &= C\|\nabla u_n\|_{L^p}^q \int_{B_r} |x|^{b_0-\frac{N-p}{p}q} dx \\ &\leqslant C\, r^{b_0+N-\frac{N-p}{p}q}\|\nabla u_n\|_{L^p}^q \\ &\leqslant Cr^{b_0+N-\frac{N-p}{p}q}\|u_n\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}.\end{aligned} \tag{2.9}$$

For $(a_0,b_0)\in A_i^0(i=3,4,5)$, $q<q^*=\frac{p^2(b_0+N)}{p(N-1)+a_0(p-1)}$ implies that

$$\beta_i^0 := b_0+N-\frac{p(N-1)+a_0(p-1)}{p^2}q>0, \quad i=4,5.$$

We choose a cutoff function ϕ such that $\phi(x)=1$ for $0\leqslant|x|\leqslant\frac{r_0}{2}$, and $\phi(x)=0$ for $|x|\geqslant r_0$. By Lemma 2.2, for $0<r\leqslant\frac{r_0}{2}$, we have

$$\begin{aligned}\int_{B_r} Q(|x|)|u_n|^q dx &\leqslant C_0\int_{B_r} |x|^{b_0}|\phi u_n|^q dx \\ &\leqslant C\|\phi u_n\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}\int_{B_r}|x|^{b_0-\frac{p(N-1)+a_0(p-1)}{p^2}q}dx \\ &= C\, r^{b_0+N-\frac{p(N-1)+a_0(p-1)}{p^2}q}\|u_n\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}.\end{aligned} \tag{2.10}$$

For $(a_0,b_0)\in A_6^0$, $q>q_{**}=\frac{p^2(b_0+N)}{p(N-1)+a_0(p-1)}$ implies that

$$\beta_5^0 := b_0+N-\frac{p(N-1)+a_0(p-1)}{p^2}q>0.$$

We choose a cutoff function ϕ such that $\phi(x)=1$ for $0\leqslant|x|\leqslant\frac{r_0}{2}$, and $\phi(x)=0$ for $|x|\geqslant r_0$. By Lemma 2.2, for $0<r<\frac{r_0}{2}$, we have

$$\begin{aligned}\int_{B_r} Q(|x|)|u_n|^q dx &\leqslant C_0\int_{B_r} |x|^{b_0}|\phi u_n|^q dx \\ &\leqslant C\|\phi u_n\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}\int_{B_r}|x|^{b_0-\frac{p(N-1)+a_0(p-1)}{p^2}q}dx \\ &= C\, r^{b_0+N-\frac{p(N-1)+a_0(p-1)}{p^2}q}\|u_n\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}.\end{aligned} \tag{2.11}$$

From (2.9), (2.10) and (2.11) we see that for any fixed $0<r\leqslant r_0/2$, (2.2) implies that

$$\int_{B_r} Q(|x|)|u_n|^q dx \to 0 \quad \text{as} \quad n\to\infty. \tag{2.12}$$

Hence (2.4) holds, which contradicts (2.3). Therefore the embedding is continuous when $q_*<q<q^*$ or $\max\{q_*,q_{**}\}<q$.

We further prove that the embedding is compact for $q_* < q < q^*$ or $\max\{q_*, q_{**}\} < q$. It suffices to prove that for any $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N; V)$,

$$u_n \rightharpoonup 0 \quad \text{in} \quad W_r^{1,p}(\mathbb{R}^N; V)$$

implies that

$$\int_{\mathbb{R}^N} Q(|x|)|u_n|^q dx \to 0, \quad \text{as} \quad n \to \infty. \tag{2.13}$$

Since $\|u_n\|_{W_r^{1,p}(\mathbb{R}^N;V)}$ is bounded and the exponents α_i^∞ of R in (2.6) and (2.7) are negative and the exponents β_i^0 of r in (2.9)-(2.11) are positive, for any $\epsilon > 0$, by choosing R large enough and r small enough, we have that

$$\int_{B_R^c} Q(|x|)|u_n|^q dx < \frac{\epsilon}{3}, \qquad \int_{B_r} Q(|x|)|u_n|^q dx < \frac{\epsilon}{3}, \quad n \in \mathbb{N}. \tag{2.14}$$

Fixing R and r and applying Lemma 2.4 on $r \leqslant |x| \leqslant R$, we see that for $n \in \mathbb{N}$ large enough,

$$\int_{B_R \setminus B_r} Q(|x|)|u_n|^q dx < \frac{\epsilon}{3}. \tag{2.15}$$

Therefore (2.13) holds and then the embedding is compact. The proof is finished. □

We give some comments and comparisons. From the proof of Theorem 1.2, one sees that the continuity of the embedding has not been obtained for $q = q^*$ and $q = q_*$ when $q^*, q_* \in (1, p)$. This is due to the convergence of integrals over B_r and B_R^c. An interesting problem is whether the embedding will be continuous, even be compact, for $q = q^*$ or $q = q_*$. The compact embedding for $q = p$ has been included in Theorem 1.1, and its proof is based on the Hardy inequality (see [13]). Combining Theorem 1.2 and the embedding results in [13], we see that in some cases, the embedding $W_r^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N; Q)$ is compact for all $q \in (1, \infty)$.

We also note here that although we have required the potential V to be positive on $(0, \infty)$, it is enough for V to be negative somewhere provided $\int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p dx$ is well-defined to be nonnegative, from which a norm on $W_r^{1,p}(\mathbb{R}^N; V)$ can be deduced. In fact, from the Hardy inequality, the condition on V is enough for this aim:

$$\gamma := \inf_{r>0} V(r)\, r^p > -\left(\frac{N-p}{p}\right)^p.$$

3. Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3. By the embedding Theorem 1.2, the functional

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p dx - \frac{1}{q} \int_{\mathbb{R}^N} Q(|x|)|u|^q dx$$

is well-defined and is of C^1 with derivative given by, for $u, v \in W_r^{1,p}(\mathbb{R}^N; V)$,

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla v + V(|x|)|u|^{p-2} uv) dx - \int_{\mathbb{R}^N} Q(|x|)|u|^{q-2} uv dx.$$

Therefore a weak solution $u \in W_r^{1,p}(\mathbb{R}^N; V)$ of (P) is exactly a critical point of Φ.

Set the functional

$$\Psi(u)=\frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p+V(|x|)|u|^p dx,\quad u\in W_r^{1,p}(\mathbb{R}^N;V).$$

Then Ψ' possesses the (S^+) property on $W_r^{1,p}(\mathbb{R}^N;V)$ (cf.[6, 11]); that is, for any sequence $\{u_n\}\subset W_r^{1,p}(\mathbb{R}^N;V)$, if $u_n\rightharpoonup u$ weakly in $W_r^{1,p}(\mathbb{R}^N;V)$ and $\limsup_{n\to\infty}\langle\Psi'(u_n),u_n-u\rangle\leqslant 0$, then $u_n\to u$ in $W_r^{1,p}(\mathbb{R}^N;V)$ strongly.

Now we begin to prove Theorem 1.3.

By the compactness of the embedding, the minimum

$$S_{r,V,Q}:=\inf_{u\in W_r^{1,p}(\mathbb{R}^N;V)}\frac{\displaystyle\int_{\mathbb{R}^N}(|\nabla u|^p+V(|x|)\ |u|^p)\ dx}{\left(\displaystyle\int_{\mathbb{R}^N}Q(|x|)\ |u|^q\ dx\right)^{\frac{p}{q}}}>0$$

is achieved at some $\bar u\in W_r^{1,p}(\mathbb{R}^N;V)$ which is positive. After rescaling, we can get one nontrivial weak solution of (P) which is given by $u^*=\lambda\bar u$ with $\lambda=(\frac{q}{p}\ S_{r,V,Q})^{\frac{1}{q-p}}$.

We prove the existence of one nontrivial solution of (P) in another way by showing that the functional Φ is coercive and bounded from below and then has a global minimum.

We first show that Φ is coercive. Denote by C_r the constant of the embedding $W_r^{1,p}(\mathbb{R}^N;V)\hookrightarrow L^q(\mathbb{R}^N;Q)$. For $u\in W_r^{1,p}(\mathbb{R}^N;V)$, we have

$$\begin{aligned}\Phi(u)=&\ \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p+V(|x|)|u|^p)dx-\frac{1}{q}\int_{\mathbb{R}^N}Q(|x|)|u|^q dx\\ =&\ \frac{1}{p}\|u\|^p_{W_r^{1,p}(\mathbb{R}^N;V)}-\frac{1}{q}\|u\|^q_{L^q(\mathbb{R}^N;Q)}\\ \geqslant&\ \frac{1}{p}\|u\|^p_{W_r^{1,p}(\mathbb{R}^N;V)}-\frac{1}{q}\ C_r^q\|u\|^q_{W_r^{1,p}(\mathbb{R}^N;V)}.\end{aligned}$$

Since $q<p$, we have

$$\Phi(u)\to+\infty\quad\text{as}\quad\|u\|_{W_r^{1,p}(\mathbb{R}^N;V)}\to\infty.\tag{3.1}$$

Next we show that

$$m:=\inf_{u\in W_r^{1,p}(\mathbb{R}^N;V)}\Phi(u)$$

is achieved. Let $\{u_n\}$ be a minimizing sequence of m such that

$$\Phi(u_n)\to m,\quad n\to\infty.\tag{3.2}$$

By the coercivity (3.1), we see that $\{u_n\}$ is bounded. Since $W_r^{1,p}(\mathbb{R}^N;V)$ is reflexive, up to a subsequence if necessary, we assume that there is $\bar u\in W_r^{1,p}(\mathbb{R}^N;V)$ such that

$$u_n\rightharpoonup\bar u,\quad\text{weakly as}\quad n\to\infty.$$

By the compactness of the embedding, we have that

$$u_n\to\bar u\quad\text{strongly in}\quad L^q(\mathbb{R}^N;Q),\quad n\to\infty.$$

Thus

$$\int_{\mathbb{R}^N}Q(|x|)|u_n|^q dx\to\int_{\mathbb{R}^N}Q(|x|)|\bar u|^q dx,\quad n\to\infty,$$

which, combined with the weakly lower semi-continuity of Ψ, we use to see that Φ is weakly lower semi-continuous and then

$$m \leqslant \Phi(\bar{u}) \leqslant \liminf_{n\to\infty} \Phi(u_n) = m.$$

Therefore

$$\Phi(\bar{u}) = m$$

and m is achieved. Since Φ is a C^1 functional and has m as a global minimum, we have that

$$\Phi'(\bar{u}) = 0$$

and $\bar{u}$ is a weak solution of (P).

Now we prove that $\bar{u} \neq 0$ by showing that $m < 0$ as $\Phi(0) = 0$.

Take a fixed function $v \in W_r^{1,p}(\mathbb{R}^N; V)$ and $v \neq 0$. Then $v \neq 0$ in $L^q(\mathbb{R}^N; Q)$. Set

$$a = \frac{1}{p}\|v\|^p_{W_r^{1,p}(\mathbb{R}^N;V)}, \quad b = \frac{1}{q}\|v\|^q_{L^q(\mathbb{R}^N;Q)}.$$

For $t > 0$, as $p > q$,

$$\Phi(tv) = at^p - bt^q < 0 \quad \text{for all} \quad 0 < t < (b/a)^{\frac{1}{p-q}}.$$

Take $u^* = t^* v$ for some $t^* \in (0, (b/a)^{\frac{1}{p-q}})$; then

$$m = \Phi(\bar{u}) \leqslant \Phi(u^*) < 0.$$

The proof of existence is finished.

To prove the multiplicity in Theorem 1.3, we employ an abstract critical point theorem built in [14].

Proposition 3.1 ([14])**.** *Let $\Phi \in C^1(X, \mathbb{R})$, where X is a Banach space. Assume that Φ satisfies the* (PS) *condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$ there exists a k-dimensional subspace X_k and $\rho_k > 0$ such that*

$$\sup_{X_k \cap S_{\rho_k}} \Phi < 0, \tag{3.3}$$

where $S_\rho = \{u \in X \mid \|u\| = \rho\}$, then Φ has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$.

It is clear that Φ is even, $\Phi(0) = 0$, and from the above arguments, we see that Φ is bounded from below. We need to verify that Φ satisfies (PS) and (3.3).

Let $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N; V)$ be such that $\Phi(u_n)$ is bounded and

$$\Phi'(u_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{3.4}$$

By the coercivity (3.1), $\{u_n\}$ is bounded. By the reflexivity of $W_r^{1,p}(\mathbb{R}^N; V)$ and the compactness of the embedding, up to a subsequence if necessary, we may assume that there is $u \in W_r^{1,p}(\mathbb{R}^N; V)$ such that

$$u_n \rightharpoonup u, \quad \text{weakly in} \quad W_r^{1,p}(\mathbb{R}^N; V), \quad n \to \infty, \tag{3.5}$$

and

$$u_n \to u, \quad \text{strongly in} \quad L^q(\mathbb{R}^N; Q), \quad n \to \infty. \tag{3.6}$$

By (3.4)–(3.6), we have that

$$\begin{aligned}\langle \Psi'(u_n), u_n - u\rangle &= \langle \Phi'(u_n), u_n - u\rangle + \int_{\mathbb{R}^N} Q(|x|)|u_n|^{q-2}u_n(u_n - u)dx \\ &\to 0, \quad n \to \infty.\end{aligned}$$

Since Ψ' possesses the (S^+) property, we have that

$$u_n \to u, \quad \text{strongly in} \quad W_r^{1,p}(\mathbb{R}^N; V), \quad n \to \infty$$

and then Φ satisfies (PS).

We verify (3.3). For any $k \in \mathbb{N}$, we choose k independent smooth functions $\phi_i \in C_{0,r}^\infty(\mathbb{R}^N)$ for $i = 1, 2, \ldots, k$ and define $X_k = \text{span}\{\phi_1, \phi_2, \ldots, \phi_k\}$. Then by the formula

$$\Phi(u) = \frac{1}{p}\|u\|^p_{W_r^{1,p}(\mathbb{R}^N;V)} - \frac{1}{q}\|u\|^q_{L^q(\mathbb{R}^N;Q)}$$

and the fact that all norms on X_k are equivalent, we get that for $\rho_k > 0$ small enough,

$$\sup_{u \in X_k \cap S_{\rho_k}} \Phi(u) < 0.$$

With all conditions of Proposition 3.1 being verified, we get the conclusion that Φ has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$. The proof is finished. $\square$

We finish the paper with some more remarks. First, one can apply the embedding theorem, Theorem 1.2, to consider general cases of (P) by replacing the term $|u|^{q-2}u$ with a general continuous function $f(u)$ satisfying the assumption

$$\begin{cases} |f(u)| \leqslant C(|u|^{q_1-1} + |u|^{q_2-1}), \ q_* < q_1 \leqslant q_2 < q^*, \text{or} \max\{q_*, q_{**}\} < q_1 \leqslant q_2 < p, \\ \\ \mu F(u) = \mu \int_0^u f(t)dt \geqslant uf(u) > 0, \quad \text{for} \quad u \neq 0, \quad 1 < \mu < p. \end{cases}$$

We leave the precise statements to interested readers.

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School of Mathematical Sciences, Capital Normal University, Beijing 100048, People's Republic of China

Department of Mathematics and Statistics, Utah State University, Logan, Utah 84322