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ON THE SET WHERE THE ITERATES OF AN ENTIRE FUNCTION ARE BOUNDED

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Abstract. We show that for a transcendental entire function the set of points whose orbit under iteration is bounded can have arbitrarily small positive Hausdorff dimension.

1. Introduction

The main objects studied in complex dynamics are the *Fatou set* $F(f)$ of a rational, entire or, more generally, meromorphic function f, defined as the set of all points where the iterates f^n of f form a normal family, and the *Julia set* $J(f)$, which is the complement of $F(f)$. In the dynamics of transcendental entire functions – and this is the case we shall be concerned with – a fundamental role is also played by the escaping set

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

The first systematic study of this set was undertaken by Eremenko [9] who, among other results, showed that $I(f) \neq \emptyset$ and in fact $I(f) \cap J(f) \neq \emptyset$ for every transcendental entire function f. Moreover, $J(f) = \partial I(f)$. In this paper we will consider the set

$$K(f) = \{z \in \mathbb{C} : (f^n(z)) \text{ is bounded}\}.$$

As repelling periodic points are dense in the Julia set [1], the properties of $I(f)$ mentioned above also hold for $K(f)$; that is, $K(f) \cap J(f) \neq \emptyset$ and $J(f) = \partial K(f)$.

For a polynomial f the set $K(f)$ is called the *filled Julia set* of f and we have $K(f) = \mathbb{C} \setminus I(f)$, but for a transcendental entire function f there are points which are neither in $K(f)$ nor in $I(f)$. For example, there are points in $J(f)$ whose orbit is dense in $J(f)$. However, there may also be points in $F(f)$ which are neither in $K(f)$ nor in $I(f)$; see [10, Example 1].

We denote the Hausdorff dimension and the packing dimension of a subset A of $\mathbb{C}$ by $\dim_{\mathrm{H}} A$ and $\dim_{\mathrm{P}} A$, respectively. We refer to Falconer's book [11] for the definition of these dimensions and further information. Here we only note that we always have $\dim_{\mathrm{H}} A \leq \dim_{\mathrm{P}} A$; see [11, p. 48]. By a result of Baker [2], the Julia set of a transcendental entire function f contains continua. In fact, even $I(f) \cap J(f)$

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contains continua and thus $\dim_H(I(f) \cap J(f)) \geq 1$; cf. [16, Theorem 5] and [18, Theorem 1.3].

A major open question in transcendental dynamics is whether $\dim_H J(f) > 1$ for every transcendental entire function f. It was proved by Stallard ([20], see also [7, 17]) that this is the case for functions in the Eremenko-Lyubich class B which consists of all transcendental entire functions for which the set of critical values and finite asymptotic values is bounded. Barański, Karpińska and Zdunik [3] showed that for $f \in B$ there exists a compact, invariant Cantor subset C of $J(f)$ with $\dim_{\mathrm{H}} C > 1$. In particular, $\dim_{\mathrm{H}}(K(f) \cap J(f)) > 1$ for $f \in B$. On the other hand, Rempe and Stallard [15] showed that there are functions $f \in B$ for which $\dim_{\mathrm{H}} I(f) = 1$.

We consider the dimensions of $K(f)$ for entire functions which need not be in the Eremenko-Lyubich class. The following result is a special case of a result of Rempe [14, Corollary 2.11], who proved that the hyperbolic dimension of an Ahlfors islands map is positive.

Theorem 1. *If f is a transcendental entire function, then* $\dim_{\mathrm{H}}(K(f) \cap J(f)) > 0$.

Theorem 1 is also implicit in Stallard's [19] proof that $\dim_{\mathrm{H}} J(f) > 0$ for transcendental meromorphic functions f. The proofs in [14, 19] are based on suitable versions of the Ahlfors islands theorem; see [12, Theorem 6.2] or, for an alternative proof, [5]. This is used to obtain an iterated function scheme (see [11]), whose invariant set is a (hyperbolic) Cantor subset of $K(f) \cap J(f)$, which can be shown to have positive Hausdorff dimension. We note, however, that for entire and meromorphic functions different versions of the Ahlfors islands theorem have to be used; see the discussion in [6, Section 6.4]. For entire functions such a hyperbolic, invariant Cantor subset of $K(f) \cap J(f)$ is also constructed in [8].

It is the purpose of this paper to show that Theorem 1 is best possible.

Theorem 2. *For every $\varepsilon > 0$ there exists a transcendental entire function f such that* $\dim_{\mathrm{H}} K(f) \leq \dim_P K(f) < \varepsilon$.

For an introduction to the dynamics of transcendental entire (and meromorphic) functions we refer to [4]. Results on the dimensions of Julia sets of transcendental functions are surveyed in [21].

2. Proof of Theorem 2

Let C be a large positive constant and define $(a_k)_{k \geq 1}$ recursively by $a_1 = 1$ and

$$a_{k+1} = 8C\, a_k \prod_{j=1}^{k-1} \frac{a_k}{a_j} \tag{1}$$

for $k \geq 1$. (Here $\prod_{j=1}^{0} a_1/a_j = 1$ so that $a_2 = 8Ca_1 = 8C$.) Induction shows that (a_k) increases and that

$$\frac{a_{k+1}}{a_k} \geq 8C \prod_{j=1}^{k-1} \frac{a_k}{a_{k-1}} \geq (8C)^k \tag{2}$$

for all k. Thus

$$f(z) = C\, z \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right)$$

defines an entire function f. For $k \geq 1$ we put

$$r_k = \frac{2k+1}{2k+2} a_k \quad \text{and} \quad s_k = 10 a_k$$

and we set $r_0 = 0$ and $s_0 = 16/C$. For large C we have $r_k < s_k < r_{k+1}$ for $k \geq 0$. We define, for $k \geq 0$,

$$A_k = \{z \in \mathbb{C} : r_k \leq |z| \leq s_k\} \quad \text{and} \quad B_k = \{z \in \mathbb{C} : s_k < |z| < r_{k+1}\}.$$

We will show that

$$f(B_k) \subset B_{k+1} \tag{3}$$

for all $k \geq 1$. In order to do so we note first that by (2) we can achieve that

$$\frac{a_{k+1}}{a_k} > 320e(k+1) \geq 2k+4 \tag{4}$$

for all $k \geq 1$ by choosing C sufficiently large. We deduce that if $1 \leq j \leq k-1$, then $(2k+2)a_j \leq (2k+2)a_{k-1} \leq a_k$ and hence

$$1 + \frac{r_k}{a_j} \leq \frac{a_k}{(2k+2)a_j} + \frac{r_k}{a_j} = \frac{a_k}{a_j} \tag{5}$$

and

$$\frac{r_k}{a_j} - 1 \geq \frac{r_k}{a_j} - \frac{a_k}{(2k+2)a_j} = \frac{k}{k+1}\frac{a_k}{a_j}. \tag{6}$$

Moreover, it follows from (2) that we can achieve that

$$\prod_{j=k+1}^{\infty} \left(1 + \frac{10a_k}{a_j}\right) \leq 2 \quad \text{and} \quad \prod_{j=k+1}^{\infty} \left(1 - \frac{10a_k}{a_j}\right) \geq \frac{9}{10} \geq \frac{1}{2} \tag{7}$$

for all $k \geq 1$ by choosing C large.

For $k \geq 1$ we deduce from (1), (5) and (7) that if $|z| = r_k$, then

$$\begin{aligned} |f(z)| &\leq C\, r_k \prod_{j=1}^{k-1} \left(1 + \frac{r_k}{a_j}\right) \cdot \left(1 + \frac{r_k}{a_k}\right) \cdot \prod_{j=k+1}^{\infty} \left(1 + \frac{r_k}{a_j}\right) \\ &\leq 4C\, a_k \prod_{j=1}^{k-1} \frac{a_k}{a_j} = \frac{1}{2} a_{k+1} < r_{k+1}. \end{aligned}$$

Similarly, (1), (4), (6) and (7) yield that if $|z| = r_k$, then

$$\begin{aligned} |f(z)| &\geq C\, r_k \prod_{j=1}^{k-1} \left(\frac{r_k}{a_j} - 1\right) \cdot \left(1 - \frac{r_k}{a_k}\right) \cdot \prod_{j=k+1}^{\infty} \left(1 - \frac{r_k}{a_j}\right) \\ &\geq C \left(\frac{k}{k+1}\right)^k a_k \prod_{j=1}^{k-1} \frac{a_k}{a_j} \cdot \frac{1}{2k+2} \cdot \frac{1}{2} \\ &\geq \frac{C}{2e(2k+2)} a_k \prod_{j=1}^{k-1} \frac{a_k}{a_j} = \frac{a_{k+1}}{32e(k+1)} > 10a_k = s_k. \end{aligned} \tag{8}$$

The last two inequalities imply that

$$f(z) \in B_k \quad \text{for } |z| = r_k \tag{9}$$

if $k \geq 1$. Next we note that if $k \geq 1$ and $|z| = s_k$, then

$$\begin{aligned} |f(z)| &\geq C\, s_k \prod_{j=1}^{k-1} \left(\frac{s_k}{a_j} - 1\right) \cdot \left(\frac{s_k}{a_k} - 1\right) \cdot \prod_{j=k+1}^{\infty} \left(1 - \frac{s_k}{a_j}\right) \\ &\geq 10C\, a_k \prod_{j=1}^{k-1} \frac{9a_k}{a_j} \cdot 9 \cdot \frac{9}{10} = \frac{9^{k+1}}{8} a_{k+1} > s_{k+1}. \end{aligned} \tag{10}$$

Similarly as in (7) we also see that if $|z| = s_0 = 16/C$, then

$$|f(z)| \geq C\, s_0 \prod_{j=1}^{\infty} \left(1 - \frac{s_0}{a_j}\right) \geq C\, s_0 \frac{9}{10} = \frac{16 \cdot 9}{10} > 10 = s_1, \tag{11}$$

provided C is chosen large enough. Also, since $s_k < r_{k+1}$ for all $k \geq 0$, we deduce from (9), with k replaced by $k+1$, that $|f(z)| < r_{k+2}$ for $|z| = s_k$. Together with (10) and (11) this yields that

$$f(z) \in B_{k+1} \quad \text{for } |z| = s_k \tag{12}$$

if $k \geq 0$. Combining this with (9) we obtain (3).

Next we show that with $L = C/(4e)$ we have

$$|f'(z)| \geq 2^k L \quad \text{for } z \in A_k. \tag{13}$$

In order to do so we note first that if p is a real polynomial with real zeros, then each interval bounded by two adjacent zeros of p contains exactly one zero of p', and besides multiple zeros of p there are no further zeros of p'. In particular, p' has only real zeros. Moreover, we see that p has no positive local minima and no negative local maxima.

Since our function f is a limit of real polynomials with real, nonnegative zeros, f' is also a limit of such polynomials. It follows that f' has no positive local minima and no negative local maxima. This implies that if a compact interval in $\mathbb{R}$ contains no zero of f', then $|f'|$ assumes its minimum in the interval at one of the endpoints of the interval. The fact that f' is a limit of real polynomials with real, nonnegative zeros also implies that $|f'|$ takes its minimum on a circle around the origin at the intersection of this circle with the positive real axis. We will see that f' has no zeros in the intervals $[r_k, s_k]$. The above arguments then imply that

$$\min_{z \in A_k} |f'(z)| = \min\{|f'(r_k)|, |f'(s_k)|\}. \tag{14}$$

In order to prove that f' has no zeros in the intervals $[r_k, s_k]$, we note that if $r_k \leq x < a_k$ and $1 \leq j \leq k-1$, then $x > 2a_j$ by (2) and hence $x/(x - a_j) < 2$. Thus

$$\begin{aligned} \frac{xf'(x)}{f(x)} &= 1 + \sum_{j=1}^{\infty} \frac{x}{x - a_j} \leq 1 + \sum_{j=1}^{k-1} \frac{x}{x - a_j} + \frac{r_k}{r_k - a_k} \\ &\leq 1 + 2(k-1) - (2k+1) = -2 < 0 \quad \text{for } r_k \leq x < a_k. \end{aligned} \tag{15}$$

On the other hand, using (2) it is not difficult to see that by choosing C large we can achieve that if $k \geq 1$, then

$$\frac{xf'(x)}{f(x)} \geq 1 - \sum_{j=k+1}^{\infty} \frac{s_k}{a_j - s_k} \geq \frac{1}{2} \quad \text{for } a_k < x \leq s_k. \tag{16}$$

With $a_0 = 0$ this also holds for $k = 0$ if C is large. It follows from (15) and (16) that f' has no zeros in the intervals $[r_k, s_k]$. Thus (14) holds. Moreover, (2), (8) and (15) yield that

$$|f'(r_k)| \geq 2\frac{|f(r_k)|}{r_k} \geq 2\frac{a_{k+1}}{32e(k+1)a_k} \geq \frac{(8C)^k}{16e(k+1)} \geq \frac{C}{4e}2^k = 2^k L \tag{17}$$

for $k \geq 1$ while (2), (10) and (16) give

$$|f'(s_k)| \geq \frac{1}{2}\frac{|f(s_k)|}{s_k} \geq \frac{1}{2}\frac{9^{k+1}a_{k+1}}{80a_k} \geq \frac{1}{2}\frac{9^{k+1}(8C)^k}{80} \geq 4C\,2^k \geq 2^k L \tag{18}$$

for $k \geq 1$. Finally, $f'(0) = C \geq L$ and (11) implies that

$$|f'(s_0)| \geq \frac{1}{2}\frac{s_1}{s_0} = \frac{10}{32}C \geq L. \tag{19}$$

Now (13) follows from (14), (17), (18) and (19).

To estimate the dimension of $K(f)$, we fix $N \in \mathbb{N}$ and put

$$K_N(f) = \{z \in \mathbb{C} : |f^n(z)| \leq s_N \text{ for all } n \in \mathbb{N}\}.$$

It follows from (3), and the fact that $r_0 = 0$, that $K_N(f)$ consists of all points z for which $f^n(z) \in \bigcup_{k=0}^N A_k$ for all $n \in \mathbb{N}$. Thus, assuming that C is chosen such that $L = C/(4e) > 1$, we deduce from (13) that $K_N(f)$ is a conformal repeller; see [13, Chapter 8] and [22, Chapter 5] for the definition and properties of conformal repellers. It follows (see [13, Corollary 8.1.7] or [22, Theorem 5.12]) that the Minkowski dimension, packing dimension and Hausdorff dimension of $K_N(f)$ all coincide and are given by Bowen's formula. This formula says that with $F = f|_{K_N(f)}$ these dimensions are equal to the unique zero of the pressure function $t \to P(F,t)$ defined by

$$P(F,t) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{z\in F^{-n}(a)} |(F^n)'(z)|^{-t}\right),$$

for some $a \in K_N(f)$.

In order to apply Bowen's formula we note that every point in $K_N(f)$ has $N+1$ preimages under F. Let $a \in A_k$. It follows from (9) and the maximum principle that F has no a-points in A_j for $0 \leq j \leq k-2$. Moreover, it follows from (9), (12) and the argument principle that F and $F - a$ have the same number of zeros in A_j for $k \leq j \leq N$. Thus F has exactly one a-point in A_j for $k \leq j \leq N$. We conclude that a has $k-1$ preimages under F in A_{k-1}. It follows from the above discussion, together with (13), that for $a \in K_N(f)$ and $t > 0$ we have

$$\sum_{b\in F^{-1}(a)} |F'(b)|^{-t} \leq \sum_{k=0}^{N} \left(2^k L\right)^{-t} \leq L^{-t}\sum_{k=0}^{\infty} 2^{-tk} = \frac{L^{-t}}{1-2^{-t}}.$$

Now

$$\begin{aligned}\sum_{z\in F^{-(n+1)}(a)} |(F^{n+1})'(z)|^{-t} &= \sum_{b\in F^{-1}(a)} \sum_{z\in F^{-n}(b)} |(F^{n+1})'(z)|^{-t} \\ &= \sum_{b\in F^{-1}(a)} |F'(b)|^{-t} \sum_{z\in F^{-n}(b)} |(F^n)'(z)|^{-t}.\end{aligned}$$

With

$$S_n(t) = \sup_{c \in K_N(f)} \sum_{z \in F^{-n}(c)} |(F^n)'(z)|^{-t},$$

we thus have

$$S_{n+1}(t) \leq \frac{L^{-t}}{1-2^{-t}} S_n(t).$$

Induction shows that

$$\sum_{z \in F^{-n}(a)} |(F^n)'(z)|^{-t} \leq S_n(t) \leq \left(\frac{L^{-t}}{1-2^{-t}} \right)^n \tag{20}$$

for all $a \in K_N(f)$. Thus

$$P(F,t) \leq \log \frac{L^{-t}}{1-2^{-t}}. \tag{21}$$

Given $t > 0$, we can achieve that the right-hand side of (21) is negative by choosing C and hence L large. Then the Minkowski, packing and Hausdorff dimensions of $K_N(f)$ are less than t for all N. Since $K(f) = \bigcup_{N=1}^{\infty} K_N(f)$, we deduce that $\dim_P K(f) \leq t$. As $t > 0$ can be chosen arbitrarily small, the conclusion follows.

Remark. The thermodynamic formalism of [13, 22] is not actually needed to obtain an *upper* bound for $\dim_{\mathrm{H}} K_N(f)$. Note that (13) implies that $K_N(f)$ does not intersect the postcritical set of F. Thus there exists $\delta > 0$ such that Koebe's distortion theorem may be applied to all inverse branches of the iterates of F on the disk $D(a,\delta) = \{z \in \mathbb{C} : |z-a| < \delta\}$. We obtain

$$F^{-n}(D(a,\delta)) \subset \bigcup_{z \in F^{-n}(a)} D\left(z, \frac{C}{|(F^n)'(z)|}\right)$$

for some constant C. Now (20) shows that $F^{-n}(D(a,\delta))$ can be covered by $(N+1)^n$ sets V_j whose diameters satisfy

$$\sum_j (\operatorname{diam} V_j)^t \leq (2C)^t \left(\frac{L^{-t}}{1-2^{-t}} \right)^n.$$

The compact set $K_N(f)$ can be covered by finitely many, say M, disks $D(a,\delta)$. Hence we obtain a covering of $K_N(f) = F^{-n}(K_N(f))$ by $M(N+1)^n$ sets W_j satisfying

$$\sum_j (\operatorname{diam} W_j)^t \leq M(2C)^t \left(\frac{L^{-t}}{1-2^{-t}} \right)^n.$$

This implies that the t-dimensional Hausdorff measure of $K_N(f)$ is 0, provided L is again chosen such that $L^{-t} < 1 - 2^{-t}$.

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References

[1] I. N. Baker, *Repulsive fixpoints of entire functions*. Math. Z. 104 (1968), 252–256. MR0226009 (37:1599)

[2] ———, *The domains of normality of an entire function*. Ann. Acad. Sci. Fenn. Ser. A I Math. 1 (1975), 277–283. MR0402044 (53:5867)

[3] K. Barański, B. Karpińska and A. Zdunik, *Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts*. Int. Math. Res. Not. IMRN 2009, 615–624. MR2480096 (2009k:37109)

[4] W. Bergweiler, *Iteration of meromorphic functions*. Bull. Amer. Math. Soc. (N.S.) 29 (1993), 151–188. MR1216719 (94c:30033)

[5] ———, *A new proof of the Ahlfors five islands theorem*. J. Anal. Math. 76 (1998), 337–347. MR1676971 (99m:30032)

[6] ———, *The role of the Ahlfors five islands theorem in complex dynamics*. Conform. Geom. Dyn. 4 (2000), 22–34. MR1741773 (2001c:37040)

[7] W. Bergweiler, P. J. Rippon and G. M. Stallard, *Dynamics of meromorphic functions with direct or logarithmic singularities*. Proc. London Math. Soc. 97 (2008), 368–400. MR2439666 (2010b:37122)

[8] J. P. R. Christensen and P. Fischer, *Ergodic invariant probability measures and entire functions*. Acta Math. Hungar. 73 (1996), 213–218. MR1415932 (97h:30041)

[9] A. E. Eremenko, *On the iteration of entire functions*. In "Dynamical Systems and Ergodic Theory". Banach Center Publications 23, Polish Scientific Publishers, Warsaw, 1989, pp. 339–345. MR1102727 (92c:30027)

[10] A. E. Eremenko and M. Ju. Ljubich, *Examples of entire functions with pathological dynamics*. J. London Math. Soc. (2) 36 (1987), 458–468. MR918638 (89e:30047)

[11] K. J. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*. John Wiley & Sons, Chichester, 1990. MR1102677 (92j:28008)

[12] W. K. Hayman, *Meromorphic Functions*. Clarendon Press, Oxford, 1964. MR0164038 (29:1337)

[13] F. Przytycki and M. Urbański, *Conformal Fractals: Ergodic Theory Methods*. London Math. Soc. Lect. Note Ser. 371. Cambridge Univ. Press, Cambridge, 2010. MR2656475 (2011g:37002)

[14] L. Rempe, *Hyperbolic dimension and radial Julia sets of transcendental functions*. Proc. Amer. Math. Soc. 137 (2009), 1411–1420. MR2465667 (2010h:37100)

[15] L. Rempe and G. M. Stallard, *Hausdorff dimensions of escaping sets of transcendental entire functions*. Proc. Amer. Math. Soc. 138 (2010), 1657-1665. MR2587450 (2011a:37097)

[16] P. J. Rippon and G. M. Stallard, *Escaping points of meromorphic functions with a finite number of poles*. J. Anal. Math. 96 (2005), 225–245. MR2177186 (2006f:30032)

[17] ———, *Dimensions of Julia sets of meromorphic functions with finitely many poles*. Ergodic Theory Dynam. Systems 26 (2006), 525–538. MR2218773 (2007c:37051)

[18] ———, *Fast escaping points of entire functions*. Preprint, arXiv:1009.5081v1.

[19] G. M. Stallard, *The Hausdorff dimension of Julia sets of meromorphic functions*. J. London Math. Soc. (2) 49 (1994), 281–295. MR1260113 (95b:58127)

[20] ———, *The Hausdorff dimension of Julia sets of entire functions. II*. Math. Proc. Cambridge Philos. Soc. 119 (1996), 513–536. MR1357062 (97a:58157)

[21] ———, *Dimensions of Julia sets of transcendental meromorphic functions*. In "Transcendental Dynamics and Complex Analysis". London Math. Soc. Lect. Note Ser. 348. Cambridge Univ. Press, Cambridge, 2008, pp. 425–446. MR2458811 (2010h:37101)

[22] M. Zinsmeister, *Thermodynamic Formalism and Holomorphic Dynamical Systems*. SMF/AMS Texts and Monographs 2. Amer. Math. Soc., Providence, RI; Soc. Math. France, Paris, 2000. MR1724307 (2000h:37063)

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