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POOR TRIVIALITY AND THE SAMENESS OF GROTHENDIECK SEMIRINGS

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Abstract. We provide a characterization of the complete theories, all of whose models have the same Grothendieck semiring as those in which a certain weaker version of piecewise definable triviality holds. We also list some examples and properties of this poor triviality.

1. Concepts and result

Fix a structure $\mathcal{A}$ in some first-order many-sorted language: in every sort s of the language, fix an element ξ_s; in one sort s_0, assume that there are two distinct elements ξ, ζ. Then let $\mathrm{Def}(\mathcal{A})$ be the category of all sets and functions which are definable *with parameters* in $\mathcal{A}$; the isomorphisms of $\mathrm{Def}(\mathcal{A})$ are the definable bijections. The *Grothendieck semiring* $\mathrm{K}_+(\mathcal{A})$ is the collection of isomorphism classes $[\cdot]$ of $\mathrm{Def}(\mathcal{A})$ endowed with the operations described below.

If X, Y are defined in the finite sequences of sorts σ, τ respectively, then $X \times Y$ is defined in $\sigma\tau$ and

$$X \amalg Y = \big(X \times \{\xi_\tau\} \times \{\xi\}\big) \cup \big(\{\xi_\sigma\} \times Y \times \{\zeta\}\big)$$

is defined in $\sigma\tau s_0$ with $\xi_{s_1 \ldots s_n} = (\xi_{s_1}, \ldots, \xi_{s_n})$. We can define

$$[X] + [Y] = [X \amalg Y],\ [X] \cdot [Y] = [X \times Y],\ 0 = [\emptyset],\ 1 = [\{\xi\}].$$

We refer to [2] for more on such concepts, but note the following: We do not require addition to be cancelative, so $\mathrm{K}_+(\mathcal{A})$ is not necessarily embeddable in a ring. Since we require enough elements to perform the above amalgamation, the underlying domain of $\mathrm{K}_+(\mathcal{A})$ consists only of the isomorphism classes $[X]$.

If $\mathcal{A} \preccurlyeq \mathcal{B}$, one can define $\mathrm{K}_+(\mathcal{A}) \to \mathrm{K}_+(\mathcal{B})$ via $[X(\mathcal{A})] \mapsto [X(\mathcal{B})]$, and this is an injective homomorphism. (For injectivity, quantify existentially over the parameters used to define a bijection in $\mathcal{B}$ to bring it down to $\mathcal{A}$.) There is no need for it to be surjective or for $\mathrm{K}_+(\mathcal{A})$ and $\mathrm{K}_+(\mathcal{B})$ to be the same semiring. We will study a property which ensures that surjectivity.

Triviality in terms of definability is presented in e.g. [1, Ch. 9], and we formalize it below. Among other things, it implies that, for a definable relation R, each fiber R_x is isomorphic to one of finitely many definable sets Z_i in a uniform way. In the above situation, a definable set in $\mathcal{B}$ can be written as $R(\mathcal{B})_b$ with 0-definable R

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and b in $\mathcal{B}$. Then, the isomorphism class $[R(\mathcal{B})_b]$ must equal some $[Z_i(\mathcal{B})]$, which is the image of $[Z_i(\mathcal{A})]$. This shows that the natural homomorphism is surjective. In this article, we study what is needed exactly of definable triviality to make this work. More generally, one proposed question in [2, Sec. 7] is the relation between Grothendieck rings of elementarily equivalent structures.

Henceforth, we omit the adjective "definable" before sets and functions whenever there is no conflict. Likewise, we say "(poor) triviality" instead of "piecewise definable (poor) triviality".

We say that the structure $\mathcal{A}$ has *poor triviality* if, for any $R \subseteq A_\sigma \times A_\tau$ (where σ, τ are also arbitrary sequences of sorts), A_σ can be partitioned into $X_1, \ldots, X_k$ for some $k \in \mathbb{N}^{\geqslant 1}$, and there are $Z_1, \ldots, Z_k \subseteq A_\theta$ for some sequence of sorts θ, and there is a formula ϕ (possibly with hidden parameters) such that for each $x \in X_i$ there is an adequate sequence of parameters a such that $\phi(x, a, \cdot, \cdot)$ defines a bijection $R_x \to Z_i$. In case there is, indeed, $F\colon A_{\sigma\tau} \to A_\theta$ such that $F(x, \cdot)$ restricts to a bijection $R_x \to Z_i$ whenever $x \in X_i$, then we say that $\mathcal{A}$ has *triviality*; the difference between the two properties lies in the dependence on the parameters a.

We say that a theory has poor triviality if all its models have poor triviality. Note that poor triviality is a property of $\mathrm{Th}(\mathcal{A})$ alone: One reduces poor triviality to the particular case of 0-definable relations (by including the parameters in the fiber indexes) and, since the latter case still allows for parameters in ϕ and in the definition of the sets X_i, Z_i, one quantifies existentially over them. Moreover, to assure our use of the property in elementary extensions, this method fixes X_i, Z_i for the given relation. Similarly, triviality is also a property of theories.

One can ask: Is triviality or poor triviality first-order axiomatizable? We believe not; the above method specifies the bijections defined by the formulas ϕ.

We saw that poor triviality yields $\mathcal{A} \preccurlyeq \mathcal{B} \Rightarrow \mathrm{K}_+(\mathcal{A}) \overset{\text{nat}}{\cong} \mathrm{K}_+(\mathcal{B})$. We prove:

Theorem 1.1. *Let T be a complete theory. The following are equivalent*:

(1) *If $\mathcal{A}, \mathcal{B} \models T$, then $\mathrm{K}_+(\mathcal{A}) \cong \mathrm{K}_+(\mathcal{B})$, and the isomorphism is natural for elementary extensions.*
(2) *T has poor triviality.*

Proof. (2)⇒(1): Let $\mathcal{A} \equiv \mathcal{B}$, take a $\mathcal{C} \succcurlyeq \mathcal{A}, \mathcal{B}$, assume that $\mathcal{A}, \mathcal{B}$ have poor triviality, and repeat the above argument up and down.

(1)⇒(2): This can be abstracted from [3, 4.1] and we transcribe it here for convenience. Let $\mathcal{A} \models T$ and $R \subseteq A_\sigma \times A_\tau$ as before. For each $r \in \mathrm{K}_+(\mathcal{A})$, fix a definable set $Z(r)$ with $[Z(r)] = r$. By hypothesis, the representatives $Z(r)$ are also representatives in any elementary extension of $\mathcal{A}$. Then, for each $x \in A_\sigma$ there is a formula ϕ_x and there are parameters a_x such that $\phi_x(a_x, \cdot, \cdot)$ defines a bijection $R_x \to Z([R_x])$. By compactness, there is $k \in \mathbb{N}^{\geqslant 1}$ and there are pairs $(r_1, x_1), \ldots, (r_k, x_k)$ and a partition $\{X_1, \ldots, X_k\}$ of A_σ so that if $x \in X_i$, there is a such that $\phi_{x_i}(a, \cdot, \cdot)$ defines a bijection $R_x \to Z(r_i)$. The usual tricks allow for a single ϕ with a σ-slot for x. □

(A corresponding result about Grothendieck rings will be supposedly more involved; there is much loss of information when passing from the semiring to the ring. For example, models of first-order Peano arithmetic have trivial ring, but we will note that they do not have poor triviality.)

2. Properties and examples

Let us list some simple properties of poor triviality and some immediate examples of theories which have it or not. In each example, the appropriate language is fixed.

Lemma 2.1. *Triviality is equivalent to poor triviality plus definable Skolem functions.*

Proof. (Use above notation.) Triviality clearly implies poor triviality. It also yields definable Skolem functions, because fixing elements $z_i \in Z_i$ as parameters, provided $Z_i \neq \emptyset$, allows one to select $F(x,\cdot)^{-1}(z_i) \in R_x$. For the converse, use a definable Skolem function to select a in the definition in terms of x. □

Lemma 2.2. *In any $\mathcal{A}$, poor triviality holds if and only if, for any $f\colon S \to X$, the target X can be partitioned into $X_1, \ldots, X_k$ for some $k \in \mathbb{N}^{\geqslant 1}$, there are $Z_1, \ldots, Z_k \subseteq A_\theta$ for some θ and there is ψ such that for each $x \in X_i$ there is a such that $\psi(x, a, \cdot, \cdot)$ defines a bijection $f^{-1}(x) \to Z_i$.*

Proof. Assuming poor triviality and given $f\colon S \to X$, take $R = \{\, (x,s) \in X \times S \mid f(s) = x \,\}$, so $R_x = f^{-1}(x)$. Conversely, given $R \subseteq A_\sigma \times A_\tau$, take $f\colon R \to A_\sigma$, $f(x,y) = x$, so $f^{-1}(x) = \{x\} \times R_x$. For this f, with notation as above, we take $\phi(x,a,u,v)$ to be $\psi(x,a,xu,v)$. □

Note. Given $f\colon S \to X$, if S can be partitioned into $S_1, \ldots, S_n$ for some $n \in \mathbb{N}^{\geqslant 1}$ such that each $f|_{S_j}$ has the property described in the previous lemma, then so does f. This is useful in cases where functions are piecewise "basic" or "continuous" (in some context) and the basic or continuous functions are poorly trivial, as in the following two examples:

2.3. The four complete extensions of the theory of dense linear orderings have poor triviality. (As none of those theories has definable Skolem functions, they do not have triviality in full.) In fact, functions are piecewise of the form $(x_1, \ldots, x_m) \mapsto (y_1, \ldots, y_n)$ where each y_j is either constant or one x_{i_j}, and the claim follows easily. The actual structure of $\mathrm{K}_+(\mathbb{Q})$ seems not to be known; $\mathbb{Q}^{\neq 0}$ and $\mathbb{Q}^{\neq 1}$ are clearly isomorphic, but $\mathbb{Q}^{>0}$ and $\mathbb{Q}^{>1}$ cannot be isomorphic: a bijection between them would induce an injective, non-surjective endofunction on $\mathbb{Q}^{>0}$, contradicting uniform local finiteness.

2.4. O-minimal expansions of ordered fields have triviality; cf. [1, Ch. 7, Sec. 3 and Ch. 9, Sec. 1]. In this case, the Grothendieck semiring is $\mathbb{N} \times \mathbb{Z} \cup \{(-\infty, 0)\}$ with operations $(\max) \times (+)$ and $(+) \times (\cdot)$, as one can abstract from [1, Ch. 4, Sec. 2, and Ch. 8, (2.11)]. (See also [2, 3.6].)

2.5. Presburger arithmetic and first-order Peano arithmetic do not have poor triviality because $R = \{\, (x,y) \in \mathbb{N}^2 \mid 0 \leqslant y \leqslant x \,\}$ has fibers of any finite cardinality.

2.6. Algebraically closed fields do not have poor triviality: For a, b ranging over all values in such a field, the curve $y^2 = x^3 + ax + b$ falls into infinitely many birational isomorphism classes. In this case, a birational isomorphism is the same as a definable isomorphism modulo finite sets. (Thanks to L. van den Dries for pointing out this example.) Providing a complete invariant for definable sets, or varieties, is certainly not simple!

Similarly, many other interesting structures do not have poor triviality: algebraically closed valued fields, pseudofinite fields, algebraically closed fields with an automorphism, differentially closed fields, and the fields of p-adic numbers. (Thanks to the referee for mentioning this list. Note that each theory requires an adequate implementation of that paradigm.)

Proposition 2.7. *$\mathcal{A}$ has poor triviality if and only if $\mathcal{A}^{\mathrm{eq}}$ has poor triviality.*

Proof. Assume that $\mathcal{A}$ has poor triviality and let E_1, E_2 be 0-definable equivalence relations on A_σ, A_τ respectively, let π, ρ be the corresponding projections onto the quotient sorts, and let $S \subseteq (A_\sigma/E_1) \times (A_\tau/E_2)$. Take $R = (\pi \times \rho)^{-1}(S)$, for which we let $X_i, Z_i, \theta, \phi, a_x$ be as in the definition of poor triviality. Then $S_y = \rho(R_x)$ for any $x \in \pi^{-1}(y)$. Let λ be the adequate sequence of sorts for the string consisting of x, a_x and the hidden parameters in ϕ. Define E on $A_{\theta\lambda}$ thus: $(u, \alpha)E(v, \beta)$ if and only if $(u, \alpha) = (v, \beta)$ or $\alpha = \beta = (x, a, p)$, say, and there are unique u', v' such that (for ϕ with such p) $\phi(x, a, u', u) \wedge \phi(x, a, v', v) \wedge u'E_2v'$ in $\mathcal{A}$. Then E is a 0-definable equivalence relation. Now, given $y \in A_\sigma/E_1$, there is (x, a, p) such that $S_y = \rho(R_x)$ and $R_x \cong Z_i$ via $\phi(x, a, \cdot, \cdot)$; then $S_y \cong (Z_i \times \{(x, a, p)\})/E$.

Conversely, assume that $\mathcal{A}^{\mathrm{eq}}$ has poor triviality. Again with the above notation, for R in the original sorts $\sigma\tau$ we obtain fiber representatives Z_i in some A_θ where θ may contain new sorts of the eq-expansion. Then, for $x \in X_i$ there is (a, x_i, a_i) such that $\exists z\, [\phi(x, a, \cdot, z) \wedge \phi(x_i, a_i, \cdot, z)]$ defines a bijection $R_x \to R_{x_i}$; indeed, we can fix $x_i \in X_i$ and take the appropriate a_i so that the first conjunct (with a free variable in place of z) defines $R_x \to Z_i$ and the second defines $R_{x_i} \to Z_i$. It suffices now to observe that $R_x \to R_{x_i}$ lies in $\mathcal{A}$ and, so, is definable there. $\square$

The argument in the last paragraph can be used to show that, in the case of a *conservative* definition or interpretation of one structure $\mathcal{A}$ in another $\mathcal{B}$, if $\mathcal{B}$ has poor triviality, then $\mathcal{A}$ has poor triviality, where the conservation hypothesis assures that the bijection $R_x \to R_{x_i}$ is definable also in $\mathcal{A}$.

The same equivalence is not always true of triviality in full (recall Lemma 2.1).

2.8. Strongly minimal groups and infinite vector spaces over arbitrary division rings have poor triviality. (For vector spaces, definable Skolem functions yield full triviality.) Indeed, their Grothendieck semiring is always $(\mathbb{Z}[t])^{\geqslant 0}$; cf. [3]. In that article, we were able to extend *naturally* the complete Euler characteristic to the eq-expansions of infinite vector spaces, but not to those of other strongly minimal groups; moreover, we saw that that extension is *not* complete. The preceding proposition yields uniform but non-constructive information about all those eq-expansions.

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