ON THE RELATIVE WEAK ASYMPTOTIC HOMOMORPHISM PROPERTY FOR TRIPLES OF GROUP VON NEUMANN ALGEBRAS

PAUL JOLISSAINT

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Abstract. A triple of finite von Neumann algebras $B \subset N \subset M$ is said to have the relative weak asymptotic homomorphism property if there exists a net of unitaries $(u_i)_{i \in I} \subset U(B)$ such that
\[
\lim_{i \in I} \|E_B(xu_i y) - E_B(E_N(x)u_i E_N(y))\|_2 = 0
\]
for all $x, y \in M$. Recently, J. Fang, M. Gao and R. Smith proved that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if $N$ contains the set of all $x \in M$ such that $Bx \subset \sum_{i=1}^n x_i B$ for finitely many elements $x_1, \ldots, x_n \in M$. Furthermore, if $H < G$ is a pair of groups, they get a purely algebraic characterization of the weak asymptotic homomorphism property for the pair of von Neumann algebras $L(H) \subset L(G)$, but their proof requires a result which is very general and whose proof is rather long. We extend the result to the case of a triple of groups $H < K < G$, we present a direct and elementary proof of the above-mentioned characterization, and we introduce three more equivalent conditions on the triple $H < K < G$, one of them stating that the subspace of $H$-compact vectors of the quasi-regular representation of $H$ on $\ell^2(G/H)$ is contained in $\ell^2(K/H)$.

1. Introduction

Let $1 \in B \subset N \subset M$ be a triple of finite von Neumann algebras endowed with a fixed, normal, finite, faithful and normalized trace $\tau$. Then $E_N$ (resp. $E_B$) denotes the $\tau$-preserving conditional expectation from $M$ onto $N$ (resp. $B$); we also set $M \cap N = \{x \in M : E_N(x) = 0\}$.

Following [3], we say that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if there exists a net of unitaries $(u_i)_{i \in I} \subset U(B)$ such that, for all $x, y \in M$,
\[
\lim_{i \in I} \|E_B(xu_i y) - E_B(E_N(x)u_i E_N(y))\|_2 = 0.
\]

Using the identity
\[
E_B(xuy) - E_B(E_N(x)uE_N(y)) = E_B([x - E_N(x)]u[y - E_N(y)])
\]
which holds for every $u \in U(B)$ and all $x, y \in M$, it is readily seen that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if one can find a net $(u_i)_{i \in I} \subset U(B)$ such that
\[
\lim_{i \in I} \|E_B(xu_iy)\|_2 = 0
\]
for all $x, y \in M \ominus N$.

The one-sided quasi-normalizer of $B$ in $M$ is the set of elements $x \in M$ for which there exist finitely many elements $x_1, \ldots, x_n \in M$ such that $Bx \subset \sum_{i=1}^n x_iB$. It is denoted by $qN_M^{(1)}(B)$.

Inspired by [2], the authors of [3] prove in Theorem 3.1 that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if $qN_M^{(1)}(B) \subset N$. Furthermore, they also study the case of group algebras that we recall now.

Let $G$ be a discrete group and let $H$ be a subgroup of $G$. Then there is a natural analogue of the one-sided quasi-normalizer for such a pair of groups: we denote by $qN_G^{(1)}(H)$ the set of elements $g \in G$ for which there exist finitely many elements $g_1, \ldots, g_n \in G$ such that $Hg \subset \bigcup_{i=1}^n g_iH$.

Thus, if $H < K < G$ is a triple of groups and if $B = L(H) \subset N = L(K) \subset M = L(G)$ denotes the triple of von Neumann algebras associated to $H < K < G$, then it is reasonable to ask whether $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if $qN_G^{(1)}(H) \subset K$. Corollary 5.4 in [2] states that this is indeed true when $K = H$, but the proof presented there relies heavily on the main theorem of the article. It is thus natural to look for a more direct and elementary proof of the above-mentioned result, and the aim of the present paper is to provide such a proof and to add three more equivalent conditions.

2. The main result

Before stating our result, let us fix some additional notation. For each element $g \in G$ we denote by $\lambda_g$ the unitary operator acting by left translation on $\ell^2(G)$, i.e., $(\lambda_g\xi)(g') = \xi(g^{-1}g')$ for every $\xi \in \ell^2(G)$ and every $g' \in G$. We denote also by $L_f(G)$ the subalgebra of all elements of $L(G)$ with finite support; i.e., $L_f(G)$ is the linear span of $\lambda(G)$ in $B(\ell^2(G))$.

We fix a triple of groups $H < K < G$ for the rest of the article.

Let $\pi$ denote the quasi-regular representation of $G$ on $\ell^2(G/H)$; we denote by $[g]$ the equivalence class $[g] = gH$, so that $\pi(g)\xi([g']) = \xi([g^{-1}g'])$ for all $g, g' \in G$ and $\xi \in \ell^2(G/H)$. Following [1], we say that a vector $\xi \in \ell^2(G/H)$ is $H$-compact if the norm closure of its $H$-orbit $\{\pi(h)\xi : h \in H\}$ is a compact subset of $\ell^2(G/H)$. The set of all $H$-compact vectors is a closed subspace of $\ell^2(G/H)$ that we denote by $\ell^2(G/H)_{c,H}$. We also set $\ell^2(G/H)^H = \{\xi \in \ell^2(G/H) : \pi(h)\xi = \xi \, \forall h \in H\}$, which is the subspace of all $H$-invariant vectors of $\ell^2(G/H)$. It is contained in $\ell^2(G/H)_{c,H}$.

**Theorem 2.1.** Let $H < K < G$ and $B = L(H) \subset N = L(K) \subset M = L(G)$ be as above. Then the following conditions are equivalent:

1. There exists a net $(h_i)_{i \in I} \subset H$ such that, for all $x, y \in M \ominus N$, one has
\[
\lim_{i \in I} \|E_B(x\lambda_{h_i}y)\|_2 = 0;
\]
i.e., the net of unitaries in the relative weak asymptotic homomorphism property may be chosen in the subgroup \(\lambda(H)\) of \(U(B)\).

(2) The triple \(B \subset N \subset M\) has the relative weak asymptotic homomorphism property.

(3) If \(g \in G\) and \(F \subset G\) finite are such that \(Hg \subset FH\), then \(g \in K\), i.e. \(qN_G^{(1)}(H) \subset K\).

(4) The subspace of \(H\)-compact vectors \(\ell^2(G/H)_{c,H}\) is contained in \(\ell^2(K/H)\).

(5) The subspace of \(H\)-fixed vectors \(\ell^2(G/H)^H\) is contained in \(\ell^2(K/H)\).

(6) For every nonempty finite set \(F \subset G \setminus K\), there exists \(h \in H\) such that \(FhF \cap H = \emptyset\).

Proof. (1) \(\Rightarrow\) (2) is obvious.

(2) \(\Rightarrow\) (3). Observe that condition (3) is equivalent to the following statement (since, if \(g \notin K\), then \(HgH \cap K = \emptyset\):

For every \(g \in G \setminus K\), and for every nonempty finite set \(F \subset G \setminus K\), there exists \(h \in H\) such that \(Fhg \cap H = \emptyset\).

Thus, let us assume that condition (3) does not hold. There exists \(g \in G \setminus K\) and a nonempty finite set \(F \subset G \setminus K\) such that \(Fhg \cap H \neq \emptyset\) for every \(h \in H\).

Then let \(u \in U(B)\). One has:

\[
\sum_{g' \in F} \|\mathbb{E}_B(\lambda_{g'}u\lambda_g)\|^2_2 = \sum_{g' \in F} \left( \sum_{h \in H, g' \in H} |u(h)|^2 \right) = \sum_{h \in H} \left( \sum_{g' \in F, g' \in H} |u(h)|^2 \right) \geq \sum_{h \in H} |u(h)|^2 = \|u\|^2_2 = 1
\]

since, for every \(h \in H\), one can find \(g'(h) \in F\) such that \(g'(h)hg \in H\). Hence there cannot exist a net \((u_t)_{t \in I} \subset U(B)\) as above, and the triple \(B \subset N \subset M\) does not have the relative weak asymptotic homomorphism property.

(3) \(\Rightarrow\) (4). We choose a set of representatives \(T \ni \epsilon\) of left classes so that \(G = \bigcup_{t \in T} tH\), and let \(\xi \neq 0\) be an \(H\)-compact vector.

Let \(s \in T\) be such that \(\epsilon := |\xi([s])| > 0\). There exist then finitely many vectors \(\xi_1, \ldots, \xi_n \in \ell^2(G/H)\) such that, for every \(h \in H\), there exists \(1 \leq j \leq n\) such that \(\|\pi(h)\xi - \xi_j\| \leq \epsilon/2\). Set \(F = \bigcup_{j=1}^n \{t \in T : |\xi_j([t])| \geq \epsilon/2\}\), which is a finite set. Then we claim that \(Hs \subset FH\). Indeed, if \(h \in H\), let \(t \in T\) be such that \([hs] = [t]\), and let \(j\) be such that \(\|\pi(h)\xi - \xi_j\| \leq \epsilon/2\). Then

\[
\epsilon - |\xi_j([t])| = |\xi([s])| - |\xi_j([t])| \leq |\xi([s]) - \xi_j([hs])| \leq \|\pi(h)\xi - \xi_j\| \leq \epsilon/2;
\]

hence \(\epsilon/2 \leq |\xi_j([t])|\) and \(t \in F\). Thus \(Hs \subset FH\), and condition (3) implies that \(s \in K\). This proves that \(\xi \in \ell^2(K/H)\).

(4) \(\Rightarrow\) (5) is obvious.

(5) \(\Rightarrow\) (6). Let us assume that the triple \(H < K < G\) satisfies condition (5) but not (6). Then there exists a finite set \(F = F^{-1} \subset G \setminus K\) such that \(FhF \cap H \neq \emptyset\)
for every \( h \in H \). Set
\[
\xi = \sum_{g \in F} \delta_{[g]}.
\]
Then \( \xi \perp \ell^2(K/H) \), and one has for every \( h \in H \):
\[
\langle \pi(h)\xi, \xi \rangle = \sum_{g,g' \in F} \langle \delta_{[hg]}, \delta_{[g']} \rangle \geq 1
\]
since the condition on \( F \) implies that for every \( h \in H \), there exist \( g,g' \in F \) such that \( hgH = g'H \). Let \( C \) be the closed convex hull of \( \{ \pi(h)\xi : h \in H \} \). Then it is easy to see that \( \langle \zeta, \xi \rangle \geq 1 \) for every \( \zeta \in C \). Let \( \eta \in C \) be the vector with minimal norm. By its uniqueness, it is \( H \)-invariant and nonzero by the above observation. Thus, \( \eta \) is supported in \( K/H \) and orthogonal to \( \ell^2(K/H) \) since \( \xi \) is. This is the expected contradiction.

(6) \( \Rightarrow \) (1). Let \( I = \{ F \subset G \setminus K : F \neq \emptyset, \text{ finite} \} \) be the directed set of all nonempty finite subsets of \( G \setminus K \). Condition (6) states that, for every \( F \in I \), there exists \( h_F \in H \) such that \( Fh_F \cap H = \emptyset \). Let \( x \) and \( y \) in \( L_f(G) \) satisfy \( E_N(x) = E_N(y) = 0 \). Then let \( F_0 \in I \) be chosen so that the supports of \( x \) and \( y \) are contained in \( F_0 \). Then \( x \lambda_{h_F} y = \sum_{g,g' \in F_0} x(g) y(g') \lambda_{gh_F g'} \) for every \( F \supset F_0 \); thus \( E_B(x \lambda_{h_F} y) = 0 \) for every \( F \supset F_0 \). This proves that the triple \( B \subset N \subset M \) satisfies condition (1) by the density of \( L_f(G) \) in \( L(G) \).

Remark 2.2. In the case of a pair of groups \( H < G \), which corresponds to \( H = K \), condition (4) means that all \( H \)-invariant vectors in \( \ell^2(G/H) \) are multiples of \( \delta_{[e]} \), and this means that the unitary representation \( \rho \) of \( H \) on the subspace \( \ell^2(G/H) \oplus \mathbb{C}\delta_{[e]} \) is ergodic in the sense of [1].

References