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ON THE CENTRALIZERS IN THE WEYL ALGEBRA

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ABSTRACT. Let P, Q be elements of the Weyl algebra W. We prove that if $[Q, P] = 1$, then the centralizer of P is the polynomial algebra $k[P]$.

INTRODUCTION

Let k be a characteristic zero field. The Weyl algebra W of index 1 over k is the unital associative k-algebra generated by elements X, Y and the relation $[Y, X] = 1$. This algebra was introduced by Hermann Weyl in order to study the Heisenberg uncertainty principle in quantum mechanics. A detailed analysis of W was made in [D]. Among other things, in this paper the author establishes many interesting properties about the centralizer $\mathrm{Z}(P)$ of an element P. Another important paper devoted to the investigation of centralizers of elements in the Weyl algebra is [B]. In this note we continue the study of $\mathrm{Z}(P)$. Our main result is that if $P, Q \in W$ satisfy $[Q, P] = 1$, then $\mathrm{Z}(P) = k[P]$. Dixmier asked in [D] if each endomorphism of W is an automorphism. An affirmative answer immediately implies our theorem by [D, Th. 9.1].

1. PRELIMINARIES

In this section we establish some notations and we recall some results from [D]. Let P and Q be nonzero elements of W.

Notations 1.1. For $P = \sum a_{ij} X^i Y^j$, we write the following:

- $v(P) := \max\{i - j : a_{ij} \neq 0\}$.
- $\ell(P) := \displaystyle\sum_{i-j=v(P)} a_{ij} X^i Y^j$.
- $\mathrm{Supp}(P) := \{(i, j) : a_{ij} \neq 0\}$.
- $w(P) := (i_0, i_0 - v(P))$ such that $i_0 = \max\{i : (i, i - v(P)) \in \mathrm{Supp}(\ell(P))\}$.
- $\ell_t(P) := a_{i_0 j_0} X^{i_0} Y^{j_0}$, where $(i_0, j_0) = w(P)$.

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- $\ell_c(P) := a_{i_0 j_0}$, where $(i_0, j_0) = w(P)$.
- $W_+ := \{P \in W : v(P) > 0\}$.
- If $\ell_c(P) = 1$, then we will say that P is *monic*.

We say that P is aligned with Q and write $P \sim Q$ if

$$km = jl \qquad \text{with } w(P) = (k, j) \text{ and } w(Q) = (l, m).$$

Note that $\sim$ is not an equivalence relation (it is so restricted to $\{P : w(P) \neq (0,0)\}$).

Proposition 1.2. *The following facts hold:*

(1) $X^k Y^j X^l Y^m = X^{k+l} Y^{j+m} + \sum_{i=1}^{\min(j,l)} i! \binom{j}{i} \binom{l}{i} X^{k+l-i} Y^{j+m-i}$.
(2) *If* $P \not\sim Q$, *then* $[P, Q] \neq 0$ *and* $w\big([P, Q]\big) = w(P) + w(Q) - (1,1)$.
(3) $[\ell(P), \ell(Q)] = 0$ *implies* $P \sim Q$.

Proof. (1) It suffices to prove item (1) when $k = m = 0$. In this case it follows easily by induction on j, using that

$$[Y, X^l] = l X^{l-1} \quad \text{and} \quad [Y^j, X^l] = [Y, X^l] Y^{j-1} + Y [Y^{j-1}, X^l].$$

(2) Let $w(P) = (k, j)$ and $w(Q) = (l, m)$. Since $P \not\sim Q$, it follows from item (1) that

$$\ell_t\big([\ell_t(P), \ell_t(Q)]\big) = \left(\binom{j}{1}\binom{l}{1} - \binom{m}{1}\binom{k}{1}\right) \ell_c(P) \ell_c(Q) X^{k+l-1} Y^{j+m-1}.$$

From this fact and item (1) it follows now that

$$\ell_t\big([P, Q]\big) = \left(\binom{j}{1}\binom{l}{1} - \binom{m}{1}\binom{k}{1}\right) \ell_c(P) \ell_c(Q) X^{k+l-1} Y^{j+m-1},$$

and so $w\big([P, Q]\big) = w(P) + w(Q) - (1,1)$.

(3) If $P \not\sim Q$, then $\ell(P) \not\sim \ell(Q)$, since $w(\ell(P)) = w(P)$ and $w(\ell(Q)) = w(Q)$. Thus, by item (2) we have $[\ell(P), \ell(Q)] \neq 0$. $\square$

From item (1) of Proposition 1.2 it follows immediately that for $P \neq 0$ and $Q \neq 0$,

$$\begin{aligned} v(PQ) &= v(P) + v(Q), \\ w(PQ) &= w(P) + w(Q), \\ \ell(PQ) &= \ell(P)\ell(Q), \\ \ell_t(PQ) &= \ell_t(\ell_t(P)\ell_t(Q)), \\ \ell_c(PQ) &= \ell_c(P)\ell_c(Q). \end{aligned}$$

Remark 1.3. Let $\overline{v}(P) := \max\{j - i : a_{ij} \neq 0\}$. All the results in this section admit symmetric versions with $\overline{w}, \overline{\ell}, \overline{\ell}_t, \overline{\ell}_c$ and $\overline{W}_+$, defined mimicking the definition of w, ℓ, ℓ_t, ℓ_c and W_+, using $\overline{v}$ instead of v. In the sequel, when we introduce any symbol denoting an object, the same symbol with an overline will denote the symmetric object.

For each $j \in \mathbb{Z}$ we set

$$W_j := \{P \in W \setminus \{0\} : P = \ell(P) \text{ and } v(P) = j\} \cup \{0\}.$$

Clearly W_j is a subspace of W and W is a $\mathbb{Z}$-graded algebra with W_j the homogeneous component of degree j. It is obvious that $\overline{W}_j = W_{-j}$.

2. The structure of the centralizer

For $P \in W$, we let $\mathrm{Z}(P)$ denote the centralizer of P, that is, the subalgebra of W consisting of all the Q's such that $PQ = QP$. The main purpose of this section is to prove that if there exists $Q \in W$ such that $[Q,P] = 1$, then $\mathrm{Z}(P) = k[P]$.

Lemma 2.1. *The following facts hold:*

(1) *For each $j \in \mathbb{N}_0$ and $f \in k[Z]$,*

$$f(XY)X^j = X^j f(XY+j) \quad \textit{and} \quad Y^j f(XY) = f(XY+j)Y^j.$$

(2) *For each $j \in \mathbb{Z}$,*

$$W_j = \begin{cases} X^j k[XY] & \text{if } j \geq 0, \\ k[XY]Y^{-j} & \text{if } j < 0. \end{cases}$$

Proof. See [D, 3.2, 3.3]. □

Theorem 2.2. *Let $P \in W_r \setminus k$. If $r = 0$, then $\mathrm{Z}(P) = k[XY]$. On the other hand, if $r \neq 0$, then $\dim_k\big(\mathrm{Z}(P) \cap W_j\big) \leq 1$ for each $j \in \mathbb{Z}$.*

Proof. First assume that $r = 0$. It is evident that $k[XY] \subseteq \mathrm{Z}(P)$. In order to check the opposite inclusion, we note that $\mathrm{Z}(P)$ is a graded subalgebra of W. Let $Q \in \mathrm{Z}(P) \setminus \{0\}$ be a homogeneous element. By item (3) of Proposition 1.2, we have $P \sim Q$ and so $v(Q) = 0$. Now assume that $r > 0$. Write $P = X^r f(XY)$ and take

$$Q \in \mathrm{Z}(P) \cap W_j \setminus \{0\}.$$

By item (3) of Proposition 1.2, we know that $P \sim Q$. So $j > 0$ and $Q = X^j g(XY)$ for some $g \in k[Z]$. A direct computation using that

$$f(XY)X^j = X^j f(XY+j) \quad \text{and} \quad g(XY)X^r = X^r g(XY+r)$$

shows that $[P,Q] = 0$ if and only if

$$\frac{f(Z+j)}{f(Z)} = \frac{g(Z+r)}{g(Z)}.$$

Hence, in order to finish the proof in the case $r > 0$, it suffices to check that if $g_1 \neq 0$ satisfies

$$\frac{g_1(Z+r)}{g_1(Z)} = \frac{g(Z+r)}{g(Z)},$$

then $g_1 = \lambda g$ for some $\lambda \in k$. Let $n_0 \in \mathbb{Z}$ be such that $g(n) \neq 0$ and $g_1(n) \neq 0$ for all $n \geq n_0$. Set $\lambda := g_1(n_0)/g(n_0)$. We have

$$g_1(n_0+r) = g_1(n_0)g(n_0+r)/g(n_0) = \lambda g(n_0+r).$$

Iterating the same argument, we obtain $g_1(n_0+ir) = \lambda g(n_0+ir)$ for all $i > 0$. So the polynomials g_1 and λg coincide. We leave the case $r < 0$ for the reader. Use the symmetric version of Proposition 1.2. □

For $P \in W_+$, there exists exactly one pair $(i,j) \in \mathbb{N}_0^2$, with $\gcd(i,j) = 1$, such that $w(P) = (ri, rj)$ for an $r > 0$. We define

$$\mathrm{Z}_l(P) = \{Q \in \mathrm{Z}(P) \setminus \{0\} : w(Q) = (li, lj)\}$$

for each $l \geq 0$. By item (3) of Proposition 1.2,

$$\mathrm{Z}(P) \setminus \{0\} = \bigcup_{l \geq 0} \mathrm{Z}_l(P). \tag{2.1}$$

In a similar way we define $\overline{\mathrm{Z}}_l(P)$ for every $P \in \overline{W}_+$ and $l \geq 0$.

Theorem 2.3. *For all $P \in W_+$ with $w(P) = (ri, rj)$ as above, there exist elements $R_l \in \mathrm{Z}_l(P)$ such that*

(1) $\mathrm{Z}(P) = \bigoplus_{l \in L} kR_l$, *with* $L = L(P) \subset \mathbb{N}_0$,
(2) $\ell_t(R_l) = X^{li}Y^{lj}$,
(3) $\ell(R_l)\ell(R_h) = \ell(R_{l+h})$.

For $P \in \overline{W}_+$, the symmetric result holds.

Proof. Let $L(P) = \{l \geq 0 : \mathrm{Z}_l(P) \neq \emptyset\}$. For all $l \in L(P)$, we fix an $R_l \in \mathrm{Z}_l(P)$ with $\ell_t(R_l) = X^{li}Y^{lj}$. Clearly $\bigoplus_{l \in L} kR_l \subseteq \mathrm{Z}(P)$. We now prove the opposite inclusion. Let $Q \in \mathrm{Z}(P) \setminus \{0\}$ with $w(Q) = (hi, hj)$. Since

$$\ell(Q), \ell(R_h) \in \mathrm{Z}(\ell(P)) \cap W_{h(i-j)},$$

by Theorem 2.2 there exists $\lambda_h \in k$ such that $\ell(Q) = \lambda_h \ell(R_h)$. If $Q = \lambda_h R_h$, then $Q \in \bigoplus_{l \in L} kR_l$. Otherwise, $Q - \lambda_h R_h \in \mathrm{Z}(P)$ and, by (2.1),

$$w(Q - \lambda_h R_h) = (h'i, h'j),$$

with $0 \leq h' < h$. The same argument yields $\lambda_{h'}$ such that $Q = \lambda_h R_h + \lambda_{h'} R_{h'}$ or

$$Q - \lambda_h R_h - \lambda_{h'} R_{h'} \in \mathrm{Z}(P) \quad \text{and} \quad w(Q - \lambda_h R_h - \lambda_{h'} R_{h'}) = (h''i, h''j),$$

with $0 \leq h'' < h'$. Iterating the argument, we obtain that $Q \in \bigoplus_{l \in L} kR_l$. So items (1) and (2) hold. Finally, since

$$\ell(R_l)\ell(R_h), \ell(R_{l+h}) \in \mathrm{Z}(\ell(P)) \cap W_{(l+h)(i-j)},$$

again, by Theorem 2.2, there exists $\lambda \in k$ such that $\ell(R_l)\ell(R_h) = \lambda \ell(R_{l+h})$. By item (2), necessarily $\lambda = 1$.

When $P \in \overline{W}_+$, the same argument works. □

Remark 2.4. We assert that $R_0 = 1$. Otherwise $R_0 - 1 \in \mathrm{Z}(P) \setminus \{0\}$, and so $R_0 - 1 \sim P$. But this is impossible since $v(P) > 0$ and $v(R_0 - 1) < 0$.

Remark 2.5. Let $P \in W \setminus \{0\}$. If $\overline{v}(P) \leq 0$ and $v(P) \leq 0$, then, necessarily, $\overline{v}(P) = 0 = v(P)$ and $P \in W_0$. In this case $\mathrm{Z}(P) = k[XY]$ if $P \notin k$ and $\mathrm{Z}(P) = W$ if $P \in k$.

Remark 2.6. Note that in general $R_l R_h \neq R_{l+h}$. So item (1) does not yield a graduation on $\mathrm{Z}(P)$. However if $P \in W_+$ is homogeneous (that is, $P = \ell(P)$), then we can assume that $R_l = \ell(R_l)$. Hence, by item (3) of the above theorem, $\mathrm{Z}(P)$ is graded and, therefore, the map $\varphi : \mathrm{Z}(P) \to k[Z]$, given by $\varphi(R_l) = Z^l$, is a monomorphism of graded algebras. Consequently $\mathrm{Z}(P)$ is a monomial algebra. A similar result holds for $P \in \overline{W}_+$ homogeneous.

Lemma 2.7. *Let L be an additive submonoid of $\mathbb{N}_0$. Let $r_0 = \min(L \setminus \{0\})$ and $d = \gcd(L)$. For $0 \leq r < r_0$, let $L_r := \{l \in L : l \equiv r \pmod{r_0}\}$. Then*

$$L_r \neq \emptyset \Leftrightarrow d \mid r \qquad \textit{and} \qquad L = \bigcup_{r=0}^{r_0 - 1} L_r.$$

Proof. First note that the second assertion is trivial and that $L_r \neq \emptyset$ clearly implies that $d \mid r$. We now prove the opposite implication. Let $n_0 := r_0/d$. Since the group generated by L is $d\mathbb{Z}$, there exist $l_0, l_1 \in L$ such that $d = l_1 - l_0$. So

$$l_0 n_0, l_0(n_0-1)+l_1, l_0(n_0-2)+2l_1, \dots, l_0+(n_0-1)l_1$$

are elements in the n_0 different L_r with $d \mid r$, as desired. □

For a fixed $P \in W_+$ we consider $d = \gcd\{l : l \in L(P)\}$ and set $\deg Q = l/d$ for $Q \in \mathrm{Z}_l(P)$. Note that

$\deg Q_1 > \deg Q_2$ if and only if $v(Q_1) > v(Q_2)$, and
for a polynomial $T \in k[S] \setminus k$, where $S \in \mathrm{Z}(P)$,

$$\deg T' = \deg T - \deg S,$$

in which T' denotes the usual derivative of the polynomial T.

In the sequel we will use these facts again and again without explicit mention.

For $P \in W_+$, choose an element S_0 of minimal degree in $\mathrm{Z}(P) \setminus k$ and set $n_0 := \deg S_0$. For each $0 < l < n_0$, set

$$\widehat{\mathrm{Z}}_l(P) := \{R \in \mathrm{Z}(P) \setminus \{0\} : \deg R \equiv l \pmod{n_0}\}.$$

Since $L(P)$ is an additive submonoid of $\mathbb{N}_0$, from Lemma 2.7 it follows that the $\widehat{\mathrm{Z}}_l(P)$'s are not empty. Fix $S_l \in \widehat{\mathrm{Z}}_l(P)$ of minimal degree.

Corollary 2.8. *We have*

$$\mathrm{Z}(P) = k[S_0] \oplus k[S_0]S_1 \oplus \cdots \oplus k[S_0]S_{n_0-1}. \tag{2.2}$$

For $P \in \overline{W}_+$ the symmetric result holds.

Proof. It is clear that

$$k[S_0] \oplus k[S_0]S_1 \oplus \cdots \oplus k[S_0]S_{n_0-1} \subseteq \mathrm{Z}(P).$$

Consider $R \in \mathrm{Z}(P) \setminus \{0\}$. We will prove by induction on $\deg R$ that R is contained in the right side of the equality (2.2). If $\deg R = 0$, then by item (1) of Theorem 2.3 and Remark 2.4, we have $R \in k \subseteq k[S_0]$. Otherwise, there exist $0 \le r < n_0$ and $l \in \mathbb{N}_0$ such that

$$\deg(S_0^l S_r) = \deg S_r + l n_0 = \deg R.$$

Hence, by Theorem 2.2, we can find $\lambda \in k$ such that $\lambda \ell(S_0^l S_r) = \ell(R)$. Then, by the inductive hypothesis, $R - \lambda S_0^l S_r$ belongs to the right side of (2.2), since

$$R - \lambda S_0^l S_r = 0 \quad \text{or} \quad \deg(R - \lambda S_0^l S_r) < \deg R.$$

This finishes the proof. □

The S_i's in the previous corollary can be chosen to be monic (i.e., with $\ell_c(S_i) = 1$), and we will do that from now on.

By [D, Corollary 4.5] we know that $\mathrm{Z}(P)$ is a commutative algebra, and so

$$\partial(T(S)) = T'(S)\partial(S)$$

for any derivation $\partial : \mathrm{Z}(P) \to \mathrm{Z}(P)$ and any $T \in k[S]$ with $S \in \mathrm{Z}(P)$.

Lemma 2.9. *Let $P \in W_+$ and let*

$$\mathrm{Z}(P) = k \oplus k[S_0]S_0 \oplus k[S_0]S_1 \oplus \cdots \oplus k[S_0]S_{n_0-1}$$

be as in Corollary 2.8. Let

$$\partial : \mathrm{Z}(P) \to \mathrm{Z}(P)$$

be a derivation and set $J := \{r : \partial(S_r) \neq 0\}$. *Then*

$$\deg S_r - \deg \partial(S_r) = \deg S_t - \deg \partial(S_t)$$

for all $r, t \in J$.

Proof. Set $g_r := \deg S_r$ and $w_r := \deg \partial(S_r)$. Note that $g_0 = n_0$. Consider the set

$$D := \{w_r - g_r : r \in J\}.$$

We must prove that $\#D = 1$. Assuming that this is not the case will lead us to a contradiction. Take r, t such that

$$w_t - g_t = \max D > w_r - g_r.$$

Since

$$\ell(S_t^{g_r}), \ell(S_r^{g_t}) \in \mathrm{Z}(\ell(P)) \quad \text{and} \quad v\big(\ell(S_t^{g_r})\big) = v(S_t^{g_r}) = v(S_r^{g_t}) = v\big(\ell(S_r^{g_t})\big),$$

by Theorem 2.2 we have that $\ell(S_t^{g_r}) = \lambda_1 \ell(S_r^{g_t})$ for some $\lambda_1 \in k \setminus \{0\}$. Hence, there exist $\lambda_0 \in k$ and $P_i \in k[S_0]$ such that

$$S_t^{g_r} = \lambda_1 S_r^{g_t} + \lambda_0 + \sum_{i=0}^{n_0-1} P_i S_i \quad \text{with } P_i S_i = 0 \text{ or } \deg(P_i S_i) < g_t g_r.$$

This implies that

$$U := \deg \partial(S_t^{g_r}) \leq \max\{\deg \partial(S_r^{g_t}), \deg(P_i \partial(S_i)), \deg(\partial(P_i) S_i)\}, \tag{2.3}$$

where we only consider nonzero terms $P_i\partial(S_i)$ and $\partial(P_i)S_i$ (note that $\partial(P_i) \neq 0$ only if $\partial(S_0) \neq 0$ and $P_i \notin k$). But (2.3) is impossible since U is strictly greater than each of the terms on the right side. In order to check this, note that

$$\begin{aligned}
\partial(S_t^{g_r}) &= g_r S_t^{g_r-1} \partial(S_t),\\
\partial(S_r^{g_t}) &= g_t S_r^{g_t-1} \partial(S_r),\\
\partial(P_i) S_i &= P_i' \partial(S_0) S_i,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\deg \partial(S_t^{g_r}) = g_r g_t + w_t - g_t = U,\\
&\deg \partial(S_r^{g_t}) = g_r g_t + w_r - g_r < U,\\
&\deg(P_i \partial(S_i)) = \deg P_i + w_i = \deg(P_i S_i) + w_i - g_i < g_t g_r + w_i - g_i \leq U,\\
&\deg(\partial(P_i) S_i) = \deg P_i - g_0 + w_0 + g_i = \deg(P_i S_i) + w_0 - g_0 < U.
\end{aligned}$$

This concludes the proof. □

Proposition 2.10. *Let* $P \in W_+$. *If* $\partial \colon Z(P) \to Z(P)$ *is a nonzero derivation, then* $\ker \partial = k$. *Moreover, using the same notation as in the previous lemma, if*

$$R = \lambda_0 + \sum_{i=0}^{n_0-1} R_i S_i \,\in\, \mathrm{Z}(P) \setminus k,$$

with $R_i \in k[S_0]$, *then*

$$\deg \partial(R) = \deg R + \deg \partial(S_j) - \deg S_j \quad \textit{for all } j.$$

Proof. Let $J := \{r : \partial(S_r) \neq 0\}$. We first prove that

$$\deg \partial(R) \leq \deg R + \deg \partial(S_j) - \deg S_j \quad \text{for all } j \in J. \tag{2.4}$$

By hypothesis $J \neq \emptyset$. As in the proof of Lemma 2.9, we set $g_j := \deg S_j$ and $w_j := \deg \partial(S_j)$ for all $j \in J$. Now fix $j \in J$. Since

$$\deg \partial(R) \leq \max\{\deg \partial(R_i S_i) : \partial(R_i S_i) \neq 0\}$$

and

$$\deg R = \max\{\deg(R_i S_i) : R_i S_i \neq 0\},$$

it suffices to show that if $\partial(R_i S_i) \neq 0$, then

$$\deg \partial(R_i S_i) \leq \deg(R_i S_i) + w_j - g_j.$$

By Lemma 2.9, if $i \in J$, then

$$\deg(R_i \partial(S_i)) = \deg(R_i S_i) + w_i - g_i = \deg(R_i S_i) + w_j - g_j \tag{2.5}$$

and if $0 \in J$ and $R_i \notin k$, then

$$\begin{aligned} \deg(R_i' \partial(S_0) S_i) &= \deg R_i - g_0 + w_0 + g_i \\ &= \deg(R_i S_i) + w_0 - g_0 \\ &= \deg(R_i S_i) + w_j - g_j. \end{aligned} \tag{2.6}$$

So,

$$\deg \partial(R_i S_i) = \deg\big(R_i \partial(S_i) + R_i' \partial(S_0) S_i\big) \leq \deg(R_i S_i) + w_j - g_j,$$

as desired. We now prove that $J = \{0, \dots, n_0 - 1\}$. Let $0 \leq h < n_0$. As in the proof of Lemma 2.9 we can write

$$S_h^{g_j} = \lambda_1 S_j^{g_h} + \lambda_0 + \sum_{i=0}^{n_0-1} P_i S_i \quad \text{with } P_i S_i = 0 \text{ or } \deg(P_i S_i) < g_j g_h. \tag{2.7}$$

Note that $\ell_c(S_h) = \ell_c(S_j) = 1$ and so $\lambda_1 = 1$. Hence, as we have seen above, if $\partial(P_i S_i) \neq 0$, then

$$\deg \partial(P_i S_i) \leq \deg(P_i S_i) + w_j - g_j < g_j g_h + w_j - g_j = \deg \partial(S_j^{g_h}). \tag{2.8}$$

Consequently,

$$g_j S_h^{g_j - 1} \partial(S_h) = \partial(S_h^{g_j}) = \partial(S_j^{g_h}) + \sum_{i=0}^{n_0-1} \partial(P_i S_i) \neq 0, \tag{2.9}$$

which implies that $h \in J$. It remains to prove that the equality holds in (2.4). For this it will be sufficient to check that

$$\partial(R_i S_i) \neq 0 \quad \text{and} \quad \deg \partial(R_i S_i) = \deg(R_i S_i) + w_j - g_j, \tag{2.10}$$

for all i such that $R_i S_i \neq 0$, which implies that if

$$\deg \partial(R_h S_h) = \max\{\deg \partial(R_i S_i) : \partial(R_i S_i) \neq 0\},$$

then also

$$\deg(R_h S_h) = \max\{\deg(R_i S_i) : R_i S_i \neq 0\}.$$

In fact, if (2.10) is true, then $\partial(R) \neq 0$ (proving that $\ker \partial = k$) and

$$\deg \partial(R) = \deg \partial(R_h S_h) = \deg(R_h S_h) + w_j - g_j = \deg R + w_j - g_j.$$

Now we are going to prove (2.10). By (2.5) and (2.6) we are reduced to showing that

$$M := \ell(R_i')\ell(\partial(S_0))\ell(S_i) + \ell(R_i)\ell(\partial(S_i)) \neq 0 \qquad \text{for all } R_i \in k[S_0] \setminus k.$$

Write

$$R_i = a_n S_0^n + \cdots + a_0 \quad \text{with } a_n \neq 0.$$

By (2.7), we know that $\ell(S_i^{g_0}) = \ell(S_0^{g_i})$ and by (2.8) and (2.9) that

$$g_0 \ell(S_i^{g_0-1})\ell(\partial(S_i)) = \ell(\partial(S_i^{g_0})) = \ell(\partial(S_0^{g_i})) = g_i \ell(S_0^{g_i-1})\ell(\partial(S_0)).$$

Hence, in the quotient field of $\mathrm{Z}(\ell(P))$,

$$\begin{aligned} \ell(\partial(S_i)) &= \frac{g_i}{g_0}\ell(\partial(S_0))\frac{\ell(S_0)^{g_i-1}\ell(S_i)}{\ell(S_i)^{g_0-1}\ell(S_i)} \\ &= \frac{g_i}{g_0}\ell(\partial(S_0))\frac{\ell(S_0)^{g_i-1}\ell(S_i)}{\ell(S_0)^{g_i}} \\ &= \frac{g_i}{g_0}\ell(\partial(S_0))\frac{\ell(S_i)}{\ell(S_0)}, \end{aligned}$$

which implies that

$$\ell(\partial(S_i))\ell(S_0) = \frac{g_i}{g_0}\ell(\partial(S_0))\ell(S_i)$$

in $Z(\ell(P))$. Therefore, since $S_0 \in W_+$ and $n > 0$,

$$\begin{aligned} M &= \ell(R_i')\ell(\partial(S_0))\ell(S_i) + \ell(R_i)\ell(\partial(S_i)) \\ &= n a_n \ell(S_0)^{n-1}\ell(\partial(S_0))\ell(S_i) + a_n \ell(S_0)^{n-1}\ell(S_0)\ell(\partial(S_i)) \\ &= a_n \ell(S_0)^{n-1}\ell(\partial(S_0))\ell(S_i)\left(n + \frac{g_i}{g_0}\right) \neq 0, \end{aligned}$$

as desired. □

Let $P, Q \in W$ be such that $[Q, P] = 1$. Since

$$[P, [Q, R]] = [[P, Q], R] + [Q, [P, R]] = [Q, [P, R]],$$

$ad_Q := [Q, -]$ defines a derivation ∂ from $\mathrm{Z}(P)$ to $\mathrm{Z}(P)$.

Theorem 2.11. *If $P, Q \in W$ satisfy $[Q, P] = 1$, then $\mathrm{Z}(P) = k[P]$.*

Proof. Using Lemma 2.1 it is easy to check that if $P \in W_0 \cap \overline{W}_0 = k[XY]$, then there is no Q such that $[Q, P] = 1$. So $P \in W_+ \cup \overline{W}_+$. Assume $P \in W_+$. Let $\partial := ad_Q$ and let S_i, g_i, w_i be as above. Since $g_i > g_0$ for $i > 0$, from Proposition 2.10 and Lemma 2.9 it follows that $w_i > w_0$ for $i > 0$. Write

$$P = \lambda_0 + \sum_{i=0}^{n_0-1} P_i S_i,$$

with $P_i \in k[S_0]$ and $n_0 = g_0$. By Proposition 2.10,

$$\deg \partial(P_i S_i) = \deg(P_i S_i) + w_i - g_i = w_i + \deg P_i \equiv w_i \pmod{n_0}$$

for each $P_i \neq 0$. Moreover, by Lemma 2.9

$$w_i \not\equiv w_j \pmod{n_0} \quad \text{for } i \neq j,$$

and so

$$0 = \deg \partial(P) = \max\{w_i + \deg P_i : P_i \neq 0\}.$$

Hence $P_i = 0$ for $i > 0$ (since $w_i > w_0 \geq 0$) and $w_0 = 0 = \deg P_0$. Consequently $P = \lambda_0 + \lambda_1 S_0$. We claim that $n_0 = 1$, which concludes the proof, by Corollary 2.8. In fact, if $n_0 > 1$, then $\partial(S_l) \in \mathrm{Z}(P)$ for $0 < l < n_0$, and so, by Lemma 2.9,

$$\deg \partial(S_l) = w_l = g_l - g_0 = g_l - n_0 \equiv l \pmod{n_0}.$$

Therefore

$$\partial(S_l) \in \widehat{\mathrm{Z}}_l(P) \quad \text{and} \quad \deg \partial(S_l) < \deg S_l = g_l,$$

which contradicts the minimality of g_l. For $P \in \overline{W}_+$, the same argument works, using the symmetric versions of Lemma 2.9 and Proposition 2.10. □

References

[D] Jacques Dixmier, *Sur les algèbres de Weyl*, Bulletin de la S.M.F., tome 96 (1968) 209–242. MR0242897 (39:4224)

[B] V. Bavula, *Dixmier's Problem 5 for the Weyl algebra,* Journal of Algebra, vol. 283 (2005) 604–621. MR2111212 (2005i:16048)

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