

## THE PRINCIPAL INVERSE OF THE GAMMA FUNCTION

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ABSTRACT. Let  $\Gamma(x)$  be the gamma function in the real axis and  $\alpha$  the maximal zero of  $\Gamma'(x)$ . We call the inverse function of  $\Gamma(x)|_{(\alpha, \infty)}$  the principal inverse and denote it by  $\Gamma^{-1}(x)$ . We show that  $\Gamma^{-1}(x)$  has the holomorphic extension  $\Gamma^{-1}(z)$  to  $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$ , which maps the upper half-plane into itself, namely a Pick function, and that  $\Gamma(\Gamma^{-1}(z)) = z$  on  $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$ .

### 1. INTRODUCTION

The Euler form of the gamma function  $\Gamma(x)$  is given by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$$

for  $x > 0$ . The Weierstrass form

$$(1.1) \quad \frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}$$

extends it to  $\mathbf{R} \setminus \{0, -1, -2, \dots\}$ , where  $\gamma$  is the Euler constant defined by

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n\right) = 0.57721 \dots$$

From this it follows that

$$(1.2) \quad \log \Gamma(x) = -\log x - \gamma x + \sum_{n=1}^{\infty} \left(\frac{x}{n} - \log\left(1 + \frac{x}{n}\right)\right),$$

$$(1.3) \quad \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+x}\right).$$

This is called the *psi function* or *digamma function*. It is clear that  $\Gamma(1) = \Gamma(2) = 1$ ,  $\Gamma'(1) = -\gamma$ ,  $\Gamma'(2) = -\gamma + 1$ . Denote the unique zero in  $(0, \infty)$  of  $\Gamma'(x)$  by  $\alpha$ . It is known that  $\alpha = 1.4616 \dots$  and  $\Gamma(\alpha) = 0.8856 \dots$ . We call the inverse function of the restriction of  $\Gamma(x)$  to  $(\alpha, \infty)$  the *principal inverse function* and denote it by  $\Gamma^{-1}$ .  $\Gamma^{-1}(x)$  is an increasing and concave function defined on  $(\Gamma(\alpha), \infty)$ . (1.1) guarantees that  $\Gamma(x)$  has the holomorphic extension which is a meromorphic function with poles at non-positive integers and (1.3) holds there. Let  $\Pi_+$  and  $\Pi_-$  be respectively the

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open upper half-plane and the open lower half-plane. A holomorphic function defined on  $\Pi_+$  is called a *Pick function* or *Nevanlinna function* if it maps  $\Pi_+$  into itself. From (1.3) it follows that  $\Gamma'(z)$  does not vanish on  $\mathbf{C} \setminus (-\infty, \alpha]$ ; in fact,  $\Gamma'(z)/\Gamma(z)$  is a Pick function. Hence for each point  $\omega_0 \in \Gamma(\mathbf{C} \setminus (-\infty, \alpha])$  there is a local inverse of  $\Gamma(z)$  in a neighborhood of  $\omega_0$ .

Some other Pick functions related to the gamma function were investigated in [1] and [2].

The main objective of this paper is to show

**Theorem 1.** *The principal inverse  $\Gamma^{-1}(x)$  of  $\Gamma(x)$  has the holomorphic extension  $\Gamma^{-1}(z)$  to  $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$ , which satisfies*

- (i)  $\Gamma^{-1}(\Pi_+) \subset \Pi_+$  and  $\Gamma^{-1}(\Pi_-) \subset \Pi_-$ ,
- (ii)  $\Gamma^{-1}(z)$  is univalent,
- (iii)  $\Gamma(\Gamma^{-1}(z)) = z$  for  $z \in \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$ .

We remark that (iii) implies that  $D := \Gamma^{-1}(\mathbf{C} \setminus (-\infty, \Gamma(\alpha)])$  is a domain including  $(\alpha, \infty)$  and  $\Gamma(D) = \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$ , and  $\Gamma^{-1}(\Gamma(z)) = z$  for  $z \in D$ .

To prove Theorem 1 we use the theory of kernel functions. Let  $K(x, y)$  be a real continuous function defined on  $I \times I$  and suppose  $K(x, y) = K(y, x)$ . Then  $K(x, y)$  is said to be a *positive semidefinite* (abbreviated to *p.s.d.*) kernel function on an interval  $I \times I$  if

$$(1.4) \quad \iint_{I \times I} K(x, y)\phi(x)\phi(y)dx dy \geq 0$$

for every real continuous function  $\phi$  with compact support in  $I$ . In this case, (1.4) holds for complex-valued functions  $\phi(x)$  as well, provided we take the complex conjugate of  $\phi(y)$ . It is clear that  $K(x, y)$  is p.s.d. if and only if for each  $n$  and for all  $n$  points  $x_i \in I$ , the  $n \times n$  matrices

$$(K(x_i, x_j))_{i,j=1}^n$$

are positive semidefinite matrices. Suppose  $K(x, y) \geq 0$  for every  $x, y$  in  $I$ . Then  $K(x, y)$  is said to be *infinitely divisible* if  $K(x, y)^a$  is p.s.d. for every  $a > 0$ .  $K(x, y)$  is said to be *conditionally (or almost) positive semidefinite* (abbreviated to *c.p.s.d.*) on  $I$  if (1.4) holds for every continuous function  $\phi$  on  $I$  such that the support of  $\phi$  is compact and the integral of  $\phi$  over  $I$  vanishes. One can see that  $K(x, y)$  is c.p.s.d. if and only if

$$(1.5) \quad \sum_{i,j=1}^n K(x_i, x_j)z_i\bar{z}_j \geq 0$$

for each  $n$ , for all  $n$  points  $x_i \in I$  and for  $n$  complex numbers  $z_i$  with  $\sum_{i=1}^n z_i = 0$ . Let  $f(x)$  be a  $C^1$ -function on  $I$ . Then the *Löwner kernel function* is defined by

$$K_f(x, y) = \begin{cases} \frac{f(x)-f(y)}{x-y} & (x \neq y), \\ f'(x) & (x = y). \end{cases}$$

The following excellent theorem is due to Löwner [7] (also see Koranyi [6] and [8]).

**Theorem A.** *Let  $f(x)$  be a  $C^1$ -function on  $I$ . Then the Löwner kernel function  $K_f(x, y)$  is p.s.d. if and only if  $f(x)$  has a holomorphic extension  $f(z)$  to  $\Pi_+$  and it is a Pick function.*

2. PROOF OF THEOREM 1

We begin with a simple fact which follows from Theorem A because  $\log x$  defined on  $(0, \infty)$  extends to a Pick function.

**Lemma 2.**

$$K_1(x, y) := \begin{cases} \frac{\log x - \log y}{x - y} & (x \neq y) \\ \frac{1}{x} & (x = y) \end{cases}$$

is p.s.d. on  $(0, \infty) \times (0, \infty)$ .

**Lemma 3.** Let  $K_2(x, y)$  be the function defined on  $(0, \infty) \times (0, \infty)$  by

$$K_2(x, y) := \begin{cases} \frac{\log \Gamma(x) - \log \Gamma(y)}{x - y} & (x \neq y) \\ \frac{\Gamma'(x)}{\Gamma(x)} & (x = y). \end{cases}$$

Then  $-K_2(x, y)$  is c.p.s.d. on  $(0, \infty)$ .

*Proof.* Suppose that the support of  $\phi(x)$  is included in  $[m, M]$  with  $m > 0$  and  $\int_m^M \phi(x) dx = 0$ . From (1.2) it follows that  $-K_2(x, y) = K_1(x, y) + \gamma - K_g(x, y)$ , where  $K_g$  is a Löwner kernel function of  $g$  defined by

$$g(x) = \sum_{k=1}^{\infty} \left( \frac{x}{k} - \log\left(1 + \frac{x}{k}\right) \right).$$

Since  $K_1(x, y)$  is p.s.d. and  $\int_0^\infty \int_0^\infty \gamma \phi(x) \phi(y) dx dy = 0$ , we have only to show that  $-K_g(x, y)$  is c.p.s.d. Put

$$g_n(x) = \sum_{k=1}^n \left( \frac{x}{k} - \log\left(1 + \frac{x}{k}\right) \right).$$

Then

$$g'_n(x) = \sum_{k=1}^n \frac{x}{k(k+x)}$$

converges uniformly to  $\sum_{k=1}^\infty \frac{x}{k(k+x)} = g'(x)$  on  $[0, M]$ . The sequence of Löwner kernel functions  $K_{g_n}(x, y)$  converges uniformly to  $K_g(x, y)$  on  $[0, M] \times [0, M]$ ; indeed,

$$K_{g_n}(x, y) - K_g(x, y) = \begin{cases} \frac{1}{x-y} \int_y^x (g'_n(t) - g'(t)) dt & (x \neq y) \\ g'_n(x) - g'(x) & (x = y). \end{cases}$$

Since

$$-K_{g_n}(x, y) = \sum_{k=1}^n \left( -\frac{1}{k} + \frac{1}{k} K_1\left(1 + \frac{x}{k}, 1 + \frac{y}{k}\right) \right)$$

is c.p.s.d., so is  $-K_g(x, y)$ . □

Note that a similar proof is given in [3], where it is proved that  $\Gamma(z)^c$  is the Mellin transform of a positive measure on  $(0, \infty)$  for each  $c > 0$ .

The following is known ([8], p. 152; [9] and [10]), but for completeness we give a proof.

**Lemma 4.** *Let  $K(x, y) > 0$  for  $x, y \in I$ . If  $-K(x, y)$  is c.p.s.d. on  $I \times I$ , then the reciprocal function  $\frac{1}{K(x, y)}$  is infinitely divisible there.*

*Proof.* Take  $x_i \in I$  ( $i = 1, 2, \dots, n$ ) and put  $a_{ij} = K(x_i, x_j)$ . Define  $b_{ij}$  by

$$b_{ij} = a_{ij} - a_{in} - a_{nj} + a_{nn} \quad (1 \leq i, j \leq n).$$

Since  $-a_{ij} = -K(x_i, x_j)$  satisfies (1.5), the matrix  $(-b_{ij})$  is positive semidefinite (see p. 134 of [8] or p. 458 of [11]). By Schur's theorem, the matrix  $(e^{-b_{ij}})$  is positive semidefinite too.

$$e^{-a_{ij}} = e^{-a_{in} + \frac{a_{nn}}{2}} e^{-b_{ij}} e^{-a_{nj} + \frac{a_{nn}}{2}}$$

implies that the matrix  $(\exp(-a_{ij}))$  is p.s.d. as well. Thus we have shown that  $\exp(-K(x, y))$  is p.s.d. We note that  $\exp(-tK(x, y))$  is also p.s.d. for  $t > 0$  since  $-tK(x, y)$  is c.p.s.d. By making use of

$$\Gamma(a) = k^a \int_0^\infty e^{-kt} t^{a-1} dt \quad (a > 0)$$

we get

$$K(x, y)^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty \exp(-tK(x, y)) t^{a-1} dt,$$

which is p.s.d. This implies that  $1/K(x, y)$  is infinitely divisible. □

**Lemma 5.** *Let  $K_3(x, y)$  be the kernel function defined on  $(\alpha, \infty) \times (\alpha, \infty)$  by*

$$K_3(x, y) = \begin{cases} \frac{x-y}{\Gamma(x)-\Gamma(y)} & (x \neq y) \\ \frac{1}{\Gamma'(x)} & (x = y). \end{cases}$$

*Then  $K_3(x, y)$  is p.s.d.*

*Proof.* Let  $K_1(x, y)$  and  $K_2(x, y)$  be the kernel functions defined in Lemma 2 and Lemma 3, respectively. Since  $K_1(x, y)$  is p.s.d. on  $(0, \infty) \times (0, \infty)$  and since  $\Gamma(x)$  is differentiable and increasing on  $(\alpha, \infty)$ ,

$$K_1(\Gamma(x), \Gamma(y)) = \begin{cases} \frac{\log \Gamma(x) - \log \Gamma(y)}{\Gamma(x) - \Gamma(y)} & (x \neq y) \\ \frac{1}{\Gamma'(x)} & (x = y) \end{cases}$$

is p.s.d. on  $(\alpha, \infty) \times (\alpha, \infty)$ . Since  $\Gamma(x)$  is increasing on  $(\alpha, \infty)$ ,  $K_2(x, y) > 0$  for  $(x, y) \in (\alpha, \infty) \times (\alpha, \infty)$ . By Lemma 3 and Lemma 4,  $\frac{1}{K_2(x, y)}$  is not only p.s.d. but also infinitely divisible. Thus the Schur product

$$K_3(x, y) = K_1(\Gamma(x), \Gamma(y)) \cdot \frac{1}{K_2(x, y)}$$

is p.s.d. on  $(\alpha, \infty) \times (\alpha, \infty)$  too. □

*Proof of Theorem 1.* The Löwner kernel  $K_{\Gamma^{-1}}(x, y)$  defined on  $(\Gamma(\alpha), \infty) \times (\Gamma(\alpha), \infty)$  by

$$K_{\Gamma^{-1}}(x, y) = \begin{cases} \frac{\Gamma^{-1}(x) - \Gamma^{-1}(y)}{x - y} & (x \neq y) \\ (\Gamma^{-1})'(x) & (x = y) \end{cases}$$

coincides with  $K_3(\Gamma^{-1}(x), \Gamma^{-1}(y))$ , which is p.s.d. Thus by Theorem A,  $\Gamma^{-1}(x)$  has the holomorphic extension  $\Gamma^{-1}(z)$  onto  $\Pi_+$ , which is a Pick function. By reflection,  $\Gamma^{-1}(x)$  also has a holomorphic extension to  $\Pi_-$  and the range is in it. We thus get (i).  $\Gamma(\Gamma^{-1}(z))$  is thus holomorphic on the simply connected domain  $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$ , and  $\Gamma(\Gamma^{-1}(x)) = x$  for  $\Gamma(\alpha) < x < \infty$ . By the uniqueness theorem,  $\Gamma(\Gamma^{-1}(z)) = z$  for  $z \in \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$ . This means (iii), which clearly yields (ii).  $\square$

In the proof of Lemma 4 we saw that if  $-K(x, y)$  is c.p.s.d., then  $e^{-K(x,y)}$  is infinitely divisible. For  $K_2(x, y)$  in Lemma 3,

$$e^{-K_2(x,y)} = \begin{cases} \left(\frac{\Gamma(y)}{\Gamma(x)}\right)^{\frac{1}{x-y}} & (x \neq y) \\ e^{-\frac{\Gamma'(x)}{\Gamma(x)}} & (x = y) \end{cases}$$

is infinitely divisible. Since  $\Gamma(x + 1) = x\Gamma(x)$ ,

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma, \quad \frac{\Gamma'(m+1)}{\Gamma(m+1)} = -\gamma + 1 + \dots + \frac{1}{m}, \quad \frac{\Gamma(n)}{\Gamma(m)} = \frac{(n-1)!}{(m-1)!}.$$

The following  $(n + 1) \times (n + 1)$  matrix is therefore not only p.s.d. but also infinitely divisible:

$$\left( e^{-K_2(i,j)} \right) = \begin{pmatrix} e^\gamma & \left(\frac{1!}{1!}\right)^{-1} & (2!)^{-\frac{1}{2}} & (3!)^{-\frac{1}{3}} & \dots & (n!)^{-\frac{1}{n}} \\ \left(\frac{1!}{1!}\right)^{-1} & e^{\gamma-1} & \left(\frac{2!}{1!}\right)^{-1} & \left(\frac{3!}{1!}\right)^{-\frac{1}{2}} & \dots & \left(\frac{n!}{1!}\right)^{-\frac{1}{n-1}} \\ (2!)^{-\frac{1}{2}} & \left(\frac{2!}{1!}\right)^{-1} & e^{\gamma-1-\frac{1}{2}} & \left(\frac{3!}{2!}\right)^{-1} & \dots & \left(\frac{n!}{2!}\right)^{-\frac{1}{n-2}} \\ (3!)^{-\frac{1}{3}} & \left(\frac{3!}{1!}\right)^{-\frac{1}{2}} & \left(\frac{3!}{2!}\right)^{-1} & e^{\gamma-1-\frac{1}{2}-\frac{1}{3}} & \dots & \left(\frac{n!}{3!}\right)^{-\frac{1}{n-3}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (n!)^{-\frac{1}{n}} & \left(\frac{n!}{1!}\right)^{-\frac{1}{n-1}} & \left(\frac{n!}{2!}\right)^{-\frac{1}{n-2}} & \left(\frac{n!}{3!}\right)^{-\frac{1}{n-3}} & \dots & e^{\gamma-1-\frac{1}{2}-\dots-\frac{1}{n}} \end{pmatrix}.$$

Since  $\Gamma^{-1}(z)$  is holomorphic on  $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$  and a Pick function, by Herglotz's theorem  $\Gamma^{-1}(x)$  has the following integral representation with the Borel measure  $\mu(t)$  ([4], [8], [12]):

**Corollary 6.**

$$(2.1) \quad \Gamma^{-1}(x) = a + bx + \int_{-\infty}^{\Gamma(\alpha)} \left(\frac{1}{x-t} - \frac{t}{t^2+1}\right) d\mu(t),$$

where  $\int_{-\infty}^{\Gamma(\alpha)} \frac{1}{t^2+1} d\mu(t) < \infty$ , and  $a, b$  are real numbers and  $b \geq 0$ .

**Question.** How can we extend the inverse function of the restricted function of  $\Gamma(x)$  to  $(0, \alpha)$ ?

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