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THE PRINCIPAL INVERSE OF THE GAMMA FUNCTION

MITSURU UCHIYAMA

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ABSTRACT. Let $\Gamma(x)$ be the gamma function in the real axis and α the maximal zero of $\Gamma'(x)$. We call the inverse function of $\Gamma(x)|_{(\alpha,\infty)}$ the principal inverse and denote it by $\Gamma^{-1}(x)$. We show that $\Gamma^{-1}(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ to $\mathbf{C}\setminus(-\infty,\Gamma(\alpha)]$, which maps the upper half-plane into itself, namely a Pick function, and that $\Gamma(\Gamma^{-1}(z)) = z$ on $\mathbf{C}\setminus(-\infty,\Gamma(\alpha)]$.

1. INTRODUCTION

The Euler form of the gamma function $\Gamma(x)$ is given by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$$

for $x > 0$. The Weierstrass form

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}(1+\frac{x}{n})e^{-\frac{x}{n}} \tag{1.1}$$

extends it to $\mathbf{R}\setminus\{0,-1,-2,\cdots\}$, where γ is the Euler constant defined by

$$\gamma = \lim_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n) = 0.57721\cdots .$$

From this it follows that

$$\log\Gamma(x) = -\log x - \gamma x + \sum_{n=1}^{\infty}\left(\frac{x}{n}-\log(1+\frac{x}{n})\right), \tag{1.2}$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty}(\frac{1}{n+1}-\frac{1}{n+x}). \tag{1.3}$$

This is called the *psi function* or *digamma function*. It is clear that $\Gamma(1) = \Gamma(2) = 1$, $\Gamma'(1) = -\gamma$, $\Gamma'(2) = -\gamma + 1$. Denote the unique zero in $(0,\infty)$ of $\Gamma'(x)$ by α. It is known that $\alpha = 1.4616\cdots$ and $\Gamma(\alpha) = 0.8856\cdots$. We call the inverse function of the restriction of $\Gamma(x)$ to (α,∞) *the principal inverse function* and denote it by Γ^{-1}. $\Gamma^{-1}(x)$ is an increasing and concave function defined on $(\Gamma(\alpha),\infty)$. (1.1) guarantees that $\Gamma(x)$ has the holomorphic extension which is a meromorphic function with poles at non-positive integers and (1.3) holds there. Let Π_+ and Π_- be respectively the

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open upper half-plane and the open lower half-plane. A holomorphic function defined on Π_+ is called a *Pick function* or *Nevanlinna function* if it maps Π_+ into itself. From (1.3) it follows that $\Gamma'(z)$ does not vanish on $\mathbf{C} \setminus (-\infty, \alpha]$; in fact, $\Gamma'(z)/\Gamma(z)$ is a Pick function. Hence for each point $\omega_0 \in \Gamma(\mathbf{C} \setminus (-\infty, \alpha])$ there is a local inverse of $\Gamma(z)$ in a neighborhood of ω_0.

Some other Pick functions related to the gamma function were investigated in [1] and [2].

The main objective of this paper is to show

Theorem 1. *The principal inverse $\Gamma^{-1}(x)$ of $\Gamma(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ to $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$, which satisfies*

(i) $\Gamma^{-1}(\Pi_+) \subset \Pi_+$ *and* $\Gamma^{-1}(\Pi_-) \subset \Pi_-$,

(ii) $\Gamma^{-1}(z)$ *is univalent,*

(iii) $\Gamma(\Gamma^{-1}(z)) = z$ *for* $z \in \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$.

We remark that (iii) implies that $D := \Gamma^{-1}(\mathbf{C} \setminus (-\infty, \Gamma(\alpha)])$ is a domain including (α, ∞) and $\Gamma(D) = \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$, and $\Gamma^{-1}(\Gamma(z)) = z$ for $z \in D$.

To prove Theorem 1 we use the theory of kernel functions. Let $K(x, y)$ be a *real* continuous function defined on $I \times I$ and suppose $K(x, y) = K(y, x)$. Then $K(x, y)$ is said to be a *positive semidefinite* (abbreviated to *p.s.d.*) kernel function on an interval $I \times I$ if

$$\iint_{I \times I} K(x, y)\phi(x)\phi(y)dxdy \geqq 0 \tag{1.4}$$

for every real continuous function ϕ with compact support in I. In this case, (1.4) holds for complex-valued functions $\phi(x)$ as well, provided we take the complex conjugate of $\phi(y)$. It is clear that $K(x, y)$ is p.s.d. if and only if for each n and for all n points $x_i \in I$, the $n \times n$ matrices

$$(K(x_i, x_j))_{i,j=1}^n$$

are positive semidefinite matrices. Suppose $K(x, y) \geqq 0$ for every x, y in I. Then $K(x, y)$ is said to be *infinitely divisible* if $K(x, y)^a$ is p.s.d. for every $a > 0$. $K(x, y)$ is said to be *conditionally (or almost) positive semidefinite* (abbreviated to *c.p.s.d.*) on I if (1.4) holds for every continuous function ϕ on I such that the support of ϕ is compact and the integral of ϕ over I vanishes. One can see that $K(x, y)$ is c.p.s.d. if and only if

$$\sum_{i,j=1}^n K(x_i, x_j) z_i \overline{z_j} \geqq 0 \tag{1.5}$$

for each n, for all n points $x_i \in I$ and for n complex numbers z_i with $\sum_{i=1}^n z_i = 0$. Let $f(x)$ be a C^1-function on I. Then the *Löwner kernel* function is defined by

$$K_f(x, y) = \begin{cases} \frac{f(x) - f(y)}{x - y} & (x \neq y), \\ f'(x) & (x = y). \end{cases}$$

The following excellent theorem is due to Löwner [7] (also see Koranyi [6] and [8]).

Theorem A. *Let $f(x)$ be a C^1-function on I. Then the Löwner kernel function $K_f(x, y)$ is p.s.d. if and only if $f(x)$ has a holomorphic extension $f(z)$ to Π_+ and it is a Pick function.*

2. Proof of Theorem 1

We begin with a simple fact which follows from Theorem A because $\log x$ defined on $(0,\infty)$ extends to a Pick function.

Lemma 2.

$$K_1(x,y) := \begin{cases} \frac{\log x - \log y}{x-y} & (x \neq y) \\ \frac{1}{x} & (x = y) \end{cases}$$

is p.s.d. on $(0,\infty)\times(0,\infty)$.

Lemma 3. *Let* $K_2(x,y)$ *be the function defined on* $(0,\infty)\times(0,\infty)$ *by*

$$K_2(x,y) := \begin{cases} \frac{\log\Gamma(x) - \log\Gamma(y)}{x-y} & (x \neq y) \\ \frac{\Gamma'(x)}{\Gamma(x)} & (x = y). \end{cases}$$

Then $-K_2(x,y)$ *is c.p.s.d. on* $(0,\infty)$.

Proof. Suppose that the support of $\phi(x)$ is included in $[m,M]$ with $m>0$ and $\int_m^M \phi(x)dx = 0$. From (1.2) it follows that $-K_2(x,y) = K_1(x,y) + \gamma - K_g(x,y)$, where K_g is a Löwner kernel function of g defined by

$$g(x) = \sum_{k=1}^{\infty}\left(\frac{x}{k} - \log(1+\frac{x}{k})\right).$$

Since $K_1(x,y)$ is p.s.d. and $\int_0^\infty\int_0^\infty \gamma\phi(x)\phi(y)dxdy = 0$, we have only to show that $-K_g(x,y)$ is c.p.s.d. Put

$$g_n(x) = \sum_{k=1}^{n}\left(\frac{x}{k} - \log(1+\frac{x}{k})\right).$$

Then

$$g_n'(x) = \sum_{k=1}^{n}\frac{x}{k(k+x)}$$

converges uniformly to $\sum_{k=1}^\infty \frac{x}{k(k+x)} = g'(x)$ on $[0,M]$. The sequence of Löwner kernel functions $K_{g_n}(x,y)$ converges uniformly to $K_g(x,y)$ on $[0,M]\times[0,M]$; indeed,

$$K_{g_n}(x,y) - K_g(x,y) = \begin{cases} \frac{1}{x-y}\int_y^x (g_n'(t) - g'(t))\,dt & (x\neq y) \\ g_n'(x) - g'(x) & (x = y). \end{cases}$$

Since

$$-K_{g_n}(x,y) = \sum_{k=1}^{n}\left(-\frac{1}{k} + \frac{1}{k}K_1(1+\frac{x}{k}, 1+\frac{y}{k})\right)$$

is c.p.s.d., so is $-K_g(x,y)$. □

Note that a similar proof is given in [3], where it is proved that $\Gamma(z)^c$ is the Mellin transform of a positive measure on $(0,\infty)$ for each $c>0$.

The following is known ([8], p. 152; [9] and [10]), but for completeness we give a proof.

Lemma 4. *Let $K(x,y)>0$ for $x,y \in I$. If $-K(x,y)$ is c.p.s.d. on $I \times I$, then the reciprocal function $\frac{1}{K(x,y)}$ is infinitely divisible there.*

Proof. Take $x_i \in I$ $(i=1,2,\cdots,n)$ and put $a_{ij}=K(x_i,x_j)$. Define b_{ij} by

$$b_{ij}=a_{ij}-a_{in}-a_{nj}+a_{nn} \quad (1 \leqq i,j \leqq n).$$

Since $-a_{ij}=-K(x_i,x_j)$ satisfies (1.5), the matrix $(-b_{ij})$ is positive semidefinite (see p. 134 of [8] or p. 458 of [11]). By Schur's theorem, the matrix $(e^{-b_{ij}})$ is positive semidefinite too.

$$e^{-a_{ij}}=e^{-a_{in}+\frac{a_{nn}}{2}}e^{-b_{ij}}e^{-a_{nj}+\frac{a_{nn}}{2}}$$

implies that the matrix $(\exp(-a_{ij}))$ is p.s.d. as well. Thus we have shown that $\exp(-K(x,y))$ is p.s.d. We note that $\exp(-tK(x,y))$ is also p.s.d. for $t>0$ since $-tK(x,y)$ is c.p.s.d. By making use of

$$\Gamma(a)=k^a\int_0^\infty e^{-kt}t^{a-1}dt \quad (a>0)$$

we get

$$K(x,y)^{-a}=\frac{1}{\Gamma(a)}\int_0^\infty \exp(-tK(x,y))t^{a-1}dt,$$

which is p.s.d. This implies that $1/K(x,y)$ is infinitely divisible. $\square$

Lemma 5. *Let $K_3(x,y)$ be the kernel function defined on $(\alpha,\infty)\times(\alpha,\infty)$ by*

$$K_3(x,y)=\begin{cases}\frac{x-y}{\Gamma(x)-\Gamma(y)} & (x\neq y)\\ \frac{1}{\Gamma'(x)} & (x=y).\end{cases}$$

Then $K_3(x,y)$ is p.s.d.

Proof. Let $K_1(x,y)$ and $K_2(x,y)$ be the kernel functions defined in Lemma 2 and Lemma 3, respectively. Since $K_1(x,y)$ is p.s.d. on $(0,\infty)\times(0,\infty)$ and since $\Gamma(x)$ is differentiable and increasing on (α,∞),

$$K_1(\Gamma(x),\Gamma(y))=\begin{cases}\frac{\log\Gamma(x)-\log\Gamma(y)}{\Gamma(x)-\Gamma(y)} & (x\neq y)\\ \frac{1}{\Gamma(x)} & (x=y)\end{cases}$$

is p.s.d. on $(\alpha,\infty)\times(\alpha,\infty)$. Since $\Gamma(x)$ is increasing on (α,∞), $K_2(x,y)>0$ for $(x,y)\in(\alpha,\infty)\times(\alpha,\infty)$. By Lemma 3 and Lemma 4, $\dfrac{1}{K_2(x,y)}$ is not only p.s.d. but also infinitely divisible. Thus the Schur product

$$K_3(x,y)=K_1(\Gamma(x),\Gamma(y))\cdot\frac{1}{K_2(x,y)}$$

is p.s.d. on $(\alpha,\infty)\times(\alpha,\infty)$ too. $\square$

Proof of Theorem 1. The Löwner kernel $K_{\Gamma^{-1}}(x,y)$ defined on $(\Gamma(\alpha),\infty)\times(\Gamma(\alpha),\infty)$ by

$$K_{\Gamma^{-1}}(x,y)=\begin{cases}\frac{\Gamma^{-1}(x)-\Gamma^{-1}(y)}{x-y} & (x\neq y)\\ (\Gamma^{-1})'(x) & (x=y)\end{cases}$$

coincides with $K_3(\Gamma^{-1}(x), \Gamma^{-1}(y))$, which is p.s.d. Thus by Theorem A, $\Gamma^{-1}(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ onto Π_+, which is a Pick function. By reflection, $\Gamma^{-1}(x)$ also has a holomorphic extension to Π_- and the range is in it. We thus get (i). $\Gamma(\Gamma^{-1}(z))$ is thus holomorphic on the simply connected domain $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$, and $\Gamma(\Gamma^{-1}(x)) = x$ for $\Gamma(\alpha) < x < \infty$. By the uniqueness theorem, $\Gamma\left(\Gamma^{-1}(z)\right) = z$ for $z \in \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$. This means (iii), which clearly yields (ii). $\square$

In the proof of Lemma 4 we saw that if $-K(x, y)$ is c.p.s.d., then $e^{-K(x,y)}$ is infinitely divisible. For $K_2(x, y)$ in Lemma 3,

$$e^{-K_2(x,y)} = \begin{cases} \left(\frac{\Gamma(y)}{\Gamma(x)}\right)^{\frac{1}{x-y}} & (x \neq y) \\ e^{-\frac{\Gamma'(x)}{\Gamma(x)}} & (x = y) \end{cases}$$

is infinitely divisible. Since $\Gamma(x + 1) = x\Gamma(x)$,

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma,\ \frac{\Gamma'(m + 1)}{\Gamma(m + 1)} = -\gamma + 1 + \cdots + \frac{1}{m},\ \frac{\Gamma(n)}{\Gamma(m)} = \frac{(n - 1)!}{(m - 1)!}.$$

The following $(n+1)\times(n+1)$ matrix is therefore not only p.s.d. but also infinitely divisible:

$$\left(e^{-K_2(i,j)}\right) = \begin{pmatrix} e^{\gamma} & (\frac{1!}{1!})^{-1} & (2!)^{-\frac{1}{2}} & (3!)^{-\frac{1}{3}} & \cdots & (n!)^{-\frac{1}{n}} \\ (\frac{1!}{1!})^{-1} & e^{\gamma-1} & (\frac{2!}{1!})^{-1} & (\frac{3!}{1!})^{-\frac{1}{2}} & \cdots & (\frac{n!}{1!})^{-\frac{1}{n-1}} \\ (2!)^{-\frac{1}{2}} & (\frac{2!}{1!})^{-1} & e^{\gamma-1-\frac{1}{2}} & (\frac{3!}{2!})^{-1} & \cdots & (\frac{n!}{2!})^{-\frac{1}{n-2}} \\ (3!)^{-\frac{1}{3}} & (\frac{3!}{1!})^{-\frac{1}{2}} & (\frac{3!}{2!})^{-1} & e^{\gamma-1-\frac{1}{2}-\frac{1}{3}} & \cdots & (\frac{n!}{3!})^{-\frac{1}{n-3}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (n!)^{-\frac{1}{n}} & (\frac{n!}{1!})^{-\frac{1}{n-1}} & (\frac{n!}{2!})^{-\frac{1}{n-2}} & (\frac{n!}{3!})^{-\frac{1}{n-3}} & \cdots & e^{\gamma-1-\frac{1}{2}-\cdots-\frac{1}{n}} \end{pmatrix}.$$

Since $\Gamma^{-1}(z)$ is holomorphic on $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$ and a Pick function, by Herglotz's theorem $\Gamma^{-1}(x)$ has the following integral representation with the Borel measure $\mu(t)$ ([4], [8], [12]):

Corollary 6.

$$\Gamma^{-1}(x) = a + bx + \int_{-\infty}^{\Gamma(\alpha)} \left(\frac{1}{x - t} - \frac{t}{t^2 + 1}\right) d\mu(t), \tag{2.1}$$

where $\int_{-\infty}^{\Gamma(\alpha)} \frac{1}{t^2+1} d\mu(t) < \infty$, *and* a, b *are real numbers and* $b \geqq 0$.

Question. How can we extend the inverse function of the restricted function of $\Gamma(x)$ to $(0, \alpha)$?

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Department of Mathematics, Interdisciplinary Faculty of Science and Engineering, Shimane University, Matsue City, Shimane 690-8504, Japan

E-mail address: `uchiyama@riko.shimane-u.ac.jp`