GENERALIZATION OF A THEOREM
OF CLUNIE AND HAYMAN

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Abstract. Clunie and Hayman proved that if the spherical derivative ∥f′∥
of an entire function satisfies ∥f′∥(z) = O(|z|^σ), then T(r, f) = O(r^{σ+1}). We generalize this to holomorphic curves in projective space of dimension n omitting n hyperplanes in general position.

Introduction

We consider holomorphic curves f : C → P^n; for the general background on the subject we refer to [7]. The Fubini–Study derivative ∥f′∥ measures the length distortion from the Euclidean metric in C to the Fubini–Study metric in P^n. The explicit expression is

∥f′∥^2 = ∥f∥^{-4} \sum_{i<j} |f'_i f_j - f_i f'_j|^2,

where (f_0, . . . , f_n) is a homogeneous representation of f (that is, the f_j are entire functions which never simultaneously vanish), and

∥f∥^2 = \sum_{j=0}^n |f_j|^2.


We recall that the Nevanlinna–Cartan characteristic is defined by

T(r, f) = \int_0^r \frac{dt}{t} \left( \frac{1}{\pi} \int_{|z|\leq t} ∥f′∥^2(z)dm(z) \right),

where dm is the area element in C. So the condition

(1) lim sup_{z→∞} |z|^{-σ} ∥f′(z)∥ ≤ K < ∞

implies

(2) lim sup_{r→∞} \frac{T(r, f)}{r^{σ+2}} < ∞.
Clunie and Hayman [4] found that for curves $C \to P^1$ omitting one point in $P^1$, a stronger conclusion follows from (1), namely

$$\limsup_{r \to \infty} \frac{T(r,f)}{r^{\sigma+1}} \leq KC(\sigma).$$

In the most important case of $\sigma = 0$, a different proof of this fact for $n = 1$ is due to Pommerenke [8]. Pommerenke’s method gives the exact constant $C(0)$. In this paper we prove that this phenomenon persists in all dimensions.

**Theorem.** For holomorphic curves $f : C \to P^n$ omitting $n$ hyperplanes in general position, condition (1) implies (3) with an explicit constant $C(n,\sigma)$.

In [6], the case $\sigma = 0$ was considered. There it was proved that holomorphic curves in $P^n$ with bounded spherical derivative and omitting $n$ hyperplanes in general position must satisfy $T(r,f) = O(r)$. With a stronger assumption that $f$ omits $n+1$ hyperplanes this was earlier established by Berteloot and Duval [2] and by Tsukamoto [9]. The proof in [6] has two drawbacks: it does not extend to arbitrary $\sigma \geq 0$, and it is non-constructive; unlike Clunie–Hayman and Pommerenke’s proofs mentioned above, it does not give an explicit constant in (3).

It is shown in [6] that the condition that $n$ hyperplanes are omitted is exact: there are curves in any dimension $n$ satisfying (1), $T(r,f) \sim cr^{2\sigma+2}$ and omitting $n-1$ hyperplanes.

**Preliminaries**

Without loss of generality we assume that the omitted hyperplanes are given in the homogeneous coordinates by the equations $\{w_j = 0\}, 1 \leq j \leq n$. We fix a homogeneous representation $(f_0, \ldots, f_n)$ of our curve, where $f_j$ are entire functions and $f_n = 1$. Then

$$u = \log \sqrt{|f_0|^2 + \ldots + |f_n|^2}$$

is a positive subharmonic function, and Jensen’s formula gives

$$T(r,f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) d\theta - u(0) = \int_0^r \frac{n(t)}{t} dt,$$

where $n(t) = \mu(\{z : |z| \leq t\})$ and $\mu = \mu_u$ is the Riesz measure of $u$, that is, the measure with the density

$$\frac{1}{2\pi} \Delta u = \frac{1}{\pi} \|f'\|^2.$$

This measure $\mu$ is also called Cartan’s measure of $f$. Positivity of $u$ and (2) imply that all $f_j$ are of order at most $2\sigma + 2$, normal type. As $f_j(z) \neq 0, 1 \leq j \leq n$, we conclude that

$$f_j = e^{P_j}, \quad 1 \leq j \leq n,$$

where

$$P_j$$

are polynomials of degree at most $2\sigma + 2$.

We need two lemmas from potential theory.

**Lemma 1** [6]. Let $v$ be a non-negative harmonic function in the closure of the disc $B(a,R)$, and assume that $v(z_1) = 0$ for some point $z_1 \in \partial B(a,R)$. Then

$$v(a) \leq 2R|\nabla v(z_1)|.$$
We include a proof, suggested by the referee, which is simpler than that given in [6]. Without loss of generality, assume that $a = 0, R = 1, z_1 = 1$. Then Harnack’s inequality gives

$$v(0) \frac{1 + r}{1 - r} \leq v(r) - v(1) \frac{1}{1 - r}. $$

Passing to the limit as $r \to 1$, we obtain the result.

Lemma 2. Let $v$ be a non-negative superharmonic function in the closure of the disc $B(a, R)$ and suppose that $v(z_1) = 0$ for some $z_1 \in \partial B(a, R)$. Then

$$|\mu_v(B(a, R/2))| \leq 3R \left| \frac{\partial v}{\partial n}(z_1) \right|. $$

By $|\partial v/\partial n|$ we mean here $\liminf |v(rz_1)|/(R(1 - r))$ as $r \to 1^-$. 

Proof. The function $v(a + Rz)$ satisfies the conditions of the lemma with $R = 1$. So it is enough to prove the lemma with $a = 0$ and $R = 1$. Let

$$w(z) = \int_{|\zeta| \leq 1/2} G(z, \zeta) d\mu_v(\zeta) $$

be the Green potential of the restriction of $\mu_v$ onto the disc $|\zeta| \leq 1/2$, that is,

$$G(z, \zeta) = \log \left| \frac{1 - \zeta z}{z - \zeta} \right|. $$

Then $w \leq v$ and $w(z_1) = v(z_1) = 0$, which implies that

$$\left| \frac{\partial v}{\partial n}(z_1) \right| \geq \left| \frac{\partial w}{\partial |z|}(z_1) \right|. $$

Minimizing $|\partial G/|z||$ over $|z| = 1$ and $|\zeta| = 1/2$, we obtain $1/3$, which proves the lemma.

Proof of the theorem

We may assume without loss of generality that $f_0$ has infinitely many zeros. Indeed, we can compose $f$ with an automorphism of $\mathbb{P}^n$; for example replace $f_0$ by $f_0 + cf_1$, $c \in \mathbb{C}$, and leave all other $f_j$ unchanged. This transformation changes neither the $n$ omitted hyperplanes nor the rate of growth of $T(r, f)$ and multiplies the spherical derivative by a bounded factor.

Let $u_j = \log |f_j|$ and

$$u^* = \max_{1 \leq j \leq n} u_j. $$

Here and in what follows max denotes the pointwise maximum of subharmonic functions.

Proposition 1. Suppose that at some point $z_1$ we have

$$u_m(z_1) = u_k(z_1) \geq u_j(z_1) $$

for some $m \neq k$ and all $j$ where $m, k, j \in \{0, \ldots, n\}$. Then

$$\|f'(z_1)\| \geq (n + 1)^{-1} |\nabla u_m(z_1) - \nabla u_k(z_1)|. $$
Suppose that in the disc for all $|z| > r$.

Proof. If $u_0(z) \leq u^*(z)$ for all sufficiently large $|z|$, then there is nothing to prove. Suppose that $u_0(a) > u^*(a)$, and consider the largest disc $B(a, R)$ centered at $a$ where the inequality $u_0(z) > u^*(z)$ persists. If $z_0$ is the zero of the smallest modulus of $f_0$, then $R \leq |a| + |z_0| < (1 + \epsilon)|a|$ when $|a|$ is large enough.

There is a point $z_1 \in \partial B(a, R)$ such that $u_0(z_1) = u^*(z_1)$. This means that there is some $k \in \{1, \ldots, n\}$ such that $u_0(z_1) = u_k(z_1) \geq u_m(z_1)$ for all $m \in \{1, \ldots, n\}$. Applying Proposition 1 we obtain

$$\|\nabla u_k(z_1) - \nabla u_0(z_1)\| \leq (n + 1)\|f'(z_1)\|.$$ 

Now $u_0(z) > u^*(z)$ for $z \in B(a, R)$, so we can apply Lemma 1 to $v = u_0 - u_k$ in the disc $B(a, R)$. This gives

$$u_0(a) - u_k(a) \leq 2R\|\nabla u_k(z_1) - \nabla u_0(z_1)\| \leq 2R(n + 1)\|f'(z_1)\|.$$ 

Now $R < (1 + \epsilon)|a|$ and $|z_1| \leq (2 + \epsilon)|a|$, so

$$u_0(a) \leq u^*(a) + K(2 + \epsilon)^{\sigma + 1}(n + 1)|a|^\sigma + 1,$$

and the result follows because $u = \max\{u_0, u^*\} + O(1)$.

Next we study the Riesz measure of the subharmonic function

$$u^* = \max\{u_1, \ldots, u_n\}.$$

We begin with the maximum of two harmonic functions. Let $u_1$ and $u_2$ be two harmonic functions in $\mathbb{C}$ of the form $u_j = \text{Re} P_j$ where $P_j \neq 0$ are polynomials. Suppose that $u_1 \neq u_2$. Then the set $E = \{z \in \mathbb{C} : u_1(z) = u_2(z)\}$ is a proper real-algebraic subset of $\mathbb{C}$ without isolated points. Apart from a finite set of ramification points, $E$ consists of smooth curves. For every smooth point $z \in E$, we denote by $J(z)$ the jump of the normal (to $E$) derivative of the function $w = \max\{u_1, u_2\}$ at the point $z$. This jump is always positive and the Riesz measure $\mu_w$ is given by the formula

$$d\mu_w = \frac{J(z)}{2\pi} |dz|,$$

which means that $\mu_w$ is supported by $E$ and has a density $J(z)/2\pi$ with respect to the length element $|dz|$ on $E$.

Now let $E_{i,j} = \{z : u_i(z) = u_j(z) \geq u_k(z), 1 \leq k \leq n\}$, and let $E = \bigcup E_{i,j}$ where the union is taken over all pairs $1 \leq i, j \leq n$ for which $u_i \neq u_j$. Then $E$ is a proper real semi-algebraic subset of $\mathbb{C}$ and $\infty$ is not an isolated point of $E$. For the elementary properties of semi-algebraic sets that we use here, see, for example,
There exists \( r_0 > 0 \) such that \( \Gamma = E \cap \{ r_0 < |z| < \infty \} \) is a union of finitely many disjoint smooth simple curves,

\[
\Gamma = \bigcup_{k=1}^{m} \Gamma_k.
\]

This union coincides with the support of \( \mu_{u^*} \) in \( \{ z : r_0 < |z| < \infty \} \).

Consider a point \( z_0 \in \Gamma \). Then \( z_0 \in \Gamma_k \) for some \( k \). As \( \Gamma_k \) is a smooth curve, there is a neighborhood \( D \) of \( z_0 \) which does not contain other curves \( \Gamma_j, j \neq k \), and which is divided by \( \Gamma_k \) into two parts, \( D_1 \) and \( D_2 \). Then there exist \( i \) and \( j \) such that \( u^*(z) = u_i(z), z \in D_1 \) and \( u^*(z) = u_j(z), z \in D_2 \), and \( u^*(z) = \max\{u_i(z), u_j(z)\} \), \( z \in D \). So the restriction of the Riesz measure \( \mu_{u^*} \) on \( D \) is supported by \( \Gamma_k \cap D \) and has density \( J(z)/(2\pi) \) where

\[
|J(z)| = |\partial u_i/\partial n - \partial u_j/\partial n|(z) = |\nabla (u_i - u_j)|(z)
\]

and \( \partial/\partial n \) is the derivation in the direction of a normal to \( \Gamma_k \). Taking into account that \( u_j = \text{Re} P_j \) where \( P_j \) are polynomials, we conclude that there exist positive numbers \( c_k \) and \( b_k \) such that

\[
J(z)/(2\pi) = (c_k + o(1))|z|^{b_k}, \quad z \to \infty, \quad z \in \Gamma_k.
\]

Let \( b = \max_k b_k \), and among those curves \( \Gamma_k \) for which \( b_k = b \) choose one with maximal \( c_k \) (which we denote by \( c_0 \)). We denote this chosen curve by \( \Gamma_0 \) and fix it for the rest of the proof.

**Proposition 3.** We have

\[
b \leq \sigma \quad \text{and} \quad c_0 \leq 3 \cdot 4^\sigma K(n+1).
\]

**Proof.** We consider two cases.

**Case 1.** There is a sequence \( z_n \to \infty, \ z_n \in \Gamma_0 \), such that \( u_0(z_n) \leq u^*(z_n) \). Then (1) and Proposition 1 imply that

\[
J(z_n) \leq (n+1)K|z_n|^{\sigma},
\]

and comparison with (8) shows that \( b \leq \sigma \) and \( c_0 \leq K(n+1)/(2\pi) \).

**Case 2.** \( u_0(z) > u^*(z) \) for all sufficiently large \( z \in \Gamma_0 \). Let \( a \) be a point on \( \Gamma_0 \), \( |a| > 3r_0 \), and \( u_0(a) > u^*(a) \). Let \( B(a, R) \) be the largest open disc centered at \( a \) in which the inequality \( u_0(z) > u^*(z) \) holds. Then

\[
R \leq |a| + O(1), \quad a \to \infty,
\]

because we assume that \( f_0 \) has zeros, so \( u_0(z_0) = -\infty \) for some \( z_0 \).

In \( B(a, R) \) we consider the positive superharmonic function \( v = u_0 - u^* \). Let us check that it satisfies the conditions of Lemma 2. The existence of a point \( z_1 \in \partial B(a, R) \) with \( v(z_1) = 0 \) follows from the definition of \( B(a, R) \). The Riesz measure of \( \mu_v \) is estimated using (7), (8):

\[
|\mu_v(B(a, R/2))| \geq |\mu_v(\Gamma_0 \cap B(a, R/2))| \geq c_0 R(|a| - R/2)^b.
\]

Now Lemma 2 applied to \( v \) in \( B(a, R) \) implies that

\[
|\nabla v(z_1)| \geq (c_0/3)(|a| - R/2)^b.
\]

On the other hand (1) and Proposition 1 imply that

\[
|\nabla v(z_1)| \leq K(n+1)(|a| + R)^\sigma.
\]
Combining these two inequalities and taking (11) into account, we obtain \( b \leq \sigma \) and \( c_0 \leq 3 \cdot 4^\sigma K(n+1) \), as required.

We denote
\[
T^*(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^*(re^{i\theta}) d\theta - u^*(0),
\]
This is the characteristic of the “reduced curve” \((f_1, \ldots, f_n)\).

**Proposition 4.**
\[
T^*(r) \leq 6 \cdot 4^\sigma K \frac{n(n+1)^2}{\sigma + 1} r^{\sigma + 1}.
\]

**Proof.** By Jensen’s formula,
\[
T^*(r) = \int_0^r \nu(t) \frac{dt}{t},
\]
where \( \nu(t) = \mu_u^* (\{ z : |z| \leq t \}) \). The number of curves \( \Gamma_k \) supporting the Riesz measure of \( u^* \) is easily seen to be at most \( 2n(n-1)(\sigma + 1) \). The density of the Riesz measure \( \mu_u^* \) on each curve \( \Gamma_k \) is given by (10), where \( c_k \leq c_0 \) and \( b_k \leq b \) and the parameters \( c_0 \) and \( b \) are estimated in Proposition 3. Combining all these data, we obtain the result.

It remains to combine Propositions 2 and 4 to obtain the final result.

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**References**


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