

SIMPLE RELATIONS IN THE CREMONA GROUP

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ABSTRACT. We give a simple set of generators and relations for the Cremona group of the plane. Namely, we show that the Cremona group is the amalgamated product of the de Jonquières group with the group of automorphisms of the plane, divided by one relation which is $\sigma\tau = \tau\sigma$, where $\tau = (x : y : z) \mapsto (y : x : z)$ and $\sigma = (x : y : z) \dashrightarrow (yz : xz : xy)$.

Let k be any fixed algebraically closed field. The *Cremona group* $\text{Bir}(\mathbb{P}^2)$ is the group of birational transformations of the projective plane $\mathbb{P}^2 = \mathbb{P}_k^2$.

The classical Noether-Castelnuovo Theorem says that $\text{Bir}(\mathbb{P}^2)$ is generated by the group $\text{Aut}(\mathbb{P}^2) \cong \text{PGL}(3, k)$, which we will denote by \mathbf{A} , and by the *standard quadratic transformation*

$$\sigma : (X : Y : Z) \dashrightarrow (YZ : XZ : XY).$$

For a proof which is valid over any algebraically closed field (in particular, in any characteristic), see for example [Sha, Chapter V, §5, Theorem 2, page 100].

A presentation of $\text{Bir}(\mathbb{P}^2)$ was given in [Giz]. The generators are all the quadratic transformations of the plane (among them, all elements of the form $a_1\sigma a_2$, where $a_1, a_2 \in \mathbf{A}$), and the relations are all those of the form $q_1q_2q_3 = 1$, where q_i is a quadratic map. The proof is quite long and uses many sophisticated tools of algebraic geometry, such as cell complexes associated to rational surfaces.

Another presentation was given in [Isk2] (and announced in [Isk1]). The surface taken here is $\mathbb{P}^1 \times \mathbb{P}^1$, and the generators used are the group $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and the de Jonquières group \mathbf{J} of birational maps of $\mathbb{P}^1 \times \mathbb{P}^1$ which preserve the first projection (see below). There is only one relation in the amalgamated product of these two groups, which is $(\rho\tau)^3 = \sigma$, where $\rho = (x, y) \mapsto (x, x/y)$ and $\tau = (x, y) \mapsto (y, x)$ in local coordinates. The proof is much shorter than the one of [Giz], and the number of relations is also much smaller, but everything is now on $\text{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$. There is also some gap in the proof (observed by S. Lamy): the author implicitly uses relations of the form $(\rho'\tau)^3 = \sigma'$ where ρ' has base-points infinitely near, without proving that they are generated by the first one (a fact not so hard to prove).

In this short paper, we give a new presentation of the Cremona group, which is as simple as the one of [Isk2], but stays on \mathbb{P}^2 . The proof is also very short and is in fact strongly inspired from the one of [Isk2]. We take care of infinitely near points and translate the idea of Iskovskikh from $\mathbb{P}^1 \times \mathbb{P}^1$ to \mathbb{P}^2 , where it becomes

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simpler. We use only classical tools of plane birational geometry (base-points and blow-ups), as mathematicians of the nineteenth century did and as in [Isk2].

The *de Jonquières group*, that we will denote by \mathbf{J} , is the subgroup of $\text{Bir}(\mathbb{P}^2)$ consisting of elements which preserve the pencil of lines passing through $p_1 = (1 : 0 : 0)$. This group can be viewed in local coordinates $x = X/Z$ and $y = Y/Z$ as

$$\mathbf{J} = \left\{ (x, y) \mapsto \left(\frac{ax + b}{cx + d}, \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)} \right) \mid \begin{array}{l} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}(2, k), \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{PGL}(2, k(x)) \end{array} \right\}.$$

It is thus naturally isomorphic to $\text{PGL}(2, k(x)) \rtimes \text{PGL}(2, k)$, where $\text{PGL}(2, k) = \text{Aut}(\mathbb{P}^1)$ acts on $\text{PGL}(2, k(x))$ via its action on $k(x) = k(\mathbb{P}^1)$.

Since $\sigma \in \mathbf{J}$, the group $\text{Bir}(\mathbb{P}^2)$ is generated by \mathbf{A} and \mathbf{J} . The aim of this paper is to prove the following result:

Theorem 1. *The Cremona group $\text{Bir}(\mathbb{P}^2)$ is the amalgamated product of $\mathbf{A} = \text{Aut}(\mathbb{P}^2)$ and \mathbf{J} along their intersection, divided by one relation, which is*

$$\sigma\tau = \tau\sigma,$$

where $\tau \in \mathbf{A}$ is given by $\tau = (X : Y : Z) \mapsto (Y : X : Z)$.

Since $\sigma\tau = \tau\sigma$ is easy to verify, it suffices to prove that no other relation holds. We prove this after proving the following simple lemma.

Lemma 1. *If $\theta \in \mathbf{J}$ is a quadratic map having $p_1 = (1 : 0 : 0)$ and q as base-points, where q is a proper point of $\mathbb{P}^2 \setminus \{p_1\}$ and $\nu \in \mathbf{A}$ exchanges p_1 and q , the map $\theta' = \nu\theta\nu^{-1}$ belongs to \mathbf{J} and the relation*

$$\nu\theta^{-1} = (\theta')^{-1}\nu$$

is generated by the relation $\sigma\tau = \tau\sigma$ in the amalgamated product of \mathbf{A} and \mathbf{J} .

Proof of Lemma 1. The relations $\theta' = \nu\theta\nu^{-1}$ and $\nu\theta^{-1} = (\theta')^{-1}\nu$ are clearly equivalent. In particular, the result is invariant under conjugation of both θ and ν by an element of $\mathbf{A} \cap \mathbf{J}$. Observe that $\mathbf{A} \cap \mathbf{J}$ is the group of automorphisms of \mathbb{P}^2 which fix the point p_1 . Choosing an element in $\mathbf{A} \cap \mathbf{J}$ which sends q onto $p_2 = (0 : 1 : 0)$, we can assume that $q = p_2$. Then ν is equal to $a\tau$, where $\tau = (X : Y : Z) \mapsto (Y : X : Z)$ and a is an element of $\mathbf{A} \cap \mathbf{J}$ which fixes p_2 . We can moreover assume that $\nu = \tau$, since this only replaces θ' with its conjugation by $a \in \mathbf{A} \cap \mathbf{J}$.

We study two cases separately, depending on the number of proper base-points of θ .

(a) Suppose that θ has exactly three proper base-points, p_1, p_2 and p_3 . Choose $a_1 \in \mathbf{A} \cap \mathbf{J}$ which fixes p_2 (and p_1) and sends p_3 onto $(0 : 0 : 1)$ (this is possible since p_3 is not collinear with p_1 and p_2). The two maps σa_1 and θ are quadratic maps with the same base-points: p_1, p_2, p_3 . This means that $\theta = a_2\sigma a_1$ for some $a_2 \in \mathbf{A}$. Since $\theta, \sigma, a_1 \in \mathbf{J}$, we see that $a_1, a_2 \in \mathbf{A} \cap \mathbf{J}$. This yields the following equality in the amalgamated product:

$$\tau\theta\tau^{-1} = \tau a_2\sigma a_1\tau^{-1} = (\tau a_2\tau^{-1})(\tau\sigma\tau^{-1})(\tau a_1\tau^{-1}).$$

Since $\tau, a_1, a_2 \in \mathbf{A}$ and $\tau a_i\tau^{-1} \in \mathbf{A} \cap \mathbf{J}$ for $i = 1, 2$, the above equality implies that $\tau\theta\tau^{-1}$ is equal to an element of \mathbf{J} modulo the relation $\sigma\tau = \tau\sigma$ and therefore yields the result.

(b) Suppose now that θ has only two proper base-points, p_1, p_2 , and that its third base-point is infinitely near to p_i for some $i \in \{1, 2\}$. Choose $a_1 \in \mathbf{A} \cap \mathbf{J}$ which fixes p_2 (and p_1) and sends this infinitely near point to the tangent direction $z = 0$ at the point p_i . Once again this is possible since the three base-points of θ are not “collinear”.

The maps θ and $\nu_i a_1$ are quadratic maps with the same base-points, where ν_1, ν_2 are the following quadratic involutions:

$$\begin{aligned} \nu_1: & (X : Y : Z) \dashrightarrow (XY : Z^2 : YZ), \\ \nu_2: & (X : Y : Z) \dashrightarrow (Z^2 : XY : XZ). \end{aligned}$$

This means that $\theta = a_1 \nu_i a_2$ for some $a_2 \in \mathbf{A}$. Once again, since $\theta, \sigma, a_1 \in \mathbf{J}$, we see that $a_1, a_2 \in \mathbf{A} \cap \mathbf{J}$.

Denoting by $\rho_1, \rho_2 \in \mathbf{A} \cap \mathbf{J}$ the maps

$$\begin{aligned} \rho_1: & (X : Y : Z) \dashrightarrow (X : Z - Y : Z), \\ \rho_2: & (X : Y : Z) \dashrightarrow (Z - X : Y : Z), \end{aligned}$$

a direct calculation shows that $\nu_i = \rho_i \sigma \rho_i \sigma \rho_i$ in \mathbf{J} for $i = 1, 2$. As above, this yields the following equality:

$$\tau \theta \tau^{-1} = (\tau a_1 \tau^{-1})(\tau \rho_i \tau^{-1})(\tau \sigma \tau^{-1})(\tau \rho_i \tau^{-1})(\tau \sigma \tau^{-1})(\tau \rho_i \tau^{-1})(\tau a_2 \tau^{-1}).$$

Using $\sigma \tau = \tau \sigma$ and the fact that $\tau \rho_i \tau^{-1} = \rho_j$ in \mathbf{A} , with $j = 3 - i$, we obtain

$$\tau \theta \tau^{-1} = (\tau a_1 \tau^{-1})(\rho_j \sigma \rho_j \sigma \rho_j)(\tau a_2 \tau^{-1}) = (\tau a_1 \tau^{-1}) \nu_j (\tau a_2 \tau^{-1}).$$

Therefore, $\tau \theta \tau^{-1}$ is again equal to an element of \mathbf{J} modulo the relation $\sigma \tau = \tau \sigma$. □

Proof of Theorem 1. Taking an element f in the amalgamated product $\mathbf{A} \star_{\mathbf{A} \cap \mathbf{J}} \mathbf{J}$ which corresponds to the identity map of $\text{Bir}(\mathbb{P}^2)$, we have to prove that f is the identity in the amalgamated product, modulo the relation $\sigma \tau = \tau \sigma$.

We write $f = j_r a_r \dots j_1 a_1$, where $a_i \in \mathbf{A}, j_i \in \mathbf{J}$ for $i = 1, \dots, r$. Note that many such writings exist, and we allow the a_i, j_i to be trivial (equal to the identity).

We denote by Λ_0 the linear system of lines of the plane, and for $i = 1, \dots, n$, we denote by Λ_i the linear system $j_i a_i \dots j_1 a_1(\Lambda_0)$ and by d_i its degree. We define

$$D = \max \{d_i \mid i = 1, \dots, r\}, n = \max \{i \mid d_i = D\} \text{ and } k = \sum_{i=1}^n (\deg(j_i) - 1).$$

Recall that each j_i is a birational transformation of the plane and that it has a degree $\deg(j_i)$, which is equal to the degree of the linear systems $j_i(\Lambda_0)$ and $(j_i)^{-1}(\Lambda_0)$. In particular, $\deg(j_i) = 1 \Leftrightarrow j_i \in \mathbf{A}$. The number k counts the complexity of the word $j_n a_n \dots j_1 a_1$, which corresponds to the birational map $j_i a_i \dots j_1 a_1$ of the highest degree.

When $D = 1$, each j_i belongs to \mathbf{A} , and the word f is equal to an element of \mathbf{A} in the amalgamated product. Since \mathbf{A} embeds into $\text{Bir}(\mathbb{P}^2)$, this case is clear. We can thus assume that $D > 1$ and prove the result by induction on the pairs (D, k) , ordered lexicographically.

If j_n belongs to \mathbf{A} , we replace $a_{n+1} j_n a_n$ by its product in \mathbf{A} . This does not change the pair (D, k) but decreases n by 1. If j_{n+1} belongs to \mathbf{A} , we replace $a_{n+2} j_{n+1} a_{n+1}$ by its product in \mathbf{A} and decrease r by 1 without changing the pair (D, k) . If $a_{n+1} \in \mathbf{J}$ we replace $j_{n+1} a_{n+1} j_n$ by its product in \mathbf{J} , decreasing the pair (D, k) .

We can thus assume that $j_{n+1}, j_n \in \mathbf{J} \setminus \mathbf{A}$ and that $a_{n+1} \in \mathbf{A} \setminus \mathbf{J}$. Remembering that $p_1 = (1 : 0 : 0)$ is the base-point of the pencil associated to \mathbf{J} , the fact that $a_{n+1} \notin \mathbf{J}$ implies that $a_{n+1}(p_1) \neq p_1$. Moreover, the fact that $j_n, j_{n+1} \notin \mathbf{A}$ implies that both have degree > 1 . We will denote these degrees by $D_L = \deg(j_{n+1})$, $D_R = \deg(j_n)$. Since j_{n+1} and j_n preserve the pencil of lines passing through p_1 , the point p_1 is a base-point of j_{n+1} and j_n of multiplicity $D_L - 1$ and $D_R - 1$ respectively. This means that the systems $(j_{n+1})^{\pm 1}(\Lambda_0)$ and $(j_n)^{\pm 1}(\Lambda_0)$ consist of curves of degree D_L and D_R having multiplicity $D_L - 1$ and $D_R - 1$ at p_1 . These systems, being the image of the system Λ_0 of lines of the plane, are homoloidal nets (systems of rational curves with exactly one free intersection point). They have thus respectively $2D_L - 2$ and $2D_R - 2$ base-points distinct from p_1 , which all have multiplicity one.

We denote by Ω_L and Ω_R the systems

$$\Omega_L = (j_{n+1}a_{n+1})^{-1}(\Lambda_0) \text{ and } \Omega_R = j_n a_n(\Lambda_0)$$

which have degree D_L and D_R respectively, since $a_n, a_{n+1} \in \mathbf{A} = \text{Aut}(\mathbb{P}^2)$ are only changes of coordinates. Moreover, the points $l_0 = (a_{n+1})^{-1}(p_1) \neq p_1$ and $r_0 = p_1$ are base-points of respectively Ω_L and Ω_R of multiplicity $D_L - 1$ and $D_R - 1$.

We use these systems to compute the degrees d_{n+1} and d_{n-1} of the systems $\Lambda_{n+1} = j_{n+1}a_{n+1}(\Lambda_n)$ and $\Lambda_{n-1} = (a_n)^{-1}(j_n)^{-1}(\Lambda_n)$. For any i , d_i is equal to the degree of Λ_i , which is the intersection of Λ_i with a general line, or the free intersection of Λ_i with Λ_0 . In particular, d_{n+1} is equal to the free intersection of $\Lambda_{n+1} = (j_{n+1}a_{n+1})(\Lambda_n)$ with Λ_0 or to the free intersection of Λ_n with Ω_L . Similarly, d_{n-1} is equal to the free intersection of $\Lambda_{n+1} = (j_{n+1}a_{n+1})(\Lambda_n)$ with Λ_0 or to the free intersection of Λ_n with Ω_L .

Writing l_1, \dots, l_{2D_L-2} and r_1, \dots, r_{2D_R-2} as the other base-points of Ω_L and Ω_R (which have all multiplicity one, as we said above), we obtain the following equalities:

$$\begin{aligned} d_{n+1} &= D_L \cdot d_n - (D_L - 1) \cdot m(l_0) - \sum_i^{2D_L-2} m(l_i) < d_n, \\ d_{n-1} &= D_R \cdot d_n - (D_R - 1) \cdot m(r_0) - \sum_i^{2D_R-2} m(r_i) \leq d_n, \end{aligned}$$

where $m(q) \geq 0$ is the multiplicity of a point q as a base-point of Λ_n . We order the points l_1, \dots, l_{2D_L-2} so that $m(l_i) \geq m(l_{i+1})$ for each $i \geq 1$ and that if l_i is infinitely near to l_j , then $i > j$. We do the same for r_1, \dots, r_{2D_R-2} . Rewriting the above inequalities as

$$\begin{aligned} d_{n+1} &= d_n + \sum_{i=1}^{D_L-1} (d_n - m(l_0) - m(l_{2i-1}) - m(l_{2i})) < d_n, \\ d_{n-1} &= d_n + \sum_{i=1}^{D_R-1} (d_n - m(r_0) - m(r_{2i-1}) - m(r_{2i})) \leq d_n, \end{aligned}$$

the order chosen implies that

$$(1) \quad \begin{aligned} m(l_0) + m(l_1) + m(l_2) &> d_n, \\ m(r_0) + m(r_1) + m(r_2) &\geq d_n. \end{aligned}$$

(a) Suppose that $m(l_0) \geq m(l_1)$ and $m(r_0) \geq m(r_1)$. We choose a point q in the set $\{l_1, l_2, r_1, r_2\} \setminus \{l_0, r_0\}$ with the maximal multiplicity $m(q)$, and so that q is a proper point of the plane or infinitely near to l_0 or r_0 (which are distinct proper points of the plane). We now prove that

$$(2) \quad m(l_0) + m(r_0) + m(q) > d_n.$$

If $l_1 = r_0$, $m(q) \geq m(l_2)$ and $m(l_0) + m(r_0) + m(q) \geq m(l_0) + m(l_1) + m(l_2) > d_n$ by (1). If $l_1 \neq r_0$, $m(q) \geq m(l_1) \geq m(l_2)$ so $m(l_0) + m(q) > 2d_n/3$. Since $m(r_0) \geq m(r_1) \geq m(r_2)$, we have $m(r_0) \geq d_n/3$, and inequality (2) is clear.

Because of inequality (2), the points l_0, r_0 and q are not aligned, and there exists a quadratic map $\theta \in \mathbf{J}$ with base-points l_0, r_0, q (recall that $r_0 = p_1$ is the point associated to the pencil of \mathbf{J}). Moreover, the degree of $\theta(\Lambda_n)$ is $2d_n - m(l_0) - m(r_0) - m(q) < d_n$. Recall that $a_{n+1} \in \mathbf{A}$ sends l_0 onto $r_0 = p_1$. Choosing $\nu \in \mathbf{A} \cap \mathbf{J}$ which (fixes $r_0 = p_1$ and) sends $a_{n+1}(r_0)$ onto l_0 and replacing respectively a_{n+1} and j_{n+1} by νa_{n+1} and $j_{n+1}\nu^{-1}$, we can assume that a_{n+1} exchanges l_0 and r_0 . Using Lemma 1, we write $\theta' = a_{n+1}\theta(a_{n+1})^{-1} \in \mathbf{J}$ and obtain the following equality modulo the relation $\sigma\tau = \tau\sigma$:

$$j_{n+1}a_{n+1}j_n = j_{n+1}a_{n+1}\theta^{-1}(\theta j_n) = (j_{n+1}(\theta')^{-1})a_{n+1}(\theta j_n),$$

and both $(j_{n+1}(\theta')^{-1})$ and (θj_n) belong to \mathbf{J} , but $a_{n+1} \in \mathbf{A}$. Since $\theta(\Lambda_n) = (\theta j_n)(\Lambda_{n-1})$ has degree $< d_n$, this rewriting decreases the pair (D, k) .

(b) Suppose now that we are in a “bad case” where $m(l_0) < m(l_1)$ or $m(r_0) < m(r_1)$. We now prove that it is possible to change the writing of f in the amalgamated product (modulo the relation) without changing (D, k) but by reversing the inequalities. We will thus be able to go back to the “good case” already studied in (a) to conclude.

Assume first that $m(r_1) > m(r_0)$. This implies that r_1 is a proper point of the plane and that there exists a quadratic map $\theta \in \mathbf{J}$ with base-points $p_1 = r_0, r_1, r_2$. Since these three points are base-points of $(j_n)^{-1}$, the degree of $\theta j_n \in \mathbf{J}$ is equal to the degree of $j_n \in \mathbf{J}$ minus 1.

Taking $\nu \in \mathbf{A}$ which exchanges r_0 and r_1 and applying Lemma 1, we write $\theta' = \nu\theta\nu^{-1} \in \mathbf{J}$ and obtain the following equality modulo the relation $\sigma\tau = \tau\sigma$:

$$a_{n+1}j_n = (a_{n+1}\nu^{-1})\nu\theta^{-1}(\theta j_n) = (a_{n+1}\nu^{-1})(\theta')^{-1}\nu(\theta j_n),$$

and both θ' and (θj_n) belong to \mathbf{J} , but $(a_{n+1}\nu^{-1})$ and ν belong to \mathbf{A} . Since $(\theta')^{-1}\nu\theta(\Lambda_n) = \nu(\Lambda_n)$, this rewriting replaces

$$\begin{aligned} (j_1, \dots, j_n, \dots, j_r) & \text{ with } (j_1, \dots, j_{n-1}, \theta j_n, (\theta')^{-1}, j_{n+1}, \dots, j_r), \\ (\Lambda_0, \dots, \Lambda_n, \dots, \Lambda_r) & \text{ with } (\Lambda_0, \dots, \Lambda_{n-1}, \theta(\Lambda_n), \nu(\Lambda_n), \Lambda_{n+1}, \dots, \Lambda_r). \end{aligned}$$

The degree of $\theta(\Lambda_n)$ is equal to $2d_n - m(r_0) - m(r_1) - m(r_2) \leq d_n$, and the degree of $\nu(\Lambda_n)$ is d_n . The new sequence thus has the same D , n is replaced with $n + 1$, and k stays the same since $\deg((\theta')^{-1}) - 1 + \deg(\theta j_n) - 1 = 2 - 1 + \deg(\theta j_n) - 1 = \deg(j_n) - 1$. The system Λ_n being replaced with $\nu(\Lambda_n)$, where $\nu \in \mathbf{A}$ exchanges r_0 and r_1 , the multiplicity of r_0 as a base-point of $\nu(\Lambda_n)$ is now the biggest among the base-points of θ' . In the new sequence, we have $m(r_0) > m(r_1)$ instead of $m(r_1) > m(r_0)$.

If $m(l_1) > m(l_0)$, the same kind of replacement exchanges the points l_0 and l_1 .

We can thus go back to case (a) after having made one or two replacements. This achieves the proof. □

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