MINIMAL VOLUME OF SLABS IN THE COMPLEX CUBE

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Abstract. We study the volume of sections and slabs in the n-dimensional cube for complex scalars. In particular, we investigate the directions of minimal volume for a small width of the slab.

1. Introduction and results

Let $\| \cdot \|_\infty$ and $| \cdot |$ denote the supremum and the euclidean norm on $\mathbb{K}^n$ where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For volume considerations, $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ and the $(2n)$-dimensional volume is used. Let

$$B_\infty := \{ x \in \mathbb{K}^n | \|x\|_\infty \leq \alpha \}$$

be the n-dimensional cube of volume 1, i.e. $\alpha = 1/2$ if $\mathbb{K} = \mathbb{R}$ and $\alpha = 1/\sqrt{\pi}$ if $\mathbb{K} = \mathbb{C}$. Given some direction $a \in \mathbb{K}^n$ of length $|a| = 1$ and $t \in \mathbb{K}$, we introduce the section of the cube

$$S(a, t) := \{ x \in \mathbb{K}^n | \|x\|_\infty \leq \alpha, \langle x, a \rangle = \alpha t \}$$

and for $t > 0$ the slab in the cube

$$S\ell(a, t) := \{ x \in \mathbb{K}^n | \|x\|_\infty \leq \alpha, |\langle x, a \rangle| \leq \alpha t \}.$$

Let

$$A(a, t) := \text{vol}_{n-1}(S(a, t)), \quad t \in \mathbb{K},$$

$$V(a, t) := \text{vol}_{n}(S\ell(a, t)), \quad t > 0,$$

denote the volumes of the section and the slab of $B_\infty$. In the complex case, one has to use $\text{vol}_{2n-2}$ and $\text{vol}_{2n}$ for $A$ and $V$. For $a = (a_k)_{k=1}^n$, let $a^*$ denote the decreasing rearrangement of the sequence $(|a_k|)_{k=1}^n$, i.e. $|a_1| \geq \cdots \geq |a_n| \geq 0$. Since the volume is invariant under coordinate permutations and sign changes, which in the complex case means invariant under permutations and rotations of coordinate disks, we have

$$A(a, t) = A(a^*, |t|), \quad V(a, t) = V(a^*, t).$$

We will therefore assume in the following that $a = (a_k)_{k=1}^n$, $a_1 \geq \cdots \geq a_n \geq 0$ and $t \geq 0$. 

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Let $f_1 = (1, 0, \ldots, 0)$, $f_2 := (1, 1, 0, \ldots, 0)/\sqrt{2}$. For central sections $1 = A(f_1, 0) \leq A(a, 0) \leq A(f_2, 0) = 2m/2$, and where $m = 1$ for $K = \mathbb{R}$, $m = 2$ for $K = \mathbb{C}$. The lower bound is due to Hadwiger [Ha], Vaaler [V] and Hensley [He]; the upper bound for $K = \mathbb{R}$ is a well-known result of K. Ball [B]. The complex case is due to Oleskiewicz-Pelczyński [OP].

For small $t > 0$ it would seem natural that the volume of slabs mimics this behaviour in the sense that for $K = \mathbb{R}$

$$t = V(f_1, t) \leq V(a, t) \leq V(f_2, t) = \sqrt{2}t - t^2/2.$$ 

In fact, it was proved by Barthe-Koldobsky [BK] that the lower bound is true provided that $t \leq 3/4$ holds. For $t > 2(\sqrt{2} - 1)$, however, $V(f_1, t) \leq V(a, t)$ is false for $a = f_2$, and for $t$ larger but close to this number $2(\sqrt{2} - 1)$, $f_2$ probably is the direction of minimal slab volume (this is true at least for $n = 2$ and 3). The lower bound

$$t^2 = V(f_1, t) \leq V(a, t)$$

remains true in the complex case for small $t$. This generalization of the result of [BK] to the complex case is the main result of this paper. Formula (2.3) states that

$$V(f_2, t) = 2t^2 - ((1 + t^2)t\sqrt{2} - t^2 + (2t^2 - 1)\arcsin(t\sqrt{2} - t^2))/\pi,$$

and for $t \geq t_1 \approx 0.867115$, $V(f_2, t) < V(f_1, t)$, and for $t \geq t_1$ but near $t_1$, $t_2$ is probably the direction of minimal volume (true for $n = 2$). Our main result states that $f_1$ is the direction of minimal volume for slabs of width $t \leq 4/5$, leaving a gap for width $4/5 < t < t_1$ similar to the real case where the gap is $3/4 < t < 2(\sqrt{2} - 1)$.

**Theorem 1.** Let $K = \mathbb{C}$, $a \in \mathbb{C}^n$ with $|a| = 1$ and $0 \leq t \leq 4/5$. Then

$$t^2 = V(f_1, t) \leq V(a, t).$$

Note that

$$V(a, t) = 2 \int_0^t A(a, s) s\, ds,$$

$$1 = \text{vol}_{2n}(B_\infty) = 2 \int_0^\infty A(a, s) s\, ds.$$

As in [BK] the main technical tool is an integral inequality of independent interest, to be applied to $f = 2A(a, \cdot)$:

**Proposition 2.** Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be in $L_1(\mathbb{R}_+, s\, ds)$, non-increasing and log-concave. Let $t > 0$. Assume that

$$\int_0^t f(s) s\, ds \leq \left(\frac{4}{\pi}\right)^2 \int_0^\infty f(s) s\, ds.\label{eq:prop2_1}$$

Then

$$2 \left(\int_0^\infty f(s) s^3\, ds\right) \left(\int_0^t f(s) s\, ds\right) \geq t^2 \left(\int_0^\infty f(s) s\, ds\right)^2.\label{eq:prop2_2}$$

Concerning the direction of the maximal volume of slabs, in the case of complex scalars, different from the case of the maximal volume of sections [OP], $V(a, t) \leq V(f_2, t)$ is false in general for any width $t > 0$: for large $n \in \mathbb{N}$, we show $V(f_n, t) > V(f_2, t)$, where $f_n := (1, \ldots, 1)/\sqrt{n} \in S^{n-1}$.
Proposition 3. Let $0 < t < 4/9$ and $n \geq 20$. Then for $K = \mathbb{C}$ and any $n \geq 1/t$, 
\[ V(f_n, t) > V(f_2, t). \]

Contrary to this, in the case of real scalars $K = \mathbb{R}$, $V(f_n, t) < V(f_2, t)$ holds provided that $t < t_0$, where $t_0$ is approximately $1/14$.

2. Volume Formulas

We mention and recall some volume formulas for sections and slabs of the cube which yield some statements and formulas before the formulation of Theorem 1.

Proposition 4. Let $a \in \mathbb{R}^n_+$, $|a| = 1$, $t \geq 0$. Then 
\begin{align*}
A(a, t) &= \begin{cases}
\frac{2}{\pi} \int_0^\infty \prod_{k=1}^n \frac{\sin(a_k s)}{a_k s} \cos(ts) \, ds & K = \mathbb{R} \\
\frac{1}{2} t \int_0^\infty \prod_{k=1}^n j_1(a_k s) J_0(ts) \, s \, ds & K = \mathbb{C}
\end{cases} \\
V(a, t) &= \begin{cases}
t \frac{2}{\pi} \int_0^\infty \prod_{k=1}^n \frac{\sin(a_k s)}{a_k s} \frac{\sin(ts)}{ts} \, ds & K = \mathbb{R} \\
t^2 \frac{1}{2} \int_0^\infty \prod_{k=1}^n j_1(a_k s) j_1(ts) \, s \, ds & K = \mathbb{C}
\end{cases},
\end{align*}

where $J_0, J_1$ are the first Bessel functions and $j_1(x) = 2 J_1(x)/x$.

Let us note that (2.1), $K = \mathbb{R}$ is a well-known result of Polya [19]. For $K = \mathbb{C}$ and $t = 0$, it is found in [OP]. Formulas (2.2) follow from (2.1) by integrating over $t$, using $J_0(s)/s = (J_1(s)/s)'$ in the complex case. The standard way to prove (2.1) is to write (with $\alpha$ as in the introduction) 
\[ A(a, t) = \int_{\langle z, a \rangle = \alpha t} \chi(||z\|_\infty) \, dz, \]

where $\chi$ is the indicator function of $[0, \alpha]$, and to take the Fourier transform in the $t$-variable, $\hat{A}(a, \cdot)$, which by coordinate independence is a constant times a product of scaled (by $a_k$) Fourier transforms of the ball $B = \{ z \in K \mid |z| \leq \alpha \}$. For $K = \mathbb{C}$ this is where $J_1$ appears. Taking the inverse transform then gives (2.1).

In particular, for $a = f_1$ and $a = f_2$, (2.2) yields 
\begin{align*}
V(f_1, t) &= t \quad (K = \mathbb{R}) , \quad V(f_1, t) = t^2 \quad (K = \mathbb{C}) , \\
V(f_2, t) &= \sqrt{2} t - t^2/2 \quad (K = \mathbb{R}) , \\
V(f_2, t) &= 2t^2 - ((1 + t^2)t \sqrt{2} - t^2) + (2t^2 - 1) \arcsin(t \sqrt{2} - t^2) / \pi \quad (K = \mathbb{C})
\end{align*}

so that $V(f_1, t_1) = V(f_2, t_1)$ for $t_1 = 2(\sqrt{2} - 1)$ in the real case and $t_1 \simeq 0.867115$ in the complex case, as mentioned in the introduction. For $0 < t < t_1$, $V(f_1, t) < V(f_2, t)$.

Using these formulas, we now give the
Proof of Proposition 3. (a) For \( f_n = (1, \ldots, 1)/\sqrt{n} \), (2.2) implies for complex scalars and \( t > 0 \):

\[
V(f_n, t) = \frac{t^2}{2} \int_0^\infty \left( j_1(s/\sqrt{n}) \right)^n j_1(ts) \, ds.
\]

Since \( \lim_{u \to 0} j_1(u) e^{u^2/8} = 1 \), \( \lim_{n \to \infty} j_1(s/\sqrt{n})^n = e^{-s^2/8} \) for any \( s \in \mathbb{R} \), and we find using the dominated convergence theorem that

\[
\lim_{n \to \infty} V(f_n, t) = \frac{t^2}{2} \int_0^\infty e^{-s^2/8} j_1(ts) \, ds
= t \int_0^\infty e^{-s^2/8} J_1(ts) \, ds = 1 - e^{-2t^2}.
\]

(b) For the properties of Bessel functions, cf. Watson [W]. We will use the following estimates for the modified Bessel functions \( j_1(t) = 2J_1(t)/t \):

\[
|j_1(t)| \leq \min(1, 1.66/t^{3/2}), \quad 0 < t,
\]

\[
|j_1(t)| \geq \exp(-t^2/8 - t^4/316), \quad 0 \leq t \leq 2.
\]

As for (2.5), \( \sqrt{t} |J_1(t)| \) has a decreasing sequence of maxima, tending to \( \sqrt{2\pi} \) by the asymptotic formula for \( J_1(t) \), with the first maximum near \( t_0 \approx 2.17 \) bounded by 0.83. To prove (2.6), consider \( f(t) := j_1(t) \exp(t^2/8 + t^4/316) \). Then

\[
g(t) := f'(t) \exp(-t^2/8 - t^4/316) = J_1(t)(1/2 + 2/79 t^2) - 2J_2(t)/t.
\]

In \([0, 2] \), \( g \) is positive for \( t < t_1 \approx 1.67 \) and negative for \( t > t_1 \), as can be checked by using the alternating power series expansion of \( J_1 \) and \( J_2 \) to approximate \( g \) by polynomials. Thus \( f \) is first increasing and then decreasing. Since \( f(0) = 1 \) and \( f(2) > 0 \), (2.6) holds for \( t \in [0, 2] \).

(c) By (2.2) and (2.4), we have for any \( t > 0 \)

\[
V(f_n, t) = (1 - e^{-2t^2}) - t^2/2 \int_0^\infty \left( e^{-s^2/8} - j_1(s/\sqrt{n})^n \right) j_1(ts) \, ds.
\]

Using (2.6) and \( |j_1(u)| \leq 1 \), we find

\[
I_1 := t^2/2 \int_0^{2\sqrt{n}} \left| e^{-s^2/8} - j_1(s/\sqrt{n})^n \right| j_1(ts) \, ds
\]

\[
\leq t^2/2 \int_0^{2\sqrt{n}} e^{-s^2/8}(1 - e^{-s^2/(316 n)}) \, ds
\]

\[
\leq t^2/2 \int_0^\infty e^{-s^2/8} s^5/(316 n) \, ds = 64/79 \, t^2 \, n.
\]

For \( s > 2\sqrt{n} \), we estimate both terms in the integrand in (2.7) separately:

\[
I_2 := t^2/2 \int_{2\sqrt{n}}^\infty e^{-s^2/8} |j_1(ts)| \, ds \leq t^2/2 \int_{2\sqrt{n}}^\infty s e^{-s^2/8} \, ds = 2/e^{n/2} \, t^2
\]
and, using (2.5),
\[ I_3 := \frac{t^2}{2} \int_0^{2\pi} |j_1(s/\sqrt{n})|^n|j_1(ts)|s\,ds = \frac{t^2}{2} n/2 \int_2^\infty |j_1(u)|^n|j_1(\sqrt{n}t\,u)|u\,du \]
\[ \leq \frac{t^2}{2} n/2 \int_2^\infty (1.66/u^{3/2})^n u\,du = 4n/(3n-4) (0.83/\sqrt{n})^n t^2 \leq 2/e^{n/2} t^2, \quad n \geq 4. \]
Therefore \( I := I_1 + I_2 + I_3 \leq (64/79 + 4n/e^n t^2) / n \leq 0.815 t^2 / n \), provided that \( n \geq 20 \).

\section{3. Slabs of minimal volume}

We now prove Theorem 1 by first using Proposition 2 and then giving the (independent) proof of Proposition 2.

\textbf{Proof of Theorem 1 from Proposition 2.} Let \( a \in S^{n-1}(C) \) and \( 0 \leq t \leq 4/5 \). For the volume of the slab and the unit cube we have in the complex case by polar integration
\[ V(a, t) = 2 \int_0^t A(a, s) s\,ds, \quad 1 = 2 \int_0^\infty A(a, s) s\,ds. \]
Assume that the direction \( a \) satisfies \( V(a, t) \leq V(f_1, t) = t^2 \). We claim that in this case \( V(a, t) = V(f_1, t) \); i.e. \( f_1 \) is a minimal volume direction for the complex slab. The assumption in Proposition 2 is satisfied for \( f = 2A(a, \cdot) \) since
\[ V(a, t) = \int_0^t f(s) s\,ds \leq t^2 \leq \left( \frac{4}{5} \right)^2 = \left( \frac{4}{5} \right)^2 \int_0^\infty f(s) s\,ds. \]
Further, for the circle \( B = \{ z \in C \mid |z| \leq \alpha = 1/\sqrt{\pi} \} \) of area 1,
\[ \int_0^\infty f(s) s^3\,ds = 2 \int_0^\infty A(a, s) s^3\,ds \]
\[ = 2\pi^2 \int_0^\infty A(a, u/\alpha) u^3\,du \]
\[ = \pi \int_B A(a, |v|/\alpha) |v|^2\,dv \]
\[ = \pi \int_B \left( \int_{\{z,a\}=|v|} dz \right) |v|^2\,dv \]
\[ = \pi \int_{B_{\infty}} |\langle w, a \rangle|^2 dw = \frac{1}{2}, \]
where the last equality follows from
\[ \int_{B} w_i w_j \, dw_i \, dw_j = 0, \quad \int_{B} |w_i|^2 \, dw_i = \frac{1}{2\pi}. \]

We thus conclude by Proposition 2
\[ 2 \left( \int_{0}^{\infty} f(s) \, s^3 \, ds \right) \left( \int_{0}^{t} f(s) \, s \, ds \right) \geq t^2 \left( \int_{0}^{\infty} f(s) \, s \, ds \right)^2; \]
i.e. \( 2 \cdot \frac{1}{2} \cdot V(a, t) \geq t^2 \cdot 1^{2}, \ V(a, t) \geq t^2 = V(f_1, t). \)

**Proof of Proposition 2.** Let \( t > 0 \) and \( 0 < v < V. \) Consider
\[ F_{t, v, V} := \{ f : \mathbb{R}_+ \to \mathbb{R}_+ \mid f \text{ non-increasing, log-concave, } \int_{0}^{\infty} f(s) \, s \, ds = V, \int_{0}^{t} f(s) \, s \, ds = v \}. \]

We want to prove that under the assumption (1.2) on \( f, \)
\[ \inf \left\{ \int_{0}^{\infty} f(s) \, s^3 \, ds \mid f \in F_{t, v, V} \right\} \geq \frac{t^2 V^2}{2v}. \]

Using the monotonicity and log-concavity of \( f, \) one proves similarly as in [BK] that (3.2)
\[ \inf \left\{ \int_{0}^{\infty} f(s) \, s^3 \, ds \mid f \in F_{t, v, V} \right\} = \inf \left\{ \int_{0}^{d} e^{-ps} s^3 \, ds \mid p \geq 0, \ d \geq t \geq 0, \ \int_{0}^{d} e^{-ps} s \, ds = V, \int_{0}^{t} e^{-ps} s \, ds = v \right\}. \]

Thus we only have to consider functions \( f(s) = \chi_{[0,d]}(s)e^{-ps}. \) The constraints in (3.2) mean that
\[ V = \int_{0}^{d} e^{-ps} s \, ds = \frac{1}{p^2} \int_{0}^{x} e^{-u} u \, du, \ v = \int_{0}^{t} e^{-ps} s \, ds = \frac{1}{p^2} \int_{0}^{y} e^{-u} u \, du, \]
with \( x := pd \geq y := pt. \) Moreover,
\[ \int_{0}^{d} e^{-ps} s^3 \, ds = \frac{1}{p^4} \int_{0}^{x} e^{-u} u^3 \, du. \]

Assumption (1.2) for these functions is thus equivalent to
\[ \int_{0}^{y} e^{-u} u \, du \leq \left( \frac{4}{5} \right)^2 \int_{0}^{x} e^{-u} u \, du, \ 0 \leq y \leq x. \]
Therefore, using (3.2), inequality (1.3) of Proposition 2 will follow provided we can show that for any $0 \leq y \leq x$ satisfying (3.3) we have

\[
(3.4) \quad \frac{2 \int_0^x e^{-u^3}du}{\left( \int_0^x e^{-u}du \right)^2} \geq \frac{y^2}{\left( \int_0^y e^{-u}du \right)^2}.
\]

The function $g, g(y) := \frac{y^2}{\left( \int_0^y e^{-u}du \right)^2}$ is increasing since

\[
2 \int_0^y e^{-u}du \geq 2 \int_0^y du = y^2,
\]

i.e.

\[
2y \int_0^y e^{-u}du \geq e^{-y^3}.
\]

Define $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by the requirement

\[
(3.5) \quad \int_0^x e^{-u}du = \left( \frac{4}{5} \right)^2 \int_0^x e^{-u}du.
\]

Hence by assumption $y \leq \varphi(x) \leq x$, and (3.4) will follow from the stronger claim

\[
(3.6) \quad \varphi(x)^2 \leq \frac{32}{25} \left( \int_0^x e^{-u}du \right)^2 \left( \int_0^x e^{-u}du \right)^2 = \psi(x).
\]

Let $x \in [0, 4]$. Since $\varphi(0) = 0$, $\lim_{x \to 0} \psi(x) = 0$, to prove (3.6) in this range of $x$ it suffices to show that $(\varphi(x)^2)' \leq \psi'(x)$. We have by (3.5)

\[
(\varphi(x)^2)' = 2\varphi(x)\varphi'(x) = \frac{32}{25} e^{\varphi(x)} x e^{-x}
\]

and

\[
\psi'(x) = \frac{32/25}{\left( \int_0^x e^{-u}du \right)^2} \left[ \left( \int_0^x e^{-u}du \right)^2 e^{-x} - x e^{-x} \left( \int_0^x u e^{-u^3}du \right) \right].
\]

Hence $(\varphi(x)^2)' \leq \psi'(x)$ holds provided that

\[
e^{\varphi(x)} \left( \int_0^x e^{-u}du \right)^2 \leq \int_0^x (x^2 u - u^3) e^{-u}du
\]
or equivalently
\[ e^{\varphi(x)} \leq f(x)/g(x), \quad f(x) := x^2(e^x + 2) - 6(e^x - (1 + x)), \]
\[ g(x) = (e^x - (1 + x))(1 - (1 + x)e^{-x}). \]

We claim that \( \varphi(x) \leq \frac{4}{5}x - \frac{4}{75}x^2 =: \gamma(x) \) holds. This will follow from
\[ \int_0^x e^{-u}udu \geq \left(\frac{4}{5}\right)^2 \int_0^x e^{-u}udu \]
by definition of \( \varphi \). (3.8) will follow from the stronger inequality for the derivatives
\[ \gamma'(x)e^{-\gamma(x)}\gamma(x) \geq \left(\frac{4}{5}\right)^2 e^{-x}, \]
i.e.
\[ (1 - \frac{2}{15}x)(1 - \frac{1}{15}x)e^{x/5+4x^2/75} \geq 1 \]
or
\[ \frac{2}{15}x + \frac{4}{75}x^2 + \ln(1 - \frac{2}{15}x) + \ln(1 - \frac{1}{15}x) \geq 0. \]

For \( x \leq 4, \frac{2}{15}x \leq \frac{3}{5} \) and for \( y \leq \frac{3}{5}, \ln(1 - y) \geq -y - y^2 \), which implies
\[ \ln(1 - \frac{2}{15}x) + \ln(1 - \frac{1}{15}x) \geq -x - x^2 \]
Hence (3.9) and (3.8) hold for \( x \leq 4 \). We conclude that
\[ e^{\varphi(x)} \leq e^{1/5x-4x^2/75} \leq 1 + \frac{4}{5}x + \frac{4}{15}x^2 + \frac{16}{375}x^3 + \frac{8}{5625}x^4, \]
the last inequality coming from the series development of the middle term. As for \( f \) and \( g \) in (3.7), one calculates the Taylor expansions
\[ f(x) = \frac{x^4}{4} \sum_{n=0}^{\infty} \frac{4(n+1)(n+6)}{(n+4)!} x^n, \]
\[ g(x) = \frac{x^4}{4} \sum_{n=0}^{\infty} (-1)^n \cdot 4 \cdot \frac{(n+4)^2 - 3(n+4) + 1 + (-1)^n}{(n+4)!} x^n. \]
The Taylor polynomial for \( f(x)/g(x) \) of order 3 yields a lower bound
\[ 1 + \frac{4}{5}x + \frac{13}{45}x^2 + \frac{11}{189}x^3 \leq f(x)/g(x). \]
Inequality (3.7) now follows from (3.10), (3.11) and
\[ \frac{4}{15}x^2 + \frac{16}{375}x^3 + \frac{8}{5625}x^4 \leq \frac{13}{45}x^2 + \frac{11}{189}x^3, \]
which is true since \( 1 + \frac{36}{225}x - \frac{8}{725}x^2 \geq 0 \) for \( 0 \leq x \leq 4 \) holds.

For \( x \geq 4 \), inequality (3.9) is verified more easily: \( \varphi \) and \( \psi \) are increasing functions with
\[ \lim_{x \to \infty} \psi(x) = \frac{192}{25} = 7.68, \quad \lim_{x \to \infty} \varphi(x)^2 \simeq 4.74. \]
For \( x \geq 4 \)
\[ \psi(x) \geq \psi(4) \simeq 4.79 \geq \varphi(\infty)^2 \geq \varphi(x)^2. \]
\[ \square \]
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