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HEREDITARY ORDERS IN THE QUOTIENT RING OF A SKEW POLYNOMIAL RING

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Abstract. Let K be a field, and let σ be an automorphism of K of finite order. Let $K(X;\sigma)$ be the quotient ring of the skew polynomial ring $K[X;\sigma]$. Then any order in $K(X;\sigma)$ which contains K and its center is a valuation ring of the center of $K(X;\sigma)$ is a crossed-product algebra A_f, where f is some normalized 2-cocycle. Associated to f is a subgroup H of $\langle\sigma\rangle$ and a graph. In this paper, we determine the connections between hereditary-ness and maximal order properties of A_f and the properties of H, f and the graph of f.

1. Introduction

If R is a ring, then $J(R)$ will denote its Jacobson radical, $Z(R)$ its center, $U(R)$ its group of multiplicative units, and $R^{\#}$ the subset of all the non-zero elements. The ring R is called *hereditary* if all left ideals are projective as left R-modules and all right ideals are projective as right R-modules. It is called *Bézout* if finitely generated one-sided ideals are principal.

Let V be a valuation ring of a field F. If Q is a finite-dimensional central simple F-algebra, then a subring R of Q is called an order in Q if $RF = Q$. If in addition $V \subseteq R$ and R is integral over V, then R is called a V-order. If a V-order R is maximal among the V-orders of Q with respect to inclusion, then R is called a *maximal V-order* (or just a *maximal order* if the context is clear). Examples of maximal orders are Azumaya algebras over a valuation ring. An order R in Q is called a *valuation ring of* Q if it is Bézout and $R/J(R)$ is a simple Artinian ring. If V is a DVR, then a subring of Q with center V is a valuation ring if and only if it is a maximal order (see [6, Example 1.15]). Now assume Q is a division ring. Then a subring R of Q is called a *total valuation ring of* Q if, given any $0 \neq \alpha \in Q$, either $\alpha \in R$ or $\alpha^{-1} \in R$. A total valuation ring R of Q is called an *invariant valuation ring of* Q if it is invariant under inner automorphisms of Q. A valuation ring R of a finite dimensional division ring is a total valuation ring if and only if $R/J(R)$ is a division algebra by [6, Theorem G]. When V is a DVR, the class of total valuation rings of Q with center V coincides with the class of invariant valuation rings of Q with center V. For further information about properties of such rings, the reader is referred to [6].

For the moment, consider the following setup: V a DVR, with quotient field F; K/F a finite Galois extension, with group G; S the integral closure of V in K;

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$f : G \times G \mapsto S^{\#}$ a normalized cocycle, i.e., a function satisfying $f^{\sigma}(\tau,\gamma)f(\sigma,\tau\gamma) = f(\sigma,\tau)f(\sigma\tau,\gamma)$ for all $\sigma,\tau,\gamma \in G$ and $f(1,\sigma) = f(\sigma,1) = 1$ for all $\sigma \in G$. One can construct a "crossed-product" V-algebra

$$A_f = \sum_{\sigma \in G} S x_\sigma,$$

with the usual rules of multiplication ($x_\sigma s = \sigma(s)x_\sigma$ for all $s \in S, \sigma \in G$ and $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$).

Then A_f is associative, with identity $1 = x_1$, and center $V = Vx_1$. Further, A_f is a V-order in the crossed product F-algebra $Q_f = \sum_{\sigma\in G} K x_\sigma$. Two such cocycles f and g are said to be cohomologous over S (respectively cohomologous over K), and we write $f \sim_S g$ (respectively $f \sim_K g$) if there are elements $\{c_\sigma \mid \sigma \in G\} \subseteq U(S)$ (respectively $\{c_\sigma \mid \sigma \in G\} \subseteq U(K)$) such that $g(\sigma,\tau) = c_\sigma \sigma(c_\tau) c_{\sigma\tau}^{-1} f(\sigma,\tau)$ for all $\sigma,\tau \in G$. Let $H_f = \{\sigma \in G \mid f(\sigma,\sigma^{-1}) \in U(S)\}$. Then H_f is a subgroup of G which we will sometimes denote simply by H if the context is clear. On G/H, the left coset space of G by H, one can define a partial ordering by the rule

$$\sigma H \leq \tau H \ \text{ if } \ f(\sigma,\sigma^{-1}\tau) \in U(S).$$

Then "$\leq$" is well-defined and depends only on the cohomology class of f over S. Further, H is the unique least element. We call this partial ordering on G/H *the graph of f.*

Such a setup was first formulated by Haile in [3], with the assumption that S is unramified over V. Haile [3] did not deal with the case when $H = G$ in his paper, but that is hardly an impediment to the beautiful theory he developed: When $H = G$, then A_f is a classical crossed product algebra. It is Azumaya over V since S/V is a Galois extension of rings.

Now suppose $H \neq G$. If A_f is Azumaya over V, then $J(A_f) = J(V)A_f$; hence, by [3, Proposition 3.1(b)], we infer that for each $\sigma \in G$, $f(\sigma,\sigma^{-1}) \notin M$ for any maximal ideal M of S. This implies that $H = G$, which is a contradiction.

Thus we have:

Sublemma. *With the notation as above and assuming S is unramified over V, $H = G$ if and only if A_f is an Azumaya algebra.*

Here is another minor fact: if $H = G$, then A_f is obviously primary. Therefore by [3, Proposition 2.1], if S is a valuation ring, then A_f is always primary whether or not $H = G$.

Let us now fix some notation to be used in the rest of the paper. Let K denote a field, and let σ be an automorphism of K of finite order n. To avoid trivialities, we will assume $n > 1$. Consider the skew polynomial ring $K[X;\sigma]$ over K with multiplication defined by the commutation rule: $Xa = \sigma(a)X$ for all $a \in K$. Let $K(X;\sigma)$ denote the quotient ring of $K[X;\sigma]$. Then $K(X;\sigma)$ is a finite dimensional division algebra.

It is known that $K(X;\sigma)$ is a cyclic algebra; e.g., see [1, §12.2, Example 5]. For the convenience of the reader, we shall outline a proof of this fact here: Let F be the fixed field of σ. Then $X^n \in Z(K(X;\sigma))$, and $K(X^n)$ is a Galois extension of $F(X^n)$ with group $G := \langle\sigma\rangle$. If $\langle K(X^n), X\rangle$ is the subalgebra of $K(X;\sigma)$ generated by $K(X^n)$ and X, then $\langle K(X^n), X\rangle \cong (K(X^n)/F(X^n), \sigma, X^n)$, a cyclic $F(X^n)$-algebra. Since $K[X;\sigma] \subseteq \langle K(X^n), X\rangle \subseteq K(X;\sigma)$, we conclude that $\langle K(X^n), X\rangle = K(X;\sigma)$.

In $K(X;\sigma)$, we also have the subring $K[X^{-1};\sigma]$ of all polynomials in X^{-1} with coefficients in K with multiplication: $X^{-1}a = \sigma^{-1}(a)X^{-1}$. Its quotient ring is also $K(X;\sigma)$ and $\langle K(X^n), X^{-1}\rangle = K(X;\sigma)$.

We will therefore identify $K(X;\sigma)$ with either

$$Q_{f_1} = (K(X^n)/F(X^n), \sigma, X^n) \text{ or } Q_{f_2} = (K(X^n)/F(X^n), \sigma^{-1}, X^{-n}),$$

where for $1 \leq i,j < n$, $f_1(\sigma^i, \sigma^j) = f_2(\sigma^{-i}, \sigma^{-j}) = 1$ if $i+j < n$; otherwise $f_1(\sigma^i, \sigma^j) = X^n$ and $f_2(\sigma^{-i}, \sigma^{-j}) = X^{-n}$. We have $f_1 \sim_K f_2$.

Let V be a valuation ring of $F(X^n)$ that contains F. Then $V \supseteq F[X^n]$ or $V \supseteq F[X^{-n}]$. Whatever is the case, V is a DVR of $F(X^n)$. Let S be the integral closure of V in $K(X^n)$. Then $S = KV$. Since K is Galois over F, there is an element $t \in K$ of trace 1. Since $t \in S$, we see that $K(X^n)/F(X^n)$ is a tamely ramified extension. We will now see that it is actually an unramified extension.

Let $W_1, \dots, W_r$ be all the extensions of V to $K(X^n)$. Assume $V \supseteq F[X^n]$. Then $J(W_i) = q_i W_i$, where q_i is a *monic* irreducible polynomial in $K[X^n]$. Given a $\tau \in G$ then, in the ring $K[X^n]$, $\tau(q_i)$ is an associate of q_j for some j. But since the q_k as well as the $\tau(q_k)$ are monic, we actually have equality: $\tau(q_i) = q_j$ for some j. Thus G permutes the q_k, and therefore $p = q_1 q_2 \cdots q_r$ is fixed by G, and hence is an element of $F[X^n]$. We conclude that the ramification index of S over V is 1. The same conclusion holds when $V \supseteq F[X^{-n}]$. Hence $K(X^n)/F(X^n)$ is an unramified extension.

Now let A be an order in $K(X;\sigma)$ containing K whose center is a valuation ring V. Then V must contain F and is therefore a DVR. Such orders are automatically V-orders. Moreover, in this case, $A \supseteq KV = S$. Since S is unramified over V, by [3, Proposition 1.3] we have

$$A = A_f = \sum_{i=o}^{n-1} S x_{\sigma^i},$$

a "crossed-product" V-algebra, where $x_{\sigma^i} = k_i X^i$ ($x_{\sigma^i} = k_i X^{-i}$ if $X^n \notin V$), $1 = k_0, k_1, \dots, k_{n-1} \in K(X^n)^{\#}$, and $f : G \times G \mapsto S^{\#}$ is a normalized 2-cocycle cohomologous to f_1 (as well as f_2) over K.

Let A_f and A_g be two V-orders in $K(X;\sigma)$ containing K. Then $f \sim_S g$ if and only if there exists a V-algebra isomorphism $\phi : A_f \mapsto A_g$ such that $\phi(s) = s$ for all $s \in S$. Suppose $f \sim_S g$ and ϕ is such an isomorphism. Then there exists a $q = c_0 + c_1 X + c_2 X^2 + \cdots + c_m X^m \in K[X;\sigma]^{\#}$ such that ϕ is conjugation by q. Since $K \subseteq S$, $\phi(t) = t$ for all $t \in K$ and hence $qt = tq$ for all $t \in K$. Thus, for each i, $c_i \sigma^i(t) = t c_i$ for all $t \in K$. So $c_i = 0$ if $i \neq 0$; hence $q = c_0 \in K^{\#} \subseteq U(A_f)$. Hence $A_g = \phi(A_f) = c_0 A_f c_0^{-1} = A_f$. Conversely, if $A_f = A_g$, then $f \sim_S g$.

We therefore have the following lemma:

Lemma 1.1. (1) *Let V be a valuation ring of $F(X^n)$ that contains F, and let S be the integral closure of V in $K(X^n)$. Then V is a DVR and S is unramified over V.*

(2) *If A is an order in $K(X;\sigma)$ containing K and whose center is a valuation ring, then the center is a DVR as in part (1) above and $A = A_f$, a crossed-product order, where $f : G \times G \mapsto S^{\#}$ is a normalized 2-cocycle cohomologous to f_1 and f_2 over K.*

(3) *If* A_f *and* A_g *are two such orders of* $K(X;\sigma)$*, then* $f \sim_S g$ *if and only if* $A_f = A_g$*. That is, if* $\{A_f\}$ *are orders in* $K(X;\sigma)$ *of this type, then there is a one-to-one correspondence between such orders and cohomology classes of* f *over* S*.*

For the rest of this paper, V will denote a valuation ring of $F(X^n)$ that contains F, and S will denote its integral closure in $K(X^n)$. Either $X^n \in V$ or $X^{-n} \in V$ (or both). If $X^n \in V$, we will let

$$A_{f_1} = \sum_{i=o}^{n-1} SX^i,$$

and if $X^{-n} \in V$, we will let

$$A_{f_2} = \sum_{i=o}^{n-1} SX^{-i}.$$

These are crossed-product V-orders in $K(X;\sigma)$ with respect to the cocycles f_1 and f_2 described above. Observe that if $X^n \in U(V)$, then $A_{f_2} = X^n A_{f_2} = A_{f_1}$ and $f_1 \sim_S f_2$.

In this paper we want to give an explicit description, in terms of crossed-product orders,[1] of all hereditary and maximal orders in $K(X;\sigma)$ that contain K and whose center is a valuation ring. In so doing, we will determine the connections between hereditary-ness and maximal order properties of A_f, and the properties of H, f, and the graph of f. Our main result is Theorem 3.2.

2. Maximal orders in $K(X;\sigma)$ containing K

Let A be a maximal V-order in $K(X;\sigma)$ containing K. Then $A = A_f$ for some normalized 2-cocycle $f : G \times G \mapsto S^\#$ by Lemma 1.1. We consider three cases:

Case 1: $X^n \in V$, but $X^{-n} \notin V$. We know that $V = F[X^n]_P$ for some maximal ideal P of $F[X^n]$. Since $X^n \notin U(V)$, $X^n \in P$; hence $X^n F[X^n] \subseteq P$, forcing $P = X^n F[X^n]$. Thus $V = F[X^n]_{X^n F[X^n]}$ and $S = K[X^n]_{X^n K[X^n]}$, which is a DVR unramified over V. In this case, $H_{f_1} = \{1\}$, $f_1(\sigma, \sigma^{-1}) = X^n \in X^n S \setminus X^{2n} S$, and the graph of f_1 is the simple chain

$$H_{f_1} \leq \sigma H_{f_1} \leq \sigma^2 H_{f_1} \leq \cdots \leq \sigma^{n-1} H_{f_1}.$$

Hence A_{f_1} is a maximal order by [3, Theorem 2.3].

It is well known that $K[X;\sigma]$ is a Bézout ring, since K is a field and σ is an automorphism of K. Thus $K[X;\sigma]$ is a prime PI Bézout ring; hence, by [2, Theorem 2.6], it is localizable at the maximal ideal $XK[X;\sigma]$, and if B is the localization of $K[X;\sigma]$ at $XK[X;\sigma]$, then B is a valuation ring of $K(X;\sigma)$. Observe that $B/J(B) = K$, a field; therefore B is a total valuation ring. Since $Z(B) = V$ is a DVR, B must be an invariant valuation ring of $K(X;\sigma)$. Thus $A_f = B = A_{f_1}$. By the sublemma, A_f is not Azumaya over V.

Case 2: $X^{-n} \in V$, but $X^n \notin V$. This is a mirror image of Case 1. We readily see that $A_f = A_{f_2}$. It is an invariant valuation ring of $K(X;\sigma)$, but not an Azumaya algebra.

Case 3: $X^n \in U(V)$. Since S/V is a Galois extension of rings and $f_1 : G \times G \mapsto U(S)$, A_{f_1} is Azumaya over V. We therefore have $A_f \cong A_{f_1} = A_{f_2}$.

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[1] Note that since f does not necessarily take values in $U(S)$, A_f may not be a classical crossed-product algebra. This is important to bear in mind throughout this paper.

Therefore we have the following theorem:

Theorem 2.1. *Let A be a V-order in $K(X;\sigma)$ containing K whose center is a valuation ring V. If A is a maximal order, then:*

(1) $A \cong A_{f_1}$ *or* $A \cong A_{f_2}$.

(2) *A is either an invariant valuation ring of $K(X;\sigma)$ or an Azumaya algebra over its center or both.*

(3) *If $X^n \in V \backslash U(V)$ (respectively $X^{-n} \in V \backslash U(V)$), then $A = A_{f_1}$ (respectively $A = A_{f_2}$), and it is an invariant valuation ring of $K(X;\sigma)$.*

(4) *A is Azumaya over its center if and only if X^n is a unit in A. When this occurs, $A_{f_1} = A_{f_2}$.*

Conversely, if $A \cong A_{f_1}$ or $A \cong A_{f_2}$, then A is a maximal order.

Remark 2.2. A characterization of maximal V-orders in $K(X;\sigma)$ in terms of H, f, and the graph of f is a straightforward application of the sublemma and [3, Theorem 2.3]. The order A_f is maximal if and only if one of the following is true:

(1) the graph of f is trivial;

(2) $H = \{1\}$, the graph of f is $H \leq \sigma H \leq \sigma^2 H \leq \cdots \leq \sigma^{n-1}H$, and $f(\sigma, \sigma^{-1}) \in J(S) \setminus J(S)^2$;

(3) or $H = \{1\}$, the graph of f is $H \leq \sigma^{-1}H \leq \sigma^{-2}H \leq \cdots \leq \sigma^{1-n}H$, and $f(\sigma^{-1}, \sigma) \in J(S) \setminus J(S)^2$,

according to whether $X^n \in U(V)$, $X^n \in V \setminus U(V)$, or $X^{-n} \in V \setminus U(V)$.

We now want to determine the number of maximal V-orders in $K(X;\sigma)$ that contain K.

Suppose V is indecomposed in $K(X^n)$. If $X^n \notin V$ or $x^{-n} \notin V$, then by Theorem 2.1 there is only one maximal V-order in $K(X;\sigma)$ and it is an invariant valuation ring and we are done. On the other hand, if $X^n \in U(V)$, then A_{f_1} is an Azumaya algebra by Theorem 2.1. Let B be a maximal V-order in $K(X;\sigma)$ containing K. Then, since $B \supseteq S$, we have $B = A_g$ for some 2-cocycle $g : G \times G \mapsto S^{\#}$. In fact, the proof of [3, Proposition 1.3] shows that there exists $1 = k_0, k_1, \ldots, k_{n-1} \in K(X^n)^{\#}$ such that

$$A_g = \sum_{i=o}^{n-1} Sk_iX^i.$$

So if $i + j < n$, then $(k_iX^i)(k_jX^j) = g(\sigma^i, \sigma^j)k_{i+j}X^{i+j}$ and $g(\sigma^i, \sigma^j) = k_i\sigma^i(k_j)/k_{i+j}$. All maximal V-orders in $K(X;\sigma)$ are conjugate, because V is a DVR; hence B must be an Azumaya algebra as well, and so by the sublemma we have $g(\sigma^i, \sigma^j) \in U(S)$ for all i, j. Let v be a valuation on $K(X^n)$ corresponding to the valuation ring S. Since $v(g(\sigma^i, \sigma^j)) = 0$ for all i, j, we have $v(k_{i+j}) = v(k_i) + v(k_j)$ if $i + j < n$; therefore

$$v(k_i) = iv(k_1), \;\; 0 \leq i < n.$$

If $v(k_1) > 0$, then $v(k_i) > 0$ for all $1 \leq i < n$; hence

$$B = \sum_{i=o}^{n-1} Sk_iX^i \subsetneqq \sum_{i=o}^{n-1} SX^i = A_{f_1},$$

a contradiction. If $v(k_1) < 0$, then $B \supsetneqq A_{f_1}$, another contradiction. So $v(k_i) = 0$ for all i; hence we must have $B = A_{f_1}$. Thus A_{f_1} is the only maximal V-order of $K(X;\sigma)$ containing K.

Now suppose V decomposes in $K(X^n)$. Let B be a maximal V-order containing K. Then B is not an invariant valuation ring; otherwise S would be a valuation ring of $K(X^n)$. By Theorem 2.1, B is an Azumaya algebra over V and $K[X^n] \subseteq B$. If $J(V) = pV$, where $p = p(X^n)$ is an irreducible polynomial in $F[X^n]$, let $p = q_1 q_2 \cdots q_r$ be the complete factorization of p in $K[X^n]$. Then $r > 1$, since V decomposes in $K(X^n)$, and the q_k are *distinct* irreducible polynomials in $K[X^n]$, as in the proof of Lemma 1.1. Hence $q_1^l \notin pK[X^n]$ for any natural number l. Since $J(B) = pB$, we have, for any natural number l,

$$q_1^l \in \ (B \setminus J(B)) \setminus U(B).$$

By [6, Lemma 3.2], $q_1^l B q_1^{-l} \neq B$ for all l. Since $q_1^l K q_1^{-l} = K$, we see that B has infinitely many conjugates containing K.

We therefore have the following result:

Theorem 2.3. *If V is indecomposed in $K(X^n)$, then there is a unique maximal V-order in $K(X;\sigma)$ containing K, namely A_{f_1} or A_{f_2}. If V is decomposed, then there are infinitely many maximal V-orders in $K(X;\sigma)$ containing K. Moreover, $X^n \in U(V)$, and these maximal orders are Azumaya.*

Remark 2.4. Even when V is indecomposed in $K(X^n)$, if the maximal V-order containing K is not an invariant valuation ring, then, by [6, Theorem G and Corollary G], there are infinitely many maximal V-orders in $K(X;\sigma)$, but the rest do not contain K.

3. Hereditary orders in $K(X;\sigma)$ containing K

We now want to determine the precise conditions for a V-order in $K(X;\sigma)$ containing K to be hereditary. But first, we need the following lemma, based on an analysis of [4, Definition 2.1(iii)] and that of the proof of [4, Theorem 2.4]. For purposes of this lemma, we need the following setup. Let D be a finite dimensional division ring with center L, and let U be a DVR of L. Let Δ be an invariant valuation ring of D with center U, and let v be a valuation on D corresponding to Δ with value group $\mathbb{Z}$. For a fixed positive integer k, let $Q = M_k(D)$, a full matrix ring over D. If a U-order A in Q contains all the standard idempotents of Q, then $A = M_k(\Delta_{ij})$, where each Δ_{ij} is a U-submodule of D. Assume that $\Delta_{ii} = \Delta$ for each i. Then each Δ_{ij} is a Δ-submodule of D, since R is a ring. We know there exists a $\delta \in \Delta$ such that $J(\Delta) = \delta\Delta$. Fix integers i and j, with $1 \leq i, j \leq k$. Then there are fixed integers a and b such that $\Delta_{ij} = \delta^a \Delta$ and $\Delta_{ji} = \delta^b \Delta$.

Suppose $a + b > 1$. Then $\alpha = \delta^{a-1}$ has the property that $\alpha \notin \delta^a \Delta = \Delta_{ij}$, and, since $1 - a < b$, we also have $\alpha^{-1} \notin \delta^b \Delta = \Delta_{ji}$.

On the other hand, suppose $a + b \leq 1$. Let $0 \neq \alpha \in D$. If $\alpha \notin \Delta_{ij}$ and $\alpha^{-1} \notin \Delta_{ji}$, then $v(\alpha) < a$ and $v(\alpha^{-1}) < b$, and so $0 = v(\alpha \cdot \alpha^{-1}) = v(\alpha) + v(\alpha^{-1}) < a + v(\alpha^{-1}) < a + b \leq 1$; hence $0 < a + v(\alpha^{-1}) < 1$, a contradiction.

Thus for condition [4, Definition 2.1(iii)] to hold in our situation, the precise requirement is that $a + b \leq 1$; that is, we need to have $\delta^a \Delta \delta^b \Delta = \delta^{a+b} \Delta \supseteq J(\Delta)$.

We thus obtain the following corollary of [4, Theorem 2.4], valid since U is a DVR:

Lemma 3.1. *With the notation as above, A is hereditary if and only if, for all i, j, $1 \leq i, j \leq k$, we have $\Delta_{ij}\Delta_{ji} \supseteq J(\Delta)$.*

Let A be an order in $K(X;\sigma)$ containing K whose center is a valuation ring V. If A is hereditary, then it is well known that it is an intersection of maximal V-orders; e.g., see [5, Theorem 40.10]. If there exists only one maximal V-order containing K, then A must be a maximal V-order.

From now on, assume there is more than one maximal V-order containing K. Therefore, by Theorem 2.3, $X^n \in U(V)$ and V decomposes in $K(X^n)$. We know that

$$A = A_f = \sum_{i=o}^{n-1} S x_{\sigma^i},$$

where $x_{\sigma^i} = k_i X^i$, $1 = k_0, k_1, \dots, k_{n-1} \in K(X^n)^{\#}$.

Note that, as M varies over the set of maximal ideals of S, the corresponding decomposition groups coincide, since G is an abelian group. We will denote this group by G^Z.

Let $M = M_1, M_2, \dots, M_r$ be a complete list of maximal ideals of S. Let $F_i : G^Z \times G^Z \mapsto S_{M_i}^{\#}$ be the restriction of f. If $(\widehat{F(X^n)}, \widehat{V})$ is the completion of $(F(X^n), V)$, then

$$S \otimes_V \widehat{V} \cong S_1 \oplus \cdots \oplus S_r \ \text{ and } \ K(X^n) \otimes_{F(X^n)} \widehat{F(X^n)} \cong K_1 \oplus \cdots \oplus K_r,$$

where (K_i, S_i) is the completion of $(K(X^n), S_{M_i})$. Let e_{ii} be the multiplicative identity of S_i. Let $g_1, g_2, \dots, g_r$ be coset representatives of G^Z in G with $g_i(e_{11}) = e_{ii}$. It is known that

$$K(X;\sigma) \otimes_{F(X^n)} \widehat{F(X^n)} \cong M_r(Q_{\hat{f}_1}),$$

a full matrix ring over $Q_{\hat{f}_1}$, where $\hat{f}_i : G^Z \times G^Z \mapsto S_i^{\#}$ is the obvious "completion" of F_i and $Q_{\hat{f}_i} := \sum_{\sigma^j \in G^Z} K_i X^j$.[2] The e_{ii} correspond to the standard orthogonal idempotents of the matrix ring above. Observe that $e_{11}, \dots, e_{rr} \in S \otimes_V \widehat{V} \subseteq A_f \otimes_V \widehat{V}$. Therefore

$$\hat{A} := A_f \otimes_V \widehat{V} \cong M_r(\Delta_{ij}),$$

where $\Delta_{ij} = e_{ii} \hat{A} e_{jj}$. We readily see that

$$\Delta_{ij} = \sum_{h \in G^Z} S_i \hat{x}_{h g_i g_j^{-1}} \ \text{ and } \ \Delta_{ii} = \sum_{h \in G^Z} S_i \hat{x}_h.$$

(Here, $\hat{x}_g$ is the image of x_g under the natural embedding of $K(X;\sigma)$ into $K(X;\sigma) \otimes_{F(X^n)} \widehat{F(X^n)}$.)

For the moment, assume that $G^Z \subseteq H$. Let v_i be a valuation on K_i corresponding to S_i. Then by [3, Lemma 3.5] we have, for all $h \in G^Z$, $v_i(\hat{f}_i(h, g_i g_j^{-1})) = v_i(\hat{f}_i(g_i g_j^{-1}, h)) = 0$. Therefore $\Delta_{ij} = \Delta_{ii} \hat{x}_{g_i g_j^{-1}} = \hat{x}_{g_i g_j^{-1}} \Delta_{jj}$.

Observe that $\Delta_{ii} = A_{\hat{f}_i} := \sum_{\sigma^j \in G^Z} S_i \hat{x}_{\sigma^j}$. Since $G^Z \subseteq H$ and S_i is unramified over $\hat{V}$, $\Delta_{ii} \cong M_s(\Delta)$, where Δ is the invariant valuation ring with center $\widehat{V}$ of the division algebra part of $K(X;\sigma) \otimes_{F(X^n)} \widehat{F(X^n)}$, and s is some fixed positive integer independent of i. Let $e_{ii} = e_{i_1 i_1} + e_{i_2 i_2} + \cdots + e_{i_s i_s}$ be the decomposition of $e_{ii} \in \Delta_{ii}$ into primitive orthogonal idempotents corresponding to the standard

[2] $\hat{f}_1$ and $\hat{f}_2$ are not to be confused with the cocycles f_1 and f_2 defined in §1.

orthogonal idempotents of $M_s(\Delta)$. Since $\widehat{V}$ is a DVR, Lemma 3.1 now shows that $\hat{A}$ is hereditary if and only if

$$e_{i_l i_l}\hat{A}e_{j_{l'}j_{l'}} \cdot e_{j_{l'}j_{l'}}\hat{A}e_{i_l i_l} \supseteq e_{i_l i_l}J(\Delta_{ii})e_{i_l i_l}$$

for all $1 \le i, j \le r$, $1 \le l, l' \le s$. But $e_{i_l i_l}\hat{A}e_{j_{l'}j_{l'}} \cdot e_{j_{l'}j_{l'}}\hat{A}e_{i_l i_l} = e_{i_l i_l}e_{ii}\hat{A}e_{jj}e_{j_{l'}j_{l'}} \cdot e_{j_{l'}j_{l'}}e_{jj}\hat{A}e_{ii}e_{i_l i_l} = e_{i_l i_l}\Delta_{ij}e_{j_{l'}j_{l'}}\Delta_{ji}e_{i_l i_l} = e_{i_l i_l}\hat{x}_{g_i g_j^{-1}}(\Delta_{jj}e_{j_{l'}j_{l'}}\Delta_{jj})\hat{x}_{g_j g_i^{-1}}e_{i_l i_l} = e_{i_l i_l}\hat{x}_{g_i g_j^{-1}}\Delta_{jj}\hat{x}_{g_j g_i^{-1}}e_{i_l i_l} = e_{i_l i_l}\Delta_{ii}\hat{x}_{g_i g_j^{-1}}\hat{x}_{g_j g_i^{-1}}e_{i_l i_l} = e_{i_l i_l}\Delta_{ii}f(g_i g_j^{-1}, g_j g_i^{-1})e_{i_l i_l} = e_{i_l i_l}\Delta_{ii}(e_{ii}f(g_i g_j^{-1}, g_j g_i^{-1})e_{ii})e_{i_l i_l}$.

Thus $\hat{A}$ is hereditary if and only if $f(g_i g_j^{-1}, g_j g_i^{-1}) \notin (M_i)^2$ for all i, j.

We now drop the assumption that $G^Z \subseteq H$. We will continue to assume that $X^n \in U(V)$ and so all maximal V-orders in $K(X;\sigma)$ are Azumaya algebras, which implies that Δ is an Azumaya algebra over $\widehat{V}$. We know that A is hereditary if and only if $\hat{A}$ is hereditary [5, Theorem 40.5].

Now suppose A is hereditary. Then $\hat{A}$, hence $A_{\hat{f}_i} = e_{ii}\hat{A}e_{ii}$, a crossed-product in the sense of Haile, since S_i is unramified over $\widehat{V}$, is hereditary. Each $A_{\hat{f}_i}$ is primary since S_i is a valuation ring of K_i, and hence $A_{\hat{f}_i}$ is a maximal $\widehat{V}$-order in $Q_{\hat{f}_i}$ and so $A_{\hat{f}_i} \cong M_s(\Delta)$. We conclude that each $A_{\hat{f}_i}$ is Azumaya over $\widehat{V}$ and thus the subgroup of G^Z associated to $\hat{f}_i$ is the whole G^Z by the sublemma. Since $f(\sigma^j, \sigma^{-j}) \notin M_i$ for all $\sigma^j \in G^Z$ and all i, we conclude that $G^Z \subseteq H$.

So when A_f is hereditary, then $G^Z \subseteq H$ and $f(g_i g_j^{-1}, g_j g_i^{-1}) \notin (M_i)^2$ for all i, j. Thus A_f is hereditary if and only if $G^Z \subseteq H$ and $f(g_i g_j^{-1}, g_j g_i^{-1}) \notin (M_i)^2 = (M^2)^{g_i}$ for all i, j.

When V decomposes in $K(X^n)$ then by Theorem 2.3 there are infinitely many hereditary orders in $K(X;\sigma)$ of the form A_f; hence there are infinitely many cohomology classes (over S) of such an f.

We can now state our main results:

Theorem 3.2. *Let A_f be a V-order as described above.*

(1) *Suppose V is indecomposed in $K(X^n)$ and X^n is a unit in V. Then A_f is hereditary if and only if $H = G$, i.e., the graph of f is trivial. In this case, A_f is Azumaya over V, and it is the unique hereditary V-order of $K(X;\sigma)$ containing K. We have $f \sim_S f_1 \sim_S f_2$.*

(2) *Suppose V is indecomposed in $K(X^n)$ and $X^n \in V \setminus U(V)$ (respectively $X^{-n} \in V \setminus U(V)$). Then A_f is hereditary if and only if $H = \{1\}$, $f(\sigma, \sigma^{-1}) \in J(S) \setminus J(S)^2$ (respectively $f(\sigma^{-1}, \sigma) \in J(S) \setminus J(S)^2$), and the graph of f is*

$$H \le \sigma H \le \sigma^2 H \le \cdots \le \sigma^{n-1} H$$

$$(\textit{respectively } H \le \sigma^{-1} H \le \sigma^{-2} H \le \cdots \le \sigma^{1-n} H).$$

In this case, A_f is an invariant valuation ring of $K(X;\sigma)$, and so it is the unique hereditary V-order in $K(X;\sigma)$ and hence is maximal. It is not Azumaya over V. We have $f \sim_S f_1$ (respectively $f \sim_S f_2$).

(3) *Suppose V decomposes in $K(X^n)$. Then A_f is hereditary if and only if, for every maximal ideal M of S with decomposition group G^Z, we have*

$G^Z \subseteq H$ and there exist coset representatives $g_1, g_2, \ldots, g_r$ of G^Z in G such that $f(g_i g_j^{-1}, g_j g_i^{-1}) \notin (M^2)^{g_i}$ for all i, j.

There are infinitely many cohomology classes (over S) of such f in this case.

We end by giving an example that illustrates some of the main results of this paper. For the background material and definitions of any undefined terms, the reader is referred to [5].

Example 3.3. Let $F = \mathbb{Q}$, $K = \mathbb{Q}(i)$, and σ be the complex conjugation on K. Let $V = F[X^2]_{(X^4+1)}$. Then $X^2 \in U(V)$. Since $X^4 + 1 = (X^2 - i)(X^2 + i)$ in $K[X^2]$, $B := A_{f_1}$ and $B' := (x^2 + i)B(X^2 + i)^{-1}$ are distinct maximal V-orders in $K(X;\sigma)$ containing K, and each one is an Azumaya algebra. Since $(X^4 + 1)B = \big[B(X^2 - i)\big] \cdot \big[(X^2 + i)B\big]$ is a factorization of $J(B)$ into integral ideals $B(X^2 - i)$ and $(X^2 + i)B$, $A := O_l(B(X^2 - i)) \cap O_l((X^2 + i)B) = B \cap B'$ is a hereditary V-order in $K(X;\sigma)$.

In terms of crossed-product orders, note that $A = S \oplus S(X^2 + i)X$; therefore $A = A_f$, where $f(1,1) = f(1,\sigma) = f(\sigma,1) = 1$ and $f(\sigma,\sigma) = (X^4 + 1)X^2$. Starting with such an A_f, we have $\{1\} = G^Z = H$ and $f(\sigma,\sigma) \notin (X^2 + i)^2 S, (X^2 - i)^2 S$, and so by Theorem 3.2, we again see that A_f must be hereditary.

Now let $B'' = (x^2+i)B'(X^2+i)^{-1}$ and $A' = B \cap B' \cap B''$. Since V decomposes in $K(X^2)$, B is not an invariant valuation ring of $K(X;\sigma)$; therefore the completion of $K(X;\sigma)$, $K(X;\sigma) \otimes_{F(X^2)} \widehat{F(X^2)}$, is a split algebra of degree 2. It follows that any hereditary V-order in $K(X;\sigma)$ has type at most 2, and therefore A', which is an intersection of three distinct maximal V-orders, cannot be hereditary.

In terms of crossed-product orders, observe that $A' = S \oplus S(X^2+i)^2 X$; therefore $A' = A_g$, where $g(1,1) = g(1,\sigma) = g(\sigma,1) = 1$ and $g(\sigma,\sigma) = (X^4 + 1)^2 X^2$. If we started with such an A_g, we have $g(\sigma,\sigma) \in (X^2 + i)^2 S, (X^2 - i)^2 S$, and so by Theorem 3.2, we again see that A_g cannot be hereditary.

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