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ON A CLASS OF MAGNETIC SCHRÖDINGER OPERATORS WITH DISCRETE SPECTRUM

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Abstract. We introduce a class of magnetic Schrödinger operators in $\mathbf{R}^n$ which exhibit pure point spectrum in a fashion that is actually easy to check. This class is an adequate generalization of the more familiar two-dimensional setting, and the proof we give for its spectral discreteness is novel, based on the use of Euclidean Dirac operators coupled to vector potentials.

1. Introduction

Let $H_a = \sum_{j=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} + a_j \right)^2$, $i = \sqrt{-1}$, be a magnetic Schrödinger operator in $\mathbf{R}^n$, $n \geq 2$, with coordinates $x = (x_1, x_2, \ldots, x_n)$, where $a = \sum_{j=1}^{n} a_j dx_j$ is a magnetic potential whose components $a_j = a_j(x)$ are real-valued functions of class C^1. It is well known [AHS, I, KS] that H_a with domain $C_0^\infty(\mathbf{R}^n, \mathbf{C})$ is an essentially selfadjoint operator in $L^2(\mathbf{R}^n, \mathbf{C})$. Its spectrum, which is non-negative, depends only on the magnetic field $B = da$ associated to a,

$$B = \sum_{j<k} B_{jk}\, dx_j \wedge dx_k, \quad B_{jk} = \frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k},$$

by gauge invariance [AHS, I, KS].

It is a matter of great interest to decide for which magnetic fields B, H_a admits a discrete spectrum, that is, a spectrum consisting only in isolated eigenvalues of finite multiplicity. Although sufficient conditions for this to happen are known, mainly involving the behavior of $|B| = \left(\sum_{j<k} B_{jk}^2 \right)^{1/2}$ at infinity under a controlled rate of change of the directions B_{jk} of B [AHS, D, HM, I, KS], no satisfactory necessary and sufficient conditions are available.

The purpose of this paper is to present yet another sufficient condition for the discreteness of the spectrum of H_a, which seems to be the natural generalization of the familiar case $n = 2$. Its advantage is that it hides the need to control the rate of change of the directions of B in a format easily verifiable in concrete situations. Also, its proof, based on the use of Euclidean Dirac operators coupled to vector potentials, differs from those presented so far in the literature.

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Theorem. *Let H_a be a magnetic Schrödinger operator in $\mathbf{R}^n$ with magnetic field B, as described above. Assume that there are distinct indices $j_1 < k_1, j_2 < k_2, \dots, j_\mu < k_\mu$, $\mu = \left[\frac{n}{2}\right]$, in the interval $[1,n]$, and a sequence of signs ± 1, $\epsilon_1, \epsilon_2, \dots, \epsilon_\mu$, such that*

$$\lim_{|x|\longrightarrow\infty} \sum_{p=1}^{\mu} \epsilon_p B_{j_p k_p}(x) = +\infty. \tag{1}$$

Then H_a has a discrete spectrum in $L^2(\mathbf{R}^n, \mathbf{C})$.

2. Euclidean Dirac operators coupled to vector potentials

On $\mathbf{R}^n$, $n \geq 2$, consider an irreducible representation of the complex Clifford algebra $Cl(\mathbf{R}^n)$ on the finite-dimensional Hermitian complex vector space $\mathbf{C}^{2^\mu}$, $\mu = \left[\frac{n}{2}\right]$. This is equivalent to the prescription of n skew-Hermitian endomorphisms of $\mathbb{C}^{2^\mu}$, $E_1, E_2, \dots, E_n$ such that for every j, $E_j^2 = -Id$, and $E_jE_k + E_kE_j = 0$ for every $j \neq k$. For instance, such a concrete representation, useful for our needs, is the following:

Define first the 2×2 complex matrices U, V, and W, by

$$U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad W = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Any two of them anti-commute, U and V are skew-adjoint while W is selfadjoint, and $U^2 = V^2 = -I$, $W^2 = I$, where I is the 2×2 identity matrix.

If n is even, $n = 2\mu$, define, for $p = 1, 2, \dots, \mu$, n complex square matrices of size 2^μ by the formulas

$$\begin{aligned} E_{2p-1} &:= \underset{1}{W} \otimes \cdots \otimes \underset{p-1}{W} \otimes \underset{p}{U} \otimes \underset{p+1}{I} \otimes \cdots \otimes \underset{\mu}{I}, \\ E_{2p} &:= \underset{1}{W} \otimes \cdots \otimes \underset{p-1}{W} \otimes \underset{p}{V} \otimes \underset{p+1}{I} \otimes \cdots \otimes \underset{\mu}{I}. \end{aligned} \tag{2}$$

If n is odd, $n = 2\mu+1$, in addition to $E_1, E_2, \dots, E_{2\mu}$ given by (2), define

$$E_{2\mu+1} := i\underset{1}{W} \otimes \cdots \otimes \underset{\mu}{W}. \tag{3}$$

Then [B] the formulas (2) and (3) indeed represent the n generators of an irreducible matrix representation of the complex Clifford algebra $Cl(\mathbf{R}^n)$ on $\mathbf{C}^{2^\mu}$. The matrices $E_1, E_2, \dots, E_n$ introduced above have the following useful, easy to check property: All the product matrices E_jE_k, $j < k$, have 0 as the first element in the first column, except for the μ matrices $E_{2p-1}E_{2p}$, $p = 1, 2, \dots, \mu$, whose first column is precisely $[i, 0, \dots, 0]^t$. Consequently, under the splitting $\mathbf{C}^{2^\mu} = \mathbf{C} \times \mathbf{C}^{2^\mu - 1}$,

$$\begin{aligned} &E_{2p-1}E_{2p}(\mathbf{C}\times 0) = i\mathbf{C}\times 0, \quad p = 1, 2, \dots, \mu, \\ &E_jE_k(\mathbf{C}\times 0) \subset 0 \times \mathbf{C}^{2^\mu-1}, \quad j<k, \ (j,k) \neq (2p-1, 2p). \end{aligned} \tag{4}$$

The Euclidean Dirac operator is then the differential operator

$$\not{D} : C^\infty(\Omega, \mathbf{C}^{2^\mu}) \longrightarrow C^\infty(\Omega, \mathbf{C}^{2^\mu}), \quad \Omega \subseteq \mathbf{R}^n \text{ open},$$

defined, for spinors $f \in C^\infty(\Omega, \mathbf{C}^{2^\mu}) = C^\infty(\Omega, \mathbf{C}) \otimes \mathbf{C}^{2^\mu}$ in column form $f = [f_1, f_2, \dots, f_{2^\mu}]^t$, $f_\alpha \in C^\infty(\Omega, \mathbf{C})$, by

$$\not{D} f = \sum_{j=1}^{n} E_j \nabla_j f,$$

where $\nabla_j = \frac{\partial}{\partial x_j} \otimes Id$ represents ordinary component-wise differentiation with respect to x_j. It is well known [LM] that $\not{D}$ with domain $C_0^\infty(\mathbf{R}^n, \mathbf{C}^{2^\mu})$ is an essentially selfadjoint first-order elliptic differential operator in $L^2(\mathbf{R}^n, \mathbf{C}^{2^\mu})$.

For any magnetic potential $a = \sum_{j=1}^n a_j dx_j$ with real-valued components $a_j(x)$ of class C^1, the operator formally defined by

$$\not{D}_a := \sum_{j=1}^n E_j \left(\nabla_j + i a_j Id\right)$$

is a particular instance of a Dirac operator on $\mathbf{R}^n$ coupled to a vector potential in the sense of [LM], that is, a Dirac operator on the trivial tensor bundle $\mathbf{C}^{2^\mu} \otimes \mathbf{C} = \mathbf{C}^{2^\mu}$ over $\mathbf{R}^n$, where $\mathbf{C}$ is equipped with the metric connection $d_a = d + ia$. As such, it is again an essentially selfadjoint first-order elliptic differential operator in $L^2(\mathbf{R}^n, \mathbf{C}^{2^\mu})$.

For the square of $\not{D}_a$, the following Bochner-Weitzenböck formula [LM] holds true on $C^2(\mathbf{R}^n, \mathbf{C}^{2^\mu})$,

$$\not{D}_a^2 = (\nabla \otimes d_a)^* (\nabla \otimes d_a) + \mathcal{R}_a, \tag{5}$$

where the connection Laplacian $(\nabla \otimes d_a)^* (\nabla \otimes d_a)$ is easily seen to equal $H_a \otimes Id$, and where $\mathcal{R}_a$ is the Hermitian curvature bundle morphism acting on $\mathbf{C}^{2^\mu}$ according to the formula

$$\mathcal{R}_a = \sum_{j<k} E_j E_k R^a_{jk}, \quad R^a_{jk} = [\nabla_j + i a_j Id, \nabla_k + i a_k Id] = i B_{jk} Id. \tag{6}$$

3. The proof of the Theorem

Proof. It is known [I, KS] that H_a has a discrete spectrum if and only if there is a function $\lambda \in C(\mathbf{R}^n, \mathbf{R})$, $\lim_{|x| \longrightarrow \infty} \lambda(x) = +\infty$, such that for any $\phi \in C_0^\infty(\mathbf{R}^n, \mathbf{C})$,

$$(H_a \phi, \phi) \geq (\lambda \phi, \phi), \tag{7}$$

where $(\cdot, \cdot)$ denotes the standard inner product in $L^2(\mathbf{R}^n, \mathbf{C})$.

Assume first that hypothesis (1) of the Theorem is satisfied for $j_p = 2p - 1$, $k_p = 2p$, $\epsilon_p = 1$, $p = 1, 2, \ldots, \mu$. Applying the Bochner-Weitzenböck identity (5) to the spinor $[\phi, 0, 0, \ldots, 0]^t \in C_0^\infty(\mathbf{R}^n, \mathbf{C}^{2^\mu})$ yields, via equations (5) and (4),

$$(H_a \phi, \phi) = \left|\left|\not{D}_a [\phi, 0, 0, \ldots, 0]^t\right|\right|^2 + \left(\left(\sum_{p=1}^\mu B_{2p-1,2p}\right) \phi, \phi\right),$$

and so the content of equation (7) is fulfilled by letting $\lambda := \sum_{p=1}^\mu B_{2p-1,2p}$.

The general case now follows by making the substitutions $E_{j_p} \longrightarrow (-1)^{\epsilon_p} E_{2p-1}$, $E_{k_p} \longrightarrow E_{2p}$ (and eventually $E_l \longrightarrow E_{2\mu+1}, l \neq j_p, k_p$), $p = 1, 2, \ldots, \mu$, in the representation (2), (3) of the Clifford algebra $Cl(\mathbf{R}^n)$ on $\mathbf{C}^{2^\mu}$. □

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