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ON LYAPUNOV EXPONENTS OF CONTINUOUS SCHRÖDINGER COCYCLES OVER IRRATIONAL ROTATIONS

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ABSTRACT. In this paper we consider continuous, $\mathrm{SL}(2,\mathbb{R})$-valued, Schrödinger cocycles over irrational rotations. We prove two generic results on the Lyapunov exponents which improve the corresponding ones contained in a paper by Bjerklöv, Damanik and Johnson.

1. Introduction

Let α be a fixed irrational number and $A : \mathbb{T} \mapsto \mathrm{SL}(2,\mathbb{R})$ be a continuous map. Then A generates a continuous, $\mathrm{SL}(2,\mathbb{R})$-valued cocycle $\{A(n,\theta)\}$ over the irrational rotations $\theta \mapsto \theta + \alpha$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (or a continuous, quasi-periodic, $\mathrm{SL}(2,\mathbb{R})$-valued cocycle with frequency α). More precisely, define

$$A(n,\theta) = \begin{cases} A(\theta + (n-1)\alpha) \ldots A(\theta), & n > 0, \\ Id, & n = 0, \\ A^{-1}(\theta - n\alpha) \ldots A^{-1}(\theta - \alpha), & n < 0. \end{cases}$$

It is clear that $\{A(n,\theta)\}$ satisfies the cocycle property:

$$A(n+m,\theta) = A(n,\theta + m\alpha)A(m,\theta), \;\; m,n \in \mathbb{Z}, \; \theta \in \mathbb{T}.$$

The cocycle admits a well-defined (maximal) Lyapunov exponent given by

$$\Lambda(A) := \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A(n,\theta)\| d\theta = \inf_{n\geq 1} \frac{1}{n} \int_{\mathbb{T}} \log \|A(n,\theta)\| d\theta;$$

i.e., the limit exists and is independent of θ. When $\Lambda(A) > 0$, the corresponding cocycle is said to be *uniformly hyperbolic* if

$$\lim_{n\to+\infty} \frac{1}{n} \log \|A(n,\theta)\| = \Lambda(A)$$

uniformly in θ and to be *non-uniformly hyperbolic* if otherwise.

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In this paper, we pay particular attention to continuous, quasi-periodic, $\mathrm{SL}(2,\mathbb{R})$-valued, Schrödinger cocycles with fixed irrational frequency α, i.e., a family $\{A_{f,E}(n,\theta) : E \in \mathbb{R}, f \in C(\mathbb{T})\}$ of quasi-periodic, $\mathrm{SL}(2,\mathbb{R})$-valued cocycles with the frequency α which is generated by the continuous, $\mathrm{SL}(2,\mathbb{R})$-valued functions

$$A_{f,E}(\theta) = \begin{pmatrix} E - f(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Such cocycles are referred to as *Schrödinger cocycles* because they arise and play important roles in the study of the spectral problem of the discrete quasi-periodic Schrödinger operator,

$$[H_f\psi](n) = (\Delta + f(\theta + (n-1)\alpha))\psi(n) = E\psi(n), \tag{1.1}$$

where $\Delta\psi(n) = \psi(n+1) + \psi(n-1)$. For simplicity, we denote $\Lambda_f(E) =: \Lambda(A_{f,E})$, $A_f(n,\cdot) =: A_{f,0}(n,\cdot)$, and $\Lambda_f =: \Lambda_f(0)$.

Related to the spectral problem especially with respect to the non-existence of an absolutely continuous spectrum, one often considers, for a fixed f, a two-parameter family $\{A_{\lambda f,E}(n,\theta)\}$ of Schrödinger cocycles, and studies the positivity of the Lyapunov exponents $\Lambda_{\lambda f}(E)$ for λ sufficiently large. In particular, when α satisfies appropriate Diophantine conditions, for a certain class of smooth f, it is known that $\Lambda_{\lambda f}(E)$ is of scale of $\log\lambda$ as $\lambda \gg 1$ uniformly in E (see, e.g., [2, 6, 11, 12, 16]). However, in a recent work of Bjerklöv, Damanik, and Johnson [3] such uniform bounds are shown to be extremely unstable within the class of continuous functions. More precisely, it is shown in [3] that for every countable set $\{\lambda_m\}_{m=1}^{\infty} \subset (0,+\infty)$, there exists a residual set of $f \in C(\mathbb{T})$ for which $\inf_{E\in\mathbb{R}} \Lambda_{\lambda_m f}(E) = 0$ for each $m \in \mathbb{N}$.

In this paper, we will show that this result can be improved as follows.

Theorem 1. *For a residual set of $f \in C(\mathbb{T})$,*

$$\inf_{E\in\mathbb{R}} \Lambda_{\lambda f}(E) = 0$$

for any $\lambda > 0$.

For general quasi-periodic, continuous, $\mathrm{SL}(2,R)$-valued cocycles, it is shown in [4] that there is a residual set $\mathcal{R} \subset C(\mathbb{T}, SL(2,\mathbb{R}))$ such that for $A \in \mathcal{R}$, either A is uniformly hyperbolic or $\Lambda(A) = 0$ (see [9, 10] for similar results that hold for a generic set of pairs (α, f); see also [1]). The same is also shown to hold for Schrödinger cocycles with $E = 0$ ([3, 5]).

Our next result proves the same phenomenon for the parametrized Schrödinger cocycles with $E = 0$.

Theorem 2. *The set*

$$\{f \in C(\mathbb{T}) : A_{\lambda f}(n,\cdot) \text{ is uniformly hyperbolic or } \Lambda_{\lambda f} = 0 \text{ for any } \lambda \in (0,\infty)\}$$

is residual.

The rest of this paper is devoted to the proof of Theorems 1 and 2. Our proofs essentially follow the approaches of [3] with necessary modifications.

2. Proofs of theorems

Throughout the rest of the paper, we let α be a fixed irrational number. For a Schrödinger operator H_f of the form (1.1) with $\theta \in \mathbb{T}$ and $f \in L^1(\mathbb{T})$, it is well known that the spectrum $\sigma(H_f)$ is independent of $\theta \in \mathbb{T}$ almost everywhere, and if $f \in C(\mathbb{T})$, then $\sigma(H_f)$ is completely independent of θ. Uniform and non-uniform hyperbolicities of the corresponding (measurable) Schrödinger cocycles $A_{f,E}(n,\cdot)$ can be defined similarly to the continuous case.

As in [3], the following result will play an important role in the proofs of the theorems.

Theorem 2.1. *Suppose $f : \mathbb{T} \mapsto \mathbb{R}$ is of the form*

$$f(\theta) = \sum_{m=1}^{M} f_m \chi_{[\beta_{m-1},\beta_m)}(\theta), \tag{2.1}$$

where $0 = \beta_0 < \beta_1 < \cdots < \beta_M = 1$ are rational numbers and $f_1, \ldots, f_M$ are real. Then $\sigma(H_f) = \{E : \Lambda_f(E) = 0\}$.

Proof. See [7, 8]. □

A crucial step in proving the above result is to show that for any f having the form (2.1),

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_{f,E}(n,\theta)\| = \Lambda_f(E)$$

for every $E \in \mathbb{R}$ uniformly in $\theta \in \mathbb{T}$ ([7, 8, 14]). The result then follows from the following.

Theorem 2.2. *For any $f \in L^1(\mathbb{T})$,*

$$\sigma(H_f) = \{E : \Lambda_f(E) = 0 \text{ or } A_{f,E}(n,\cdot) \text{ is non-uniformly hyperbolic}\}.$$

Proof. See [15, 13]. □

Lemma 2.3. *For any non-empty compact subset $K \subseteq (0,+\infty)$, the set*

$$M_{K,0} := \{f \in C(\mathbb{T}) : \inf_{E\in\mathbb{R}} \Lambda_{\lambda f}(E) = 0 \text{ for any } \lambda \in K\}$$

is residual in $C(\mathbb{T})$.

Proof. Let $K \subseteq (0,+\infty)$ be a non-empty compact subset. We consider the family of sets

$$M_{K,\delta} = \{f \in C(\mathbb{T}) : \forall\lambda \in K\ \exists E_\lambda \in \mathbb{R} \text{ such that } \Lambda_{\lambda f}(E_\lambda) < \delta\},\ \ \delta > 0.$$

We will show that each $M_{K,\delta}$ is open and dense, and hence $M_{K,0} = \bigcap_{\delta>0} M_{K,\delta}$ is residual.

First we show that $M_{K,\delta}$ is open, i.e., $C(\mathbb{T}) \setminus M_{K,\delta}$ is closed. Let $\{f_n\} \subset C(\mathbb{T}) \setminus M_{K,\delta}$, $f \in C(\mathbb{T})$ be such that $\|f_n - f\|_\infty \to 0$. Then for each $n \in \mathbb{N}$ there exists a $\lambda_n \in K$ with $\Lambda_{\lambda_n f_n}(E) \geq \delta$ for all $E \in \mathbb{R}$. Since K is compact, there exists a subsequence $\{n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ such that $\lim_{i\to\infty} \lambda_{n_i} = \lambda_0$ for some $\lambda_0 \in K$. It follows from the upper-semicontinuity of Lyapunov exponents $\Lambda_{\lambda f}(E)$ in λ that

$$\Lambda_{\lambda_0 f}(E) \geq \limsup_{i\to\infty} \Lambda_{\lambda_{n_i} f_{n_i}}(E) \geq \delta$$

for any $E \in \mathbb{R}$. Hence $f \in C(\mathbb{T}) \setminus M_{K,\delta}$. This shows that $C(\mathbb{T}) \setminus M_{K,\delta}$ is closed.

Next we show that $M_{K,\delta}$ is dense. Let $\epsilon > 0$ and $g \in C(\mathbb{T})$ be given. In the $\frac{\epsilon}{2}$-neighborhood of g with respect to the L^∞ topology, we choose a step function s of the form (2.1); i.e., s has finitely many points of discontinuity, all of which are rational, and the jumps of s are bounded by $\frac{\epsilon}{2}$. It then follows from Theorem 2.1 that for any $\lambda \in K$, $\Lambda_{\lambda s}$ vanishes on the spectrum $\sigma(H_{\lambda s})$ of $H_{\lambda s}$; i.e., there exists an $E_\lambda \in \sigma(H_{\alpha,\lambda s})$ such that $\Lambda_{\lambda s}(E_\lambda) = 0$. By the upper-semicontinuity of Lyapunov exponents, there exists a $\delta_\lambda > 0$, for each $\lambda \in K$, such that $\Lambda_{us}(E_\lambda) < \delta$ for any $u \in B(\lambda, \delta_\lambda) := \{t \in \mathbb{R} : |t - \lambda| < \delta_\lambda\}$. As K is compact, there exist $u_1, \dots, u_\ell \in K$ such that $K \subseteq \bigcup_{i=1}^{\ell} B(u_i, \frac{\delta_{u_i}}{2})$. Then

$$\Lambda_{\lambda s}(E_{u_i}) < \delta \text{ for all } 1 \le i \le \ell \text{ and } \lambda \in B(u_i, \delta_{u_i}) \cap K.$$

Let $\{f_n\} \subset C(\mathbb{T})$ be such that $\int_{\mathbb{T}} |s(\theta) - f_n(\theta)| d\theta < \frac{1}{n}$ and $\|s - f_n\|_\infty < \frac{\epsilon}{2}$ for all $n \in \mathbb{N}$. We claim that there exists an $n_* \in \mathbb{N}$ such that $\Lambda_{\lambda f_{n_*}}(E_{u_i}) < \delta$ for all $1 \le i \le \ell$ and $\lambda \in B(u_i, \frac{\delta_{u_i}}{2}) \cap K$; i.e., $f =: f_{n_*}$ has the desired properties that $f \in M_{K,\delta}$ and $\|f - g\|_\infty < \epsilon$.

If the claim is not true, then for each $n \in \mathbb{N}$ there exist $i_n \in \{1, 2, \dots, \ell\}$ and $\lambda_n \in B(u_{i_n}, \frac{\delta_{u_{i_n}}}{2}) \cap K$ such that $\Lambda_{\lambda_n f_n}(E_{u_{i_n}}) \ge \delta$. Without loss of generality, we assume that $\lim_{n\to\infty} \lambda_n = \lambda_0$ for some $\lambda_0 \in K$ and $i_n \equiv i_0 \in \{1, 2, \dots, \ell\}$ for all $n \in \mathbb{N}$. It is clear that $\lambda_0 \in B(u_{i_0}, \delta_{u_{i_0}}) \cap K$ and $\lim_{n\to\infty} \int_{\mathbb{T}} |\lambda_0 s(\theta) - \lambda_n f_n(\theta)| d\theta = 0$. Hence by the upper semi-continuity of Lyapunov exponents, we have

$$\delta > \Lambda_{\lambda_0 s}(E_{u_{i_0}}) \ge \limsup_{n\to\infty} \Lambda_{\lambda_n f_n}(E_{u_{i_0}}) \ge \delta,$$

a contradiction. □

Proof of Theorem 1. Let $K_n = [\frac{1}{n}, n]$, $n \in \mathbb{N}$. Then by Lemma 2.3,

$$\{f \in C(\mathbb{T}) : \inf_{E \in \mathbb{R}} \Lambda_{\lambda f}(E) = 0 \text{ for any } \lambda > 0\} = \bigcap_{n=1}^{\infty} M_{K_n,0}$$

is residual. □

Proof of Theorem 2. It is sufficient to show that for any non-empty compact set $K \subseteq (0, \infty)$, the set

$$N_K = \{f \in C(\mathbb{T}) : \exists\, \lambda \in K \text{ s.t. } A_{\lambda f}(n, \cdot) \text{ is non-uniformly hyperbolic}\}$$

is a meagre set, i.e., a countable union of nowhere-dense sets. This will follow once we prove that

$$N_{K,\gamma} = \{f \in C(\mathbb{T}) : \exists\, \lambda \in K \text{ s.t. } A_{\lambda f}(n, \cdot) \text{ is non-uniformly hyperbolic and } \Lambda_{\lambda f} \ge \gamma\}$$

is nowhere dense for every $\gamma > 0$.

Let $\gamma > 0$ be given. We first show that $N_{K,\gamma}$ is closed. Let $\{f_i\} \subset N_{K,\gamma}$ and $f_0 \in C(\mathbb{T})$ be such that $\lim_{i\to\infty} \|f_i - f_0\|_\infty = 0$. Then for each $i \in \mathbb{N}$, there exists a $\lambda_i \in K$ such that $A_{\lambda_i f_i}(n, \cdot)$ is non-uniformly hyperbolic and $\Lambda_{\lambda_i f_i} \ge \gamma$. Without loss of generality, we assume that $\lim_{i\to\infty} \lambda_i = \lambda_0$ for some $\lambda_0 \in K$. Then $\lim_{i\to\infty} \|\lambda_i f_i - \lambda_0 f_0\|_\infty = 0$, and hence $\Lambda_{\lambda_0 f_0} \ge \limsup_{i\to\infty} \Lambda_{\lambda_i f_i} \ge \gamma$ according to the upper semi-continuity of Lyapunov exponents. Since uniform hyperbolicity is an open property, $A_{\lambda_0 f_0}(n, \cdot)$ is non-uniformly hyperbolic. This shows that $f_0 \in N_{K,\gamma}$. Hence $N_{K,\gamma}$ is closed.

Next we show that $N_{K,\gamma}$ has no interior. This amounts to showing that for any given $f \in N_{K,\gamma}$ and $\epsilon > 0$ there exists a function $g \in C(\mathbb{T})$ such that $\|f - g\|_\infty < \epsilon$ and $g \notin N_{K,\gamma}$. For the given $f \in N_{K,\gamma}$, we let $\lambda_* \in K$ be such that $A_{\lambda_* f}(n, \cdot)$ is non-uniformly hyperbolic and $\Lambda_{\lambda_* f} \geq \gamma$. Also let $\{s_m\}$ be a sequence of step functions of the form (2.1) in the $\frac{\epsilon}{4}$-neighborhood of f that converge to f in the L^∞ topology. Then for each $\theta \in \mathbb{T}$, the operators $H_m = \triangle + \lambda_* s_m(\cdot\alpha + \theta)$ converge strongly to $H = \triangle + \lambda_* f(\cdot\alpha + \theta)$. Since $A_{\lambda_* f}(n, \cdot)$ is non-uniformly hyperbolic, we have by Theorem 2.2 that $0 \in \sigma(H)$. By the strong convergence of H_m, we also have a sequence $\{E_m\} \subset \sigma(H_m)$ such that $E_m \to 0$. Now let $m \gg 1$ be fixed such that $|E_m| < \frac{\epsilon}{4}$. Then $s = s_m - E_m$ is a step function of the form (2.1) in the $\frac{\epsilon}{2}$-neighborhood of f such that 0 belongs to the spectrum of $\overline{H} = \triangle + \lambda_* s(\cdot\alpha + \theta)$. It follows from Theorem 2.1 that $\Lambda_{\lambda_* s} = 0$. Now consider a sequence of continuous functions $\{g_i\} \subset C(\mathbb{T})$ with $\int_{\mathbb{T}} |s(\theta) - g_i(\theta)| d\theta < \frac{1}{i}$ and $\|s - g_i\|_\infty < \frac{\epsilon}{2}$ for all $i \in \mathbb{N}$. We have by the upper semi-continuity of Lyapunov exponents that

$$0 = \Lambda_{\lambda_* s} \geq \lim_{i \to \infty} \Lambda_{\lambda_* g_i}.$$

Hence we can choose a $k \in \mathbb{N}$ such that the function $g = g_k$ has the desired properties that $\Lambda_{\lambda_* g} < \gamma$ and $\|f - g\|_\infty < \epsilon$. □

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