A NECESSARY AND SUFFICIENT CONDITION FOR RICCI SHRINKERS TO HAVE POSITIVE AVR

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(Communicated by Jianguo Cao)

Abstract. In this short paper we observe that a recent result of C.-W. Chen meshes well with earlier work of H.-D. Cao and D.-T. Zhou, O. Munteanu, J. Carrillo and L. Ni, and S.-J. Zhang. We give a necessary and sufficient condition for complete noncompact shrinking gradient Ricci solitons to have positive asymptotic volume ratio.

Let \((\mathcal{M}^n, g, f)\) denote a complete shrinking gradient Ricci soliton (shrinker for short) with \(R_{ij} + \nabla_i \nabla_j f - \frac{1}{2} g_{ij} = 0\). Throughout we shall assume that \(f\) is the normalized potential function in the sense that \(R + |\nabla f|^2 - f = 0\) holds on \(\mathcal{M}\).

It was proved by B.-L. Chen \[3\] that complete ancient solutions to the Ricci flow, and in particular shrinkers, must have nonnegative scalar curvature. As a consequence, the potential function \(f\) satisfies the estimate:

\[
0 \leq f(x) \leq \left( \frac{1}{2} r(x) + f(O)^{\frac{1}{2}} \right)^2,
\]

where \(r(x)\) denotes the distance function to a fixed point \(O\) in \(\mathcal{M}\). H.-D. Cao and Detang Zhou \[1\] proved that there exists a positive constant \(C\) which depends on the dimension \(n\), \(\sup_{y \in B(O,1)}|\nabla f|(y)\), and the minimum of the Ricci curvature \(R_{cg}\) in the ball \(B(O,1)\) such that \(f\) satisfies the lower estimate

\[
f(x) \geq \frac{1}{4} (r(x) - C)^2
\]

for \(x \in \mathcal{M} - B(O, C)\) (see Fang, Man, and Zhang \[5\] for related estimates). In fact, carefully following the proof of \[1\] and integrating by parts yield:

\[
f(x) \geq \frac{1}{4} \left[ (r(x) - 4n - 2f(O)^{\frac{1}{2}} + \frac{4}{3} c_x)^+ \right]^2,
\]

where \(c_x \doteq \max(c, 0)\). Recently Haslhofer and Müller \[6\] further observed that if the reference point \(O\) is chosen to be a global minimum point of \(f\) (its existence is ensured by \[1\] and \[2\]), then one obtains improved estimates with constants depending only on \(n\):

\[
\frac{1}{4} [(r(x) - 5n)^+]^2 \leq f(x) \leq \frac{1}{4} (r(x) + \sqrt{2n})^2.
\]
Define the functions

\[ V : \mathbb{R} \to [0, \infty), \quad R : \mathbb{R} \to [0, \infty) \]

by

\[ V(c) \triangleq \int_{\{f < c\}} d\mu, \quad R(c) \triangleq \int_{\{f < c\}} R d\mu. \]

In [1], the following ODE relating \( V(c) \) and \( R(c) \) was established:

\[ 0 \leq \frac{n}{2} V(c) - R(c) = c V'(c) - R'(c). \]

Recall that the asymptotic volume ratio (AVR) of a complete noncompact Riemannian manifold \((N^n, h)\) is defined by

\[ \text{AVR}(h) \triangleq \lim_{r \to \infty} \frac{\text{Vol} B(p, r)}{\omega_n r^n} \]

if the limit exists, where \( B(p, r) \) denotes the geodesic ball in \( N \) with center \( p \) and radius \( r \) and where \( \omega_n \) is the volume of the unit Euclidean \( n \)-ball. It is easy to check that the AVR\((h)\) is independent of the choice of \( p \). Moreover, if \( h \) has nonnegative Ricci curvature, then this limit exists by the Bishop–Gromov volume comparison theorem.


**Theorem 1.** Any complete noncompact shrinking gradient Ricci soliton must have at most Euclidean volume growth; i.e., \( \limsup_{r \to \infty} \frac{\text{Vol} B(O, r)}{\omega_n r^n} \) is finite.

Note that an earlier result by Carrillo and Ni [2] states that any nonflat shrinker with nonnegative Ricci curvature must have zero AVR. Based on Cao and Zhou’s work, Zhang [8] proved a sharp upper bound on the volume growth of shrinkers under the assumption that \( R \geq \delta \) for some constant \( \delta > 0 \). More recently, C.-W. Chen [4] proved that the AVR of a shrinker is bounded from below by some \( c > 0 \) if the average scalar curvature satisfies \( \frac{1}{\text{Vol} B(O, r)} \int_{B(O, r)} R d\mu \leq r^\alpha \), where \( \alpha \) is a negative constant (see also [1] for a similar result in the case where \( \alpha = 0 \)).

We observe that the results in [2], [1], [7], [8], and [4] lead to a necessary and sufficient condition for noncompact shrinkers to have positive AVR.

**Theorem 2.** Let \((M^n, g, f)\) be a complete noncompact shrinking gradient Ricci soliton. Then \( \text{AVR}(g) \) exists and is finite (by [6], it is bounded by a constant depending only on \( n \)). Moreover, \( \text{AVR}(g) > 0 \) if and only if \( \int_{n+2}^{\infty} \frac{R(c)}{c V(c)} dc < \infty \).

**Proof.** Let \( P(c) \triangleq \frac{V(c)}{c^\frac{n}{2}} - \frac{R(c)}{c^{\frac{n+2}{2}}} \) and \( N(c) \triangleq \frac{R(c)}{c V(c)} \). Note that \( \frac{R(c)}{V(c)} \) is the average scalar curvature over the set \( \{f < c\} \). The ODE (5) implies that

\[ P'(c) = -\left(1 - \frac{n + 2}{2c}\right) \frac{R(c)}{c^{\frac{n+2}{2}}} = -\left(1 - \frac{n + 2}{2c}\right) \frac{N(c)}{1 - N(c)} \cdot P(c). \]

Since \( 0 \leq R(c) \leq \frac{n}{2} V(c) \) by (5), we have

\[ \left(1 - \frac{n}{2c}\right) \frac{V(c)}{c^\frac{n}{2}} \leq P(c) \leq \frac{V(c)}{c^\frac{n}{2}}. \]
Hence, by the bounds (1) and (2) for $f$,
\[2^n \omega_n \text{AVR}(g) = \lim_{c \to \infty} \frac{V(c)}{e^{n/2}} = \lim_{c \to \infty} P(c),\]
which exists by (7).

Integrating (7) yields
\[(9) \quad P(c) = P(n + 2) e^{-\int_{n+2}^{c} \frac{(1 - n + 2)}{1 - N(c)} N(c) \, dc}\]
for $c \geq n + 2$. From $\frac{R(c)}{V(c)} \leq \frac{2}{n}$ it is easy to see that for any $c \in [n + 2, \infty)$ we have
\[(10) \quad \frac{1}{2} \int_{n+2}^{c} N(c) \, dc \leq \int_{n+2}^{c} \left(1 - \frac{n + 2}{2c}\right) \frac{N(c)}{1 - N(c)} \, dc \leq 2 \int_{n+2}^{c} N(c) \, dc.\]

If $\int_{n+2}^{\infty} N(c) \, dc = \infty$, then by (10) we have $\text{AVR}(g) = \frac{1}{2^n \omega_n} \lim_{c \to \infty} P(c) = 0$.

If $\int_{n+2}^{\infty} N(c) \, dc < \infty$, then by (9) and (10), we have
\[P(c) \geq P(n + 2) e^{-2 \int_{n+2}^{\infty} N(c) \, dc} > 0.\]
Hence $\text{AVR}(g) > 0$. □

Acknowledgment

We would like to thank Lei Ni for his interest and encouragement.

References


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