HOLOMORPHIC FAMILIES OF LONG $\mathbb{C}^2$'S

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Abstract. We construct a holomorphically varying family of complex surfaces $X_s$, parametrized by the points $s$ in any Stein manifold, such that every $X_s$ is a long $\mathbb{C}^2$ which is biholomorphic to $\mathbb{C}^2$ for some but not all values of $s$.

1. The main result

A complex manifold $X$ of dimension $n$ is a long $\mathbb{C}^n$ if $X = \bigcup_{j=1}^{\infty} X^j$, where $X^1 \subset X^2 \subset X^3 \subset \ldots$ is an increasing sequence of open domains exhausting $X$ such that each $X^j$ is biholomorphic to $\mathbb{C}^n$. Clearly every long $\mathbb{C}$ is biholomorphic to $\mathbb{C}$. On the other hand, for every $n > 1$ there exists a long $\mathbb{C}^n$ which is not a Stein manifold, and in particular is not biholomorphic to $\mathbb{C}^n$. Such manifolds have been constructed recently by E. F. Wold [12] using his example of a non-Runge Fatou-Bieberbach domain in $\mathbb{C}^2$ [11], thereby solving a problem posed by J. E. Fornæss [3]. Previously, Fornæss [2] used Wermer’s example of a non-Runge embedded polydisc in $\mathbb{C}^3$ [10] to construct for every $n \geq 3$ an $n$-dimensional non-Stein complex manifold that is exhausted by biholomorphic images of the polydisc.

Recently L. Meersseman asked in a private communication whether it is possible to holomorphically deform the standard $\mathbb{C}^n$ to a long $\mathbb{C}^n$ that is not biholomorphic to $\mathbb{C}^n$. This question arose naturally in certain problems concerning deformations of foliations that he had been considering. Here we give a positive answer and show that the behavior of long $\mathbb{C}^n$’s in a holomorphic family can be rather chaotic.

Theorem 1.1. Fix an integer $n > 1$. Assume that $S$ is a Stein manifold, $A = \bigcup_j A_j$ is a finite or countable union of closed complex subvarieties of $S$, and $B = \{b_j\}$ is a countable set in $S \setminus A$. Then there exists a complex manifold $X$ and a holomorphic submersion $\pi: X \to S$ onto $S$ such that

(i) the fiber $X_s = \pi^{-1}(s)$ is a long $\mathbb{C}^n$ for every $s \in S$,
(ii) $X_s$ is biholomorphic to $\mathbb{C}^n$ for every $s \in A$, and
(iii) $X_s$ is non-Stein for every $s \in B$.

In particular, for any two disjoint countable sets $A, B \subset \mathbb{C}$ there is a holomorphic family $\{X_s\}_{s \in \mathbb{C}}$ of long $\mathbb{C}^2$’s such that $X_s$ is biholomorphic to $\mathbb{C}^2$ for all $s \in A$ and is non-Stein for all $s \in B$. This is particularly striking if the sets $A$ and $B$ are chosen to be everywhere dense in $\mathbb{C}$.
The conclusion of Theorem 1.1 can be strengthened by adding to the set $B$ a closed complex subvariety of $X$ contained in $X\setminus A$. We do not know whether the same holds if $B$ is a countable union of subvarieties of $X$.

Several natural questions appear:

**Problem 1.2.** Given a holomorphic family $\{X_s\}_{s \in S}$ of long $\mathbb{C}^n$'s for some $n > 1$, what can be said about the set of points $s \in S$ for which the fiber $X_s$ is (or is not) biholomorphic to $\mathbb{C}^n$? Are these sets necessarily a $G_\delta$, an $F_\sigma$, of the first, resp. of the second category, etc.?

A more ambitious project would be to answer the following question:

**Problem 1.3.** Is there a holomorphic family $X_s$ of long $\mathbb{C}^2$'s, parametrized by the disc $\mathbb{D} = \{s \in \mathbb{C} : |s| < 1\}$ or the plane $\mathbb{C}$, such that $X_s$ is not biholomorphic to $X_s'$ whenever $s \neq s'$?

We do not know of any criteria to distinguish two long $\mathbb{C}^n$'s from each other, except if one of them is the standard $\mathbb{C}^n$ and the other one is non-Stein. Apparently there is no known example of a Stein long $\mathbb{C}^n$ other than $\mathbb{C}^n$. It is easily seen that any two long $\mathbb{C}^n$'s are smoothly diffeomorphic to each other, so the gauge-theoretic methods do not apply.

To prove Theorem 1.1 we follow Wold’s construction of a non-Stein long $\mathbb{C}^2$ [12], but doing all the key steps with families of Fatou-Bieberbach maps depending holomorphically on the parameter in a given Stein manifold $S$. (The same proof applies for any $n \geq 2$.) By using the Andersén-Lempert theory [11] [5] [9] we insure that in a holomorphically varying family of injective holomorphic maps $\phi_s : \mathbb{C}^2 \to \mathbb{C}^2$ $(s \in S)$ the image domain $\phi_s(\mathbb{C}^2)$ is Runge for some but not all values of the parameter. In the limit manifold $X$ we thus get fibers $X_s$ that are biholomorphic to $\mathbb{C}^2$, as well as fibers that are not holomorphically convex, and hence non-Stein.

2. Constructing holomorphic families of long $\mathbb{C}^n$'s

Let $S$ be a complex manifold that will be used as the parameter space. We recall how one constructs a complex manifold $X$ and a holomorphic submersion $\pi : X \to S$ such that the fiber $X_s = \pi^{-1}(s)$ is a long $\mathbb{C}^n$ for each $s \in S$. (This is a parametric version of the construction in [2] or [12, §2].)

Assume that we have a sequence of injective holomorphic maps

\begin{equation}
\Phi^k : X^k = S \times \mathbb{C}^n \to X^{k+1} = S \times \mathbb{C}^n, \quad \Phi^k(s, z) = (s, \phi^k_s(z)),
\end{equation}

where $s \in S$, $z \in \mathbb{C}^n$, and $k = 1, 2, \ldots$. Set $\Omega^k = \Phi^k(X^k) \subset X^{k+1}$. Thus for every fixed $k \in \mathbb{N}$ and $s \in S$ the map $\phi^k_s : \mathbb{C}^n \to \mathbb{C}^n$ is biholomorphic onto its image $\phi^k_s(\mathbb{C}^n) = \Omega^k_s \subset \mathbb{C}^n$ and it depends holomorphically on the parameter $s \in S$. In particular, if $\Omega^k_s$ is a proper subdomain of $\mathbb{C}^n$, then $\phi^k_s$ is a Fatou-Bieberbach map. Let $X$ be the disjoint union of all $X^k$ for $k \in \mathbb{N}$ modulo the following equivalence relation. A point $x \in X^i$ is equivalent to a point $x' \in X^k$ if and only if one of the following hold:

(a) $i = k$ and $x = x'$,
(b) $k > i$ and $\Phi^{k-1} \circ \cdots \circ \Phi^i(x) = x'$, or
(c) $i > k$ and $\Phi^{i-1} \circ \cdots \circ \Phi^k(x') = x$. 
For each $k \in \mathbb{N}$ we have an injective map $\Psi^k: X^k \hookrightarrow X$ onto the subset $\tilde{X}^k = \Psi^k(X^k) \subset X$ which sends any point $x \in X^k$ to its equivalence class $[x] \in X$. Denoting by $\iota^k: \tilde{X}^k \hookrightarrow \tilde{X}^{k+1}$ the inclusion map, we have
\begin{equation}
\iota^k \circ \Psi^k = \Psi^{k+1} \circ \Phi^k, \quad k = 1, 2, \ldots.
\end{equation}
The inverse maps $(\Psi^k)^{-1}: \tilde{X}^k \xrightarrow{\sim} X^k = S \times \mathbb{C}^n$ provide local charts on $X$. It is easily verified that this endows $X$ with the structure of a Hausdorff, second countable complex manifold. Since each of the maps $\Phi^k$ respects the fibers over $S$, we also get a natural projection $\pi: X \to S$ which is clearly a submersion. For every $s \in S$ the fiber $X_s$ is the increasing union of open subsets $\tilde{X}^k_s$ biholomorphic to $\mathbb{C}^n$. Observe that we get the same limit manifold $X$ by starting with any term of the sequence \((\Psi^k)^{-1}\).

The next lemma follows from the Andersén-Lempert theory [1]; cf. [12, Theorem 1.2].

**Lemma 2.1.** Let $\pi: X \to S$ be as above. Assume that for some $s \in S$ there exists an integer $k_s \in \mathbb{N}$ such that for every $k \geq k_s$, the domain $\Omega^k_s = \phi^k_s(\mathbb{C}^n) \subset \mathbb{C}^n$ is Runge in $\mathbb{C}^n$. Then $X_s$ is biholomorphic to $\mathbb{C}^n$.

**Proof.** The main point is that any biholomorphic map $\mathbb{C}^n \xrightarrow{\sim} \Omega$ onto a Runge domain $\Omega \subset \mathbb{C}^n$ can be approximated, uniformly on compact sets, by holomorphic automorphisms of $\mathbb{C}^n$. This observation allows one to renormalize the sequence of biholomorphisms $(\Psi^k_s)^{-1}: \tilde{X}^k_s \xrightarrow{\sim} \mathbb{C}^n$ for $k \geq k_s$ so that the new sequence converges uniformly on compact sets in $X_s$ to a biholomorphic map $X_s \xrightarrow{\sim} \mathbb{C}^n$; we leave out the straightforward details. \(\square\)

## 3. Entire Families of Holomorphic Automorphisms

Let $\mathfrak{N}_O(X)$ denote the complex Lie algebra of all holomorphic vector fields on a complex manifold $X$.

A vector field $V \in \mathfrak{N}_O(X)$ is said to be $\mathbb{C}$-complete, or completely integrable, if its flow $\{\phi_t\}_{t \in \mathbb{C}}$ exists for all complex values $t \in \mathbb{C}$, starting at an arbitrary point $x \in X$. Thus $\{\phi_t\}_{t \in \mathbb{C}}$ is a complex one-parameter subgroup of the holomorphic automorphism group $\text{Aut} \ X$. The manifold $X$ is said to enjoy the (holomorphic) density property if the Lie subalgebra $\text{Lie}(X)$ of $\mathfrak{N}_O(X)$, generated by the $\mathbb{C}$-complete holomorphic vector fields, is dense in $\mathfrak{N}_O(X)$ in the topology of uniform convergence on compact sets in $X$ (see Varolin [3, 9]). More generally, a complex Lie subalgebra $\mathfrak{g}$ of $\mathfrak{N}_O(X)$ enjoys the density property if $\mathfrak{g}$ is densely generated by the $\mathbb{C}$-complete vector fields that it contains. This property is very restrictive on open manifolds. The main result of the Andersén-Lempert theory [1] is that $\mathbb{C}^n$ for $n > 1$ enjoys the density property; in fact, every polynomial vector field on $\mathbb{C}^n$ is a finite sum of complete polynomial vector fields (the shear fields).

Varolin proved [8] that any domain of the form $(\mathbb{C}^*)^k \times \mathbb{C}^l$ with $k + l \geq 2$ and $l \geq 1$ enjoys the density property; we shall need this for the manifold $\mathbb{C}^* \times \mathbb{C}$. (Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)

**Lemma 3.1.** Assume that $X$ is a Stein manifold with the density property. Choose a distance function $\text{dist}_X$ on $X$. Let $\psi_1, \ldots, \psi_k \in \text{Aut} \ X$ be such that for each $j = 1, \ldots, k$ there exists a $C^2$ path $\theta_{j,t} \in \text{Aut} \ X$ ($t \in [0, 1]$) with $\theta_{j,0} = \text{Id}_X$ and $\theta_{j,1} = \psi_j$. Given distinct points $a_1, \ldots, a_k \in \mathbb{C}^*$, a compact set $K \subset X$ and a
number \( \epsilon > 0 \), there exists a holomorphic map \( \Psi: \mathbb{C} \times X \to X \) satisfying the following properties:

(i) \( \Psi_\zeta = \Psi(\zeta, \cdot) \in \text{Aut} X \) for all \( \zeta \in \mathbb{C} \),
(ii) \( \Psi_0 = \text{Id}_X \),
(iii) \( \sup_{x \in K} \dist_X (\Psi(a_j, x), \psi_j(x)) < \epsilon \) for \( j = 1, \ldots, k \).

A holomorphic map \( \Psi \) satisfying property (i) will be called an entire curve of holomorphic automorphisms of \( X \). Here \( \text{Id}_X \) denotes the identity on \( X \).

Proof. Consider a \( C^2 \) path \([0,1] \ni t \mapsto \gamma_t \in \text{Aut} X \). Pick a Stein Runge domain \( U \subset X \) containing the set \( K \). Then \( U_t = \gamma_t(U) \subset X \) is Runge in \( X \) for all \( t \in [0,1] \).

By \([1] \) or, more explicitly, by (the proof of) \([4 \text{, Theorem 1.1}] \) there exist finitely many complete holomorphic vector fields \( V_1, \ldots, V_m \) on \( X \), with flows \( \theta_{t_j} \), and numbers \( c_1 > 0, \ldots, c_m > 0 \) such that the composition \( \theta_{c_1} \circ \cdots \circ \theta_{c_1} \in \text{Aut} X \) approximates the automorphism \( \psi = \gamma_1 \) within \( \epsilon \) on the set \( K \). (The proof in \([4 \text{, Theorem 1.1}] \) is written for \( X = \mathbb{C}^n \), but it applies in the general case stated here. We first approximate \( \gamma_t: U \to U_t \) by compositions of short time flows of globally defined holomorphic vector fields on \( X \); here we need the Runge property of the sets \( U_t \). Since \( X \) enjoys the density property, these vector fields can be approximated by Lie combinations (using sums and commutators) of complete holomorphic vector fields. This approximates \( \gamma_t \) for each \( t \in [0,1] \), uniformly on \( K \), by compositions of flows of complete holomorphic vector fields on \( X \).)

Consider \( t^j = (t_1, \ldots, t_m) \) as complex coordinates on \( \mathbb{C}^m \). The map

\[
(\mathbb{C}^m \ni (t_1, \ldots, t_m) \mapsto \Theta(t_1, \ldots, t_m) = \theta_{t_1} \circ \cdots \circ \theta_{t_m} \in \text{Aut} X
\]

is entire, its value at the origin \( 0 \in \mathbb{C}^m \) is \( \text{Id}_X \), and its value at the point \((c_1, \ldots, c_m)\) is an automorphism that is \( \epsilon \)-close to \( \psi = \gamma_1 \) on \( K \).

Using this argument we find for every \( j = 1, \ldots, k \) an integer \( m_j \in \mathbb{N} \) and an entire map \( \Theta_j: C^{m_j} \to \text{Aut} X \) such that \( \Theta_j(0) = \text{Id}_X \) and \( \Theta_j(c_{j_1}^{(1)}, \ldots, c_{j_m}^{(1)}) \) is \( \epsilon \)-close to \( \psi_j \) on \( K \) at some point \( c_j = (c_{j_1}^{(1)}, \ldots, c_{j_m}^{(1)}) \in C^{m_j} \). Let \( t = (t_1^j, \ldots, t_k^j) \) be the complex coordinates on \( C^M = C^{m_1} \oplus \cdots \oplus C^{m_k} \), where \( t^j = (t_1^j, \ldots, t_{m_j}^j) \in C^{m_j} \).

The composition

\[
C^M \ni t \mapsto \Theta(t_1^1, \ldots, t_k^k) = \Theta^1(t_k^k) \circ \cdots \circ \Theta^1(t_1^1) \in \text{Aut} X
\]

is an entire map satisfying \( \Theta(0) = \text{Id}_X \) such that \( \Theta(0, \ldots, 0, c^j_1, 0, \ldots, 0) \) is \( \epsilon \)-close to \( \psi_j \) on \( K \) for each \( j = 1, \ldots, k \).

Choose an entire map \( g: \mathbb{C} \to C^M \) with \( g(a_j) = (0, \ldots, c^j_1, \ldots, 0) \) for \( j = 1, \ldots, k \) and \( g(0) = 0 \). Then the map \( C \ni \zeta \mapsto \Psi(\zeta) = \Theta(g(\zeta)) \in \text{Aut} X \) satisfies the conclusion of the lemma. \( \square \)

4. Proof of Theorem \([11 \text{, §2}] \)

We shall need the following result from \([11 \text{, §2}] \). This construction is due to Stolzenberg \([6 \text{, see also } 7 \text{, pp. 392–396}] \).

**Lemma 4.1.** There exists a compact set \( Y \subset C^* \times C \) (a union \( Y = D_1 \cup D_2 \) of two embedded, disjoint, polynomially convex discs) such that

(i) \( Y \) is \( O(C^* \times C) \)-convex,
(ii) the polynomial hull \( \overline{Y} \) contains the origin \((0,0) \in C^2\), and
(iii) for any nonempty open set $U \subset \mathbb{C}^* \times \mathbb{C}$ there exists a holomorphic automorphism $\psi \in \text{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that $Y \subset \psi(U)$.

Property (iii) is [11, Lemma 3.1]: Since $\mathbb{C}^* \times \mathbb{C}$ enjoys the density property according to Varolin [8], the isotopy that shrinks each of the two discs $D_1, D_2 \subset Y$ to a point in $U$ can be approximated by an isotopy of automorphisms of $\mathbb{C}^* \times \mathbb{C}$ by using the methods in [4].

**Proof of Theorem** [1.1]. We give the proof for $n = 2$. Let $B = \{b_1, b_2, \ldots\}$ be as in the theorem. Choose a set $Y \subset \mathbb{C}^* \times \mathbb{C}$ satisfying Lemma 1.1. Pick a closed ball $K \subset \mathbb{C}^2$ (or any compact set with nonempty interior).

We shall inductively construct a sequence of injective holomorphic maps $\Phi^k: S \times \mathbb{C}^2 \hookrightarrow S \times \mathbb{C}^2$ ($k = 1, 2, \ldots$) of the form

$$\Phi^k(s, z) = (s, \phi^k_s(z)), \quad s \in S, \quad z \in \mathbb{C}^2,$$

such that, setting

$$K^k_s = \phi^k_s(K) \subset \mathbb{C}^2,$$

the following properties hold for all $k \in \mathbb{N}$:

(i) $\Omega^k := \Phi^k(S \times \mathbb{C}^2) \subset S \times (\mathbb{C}^* \times \mathbb{C})$,

(ii) the fiber $\Omega^k_s = \phi^k_s(C^2)$ is Runge in $\mathbb{C}^2$ for all $s \in A_1 \cup \cdots \cup A_k$, and

(iii) $Y \subset \text{Int} K^k_s$ for each $s \in \{b_1, \ldots, b_k\}$. In particular, the polynomial hull of the set $K^k_s$ contains the origin for every such $s$.

Suppose for the moment that we have such a sequence. Let $X$ denote the limit manifold and let $\Psi^k: X^k = S \times \mathbb{C}^2 \hookrightarrow X$ be the induced inclusions (see §2).

If $s \in \bigcup_k A_k = A$, then property (ii) insures, in view of Lemma 2.1, that the fiber $X_s$ is biholomorphic to $\mathbb{C}^2$.

Suppose now that $s = b_j$ for some $j \in \mathbb{N}$. Property (iii) shows that for every integer $k \geq j$ the polynomial hull of the set $K^k_s$ contains the origin of $\mathbb{C}^2$; in particular, $\tilde{X}^k_s$ is not contained in $\Omega^k_s \subset \mathbb{C}^* \times \mathbb{C}$. For the corresponding subsets of the limit manifold $X_s$ we get in view of (2.2) that

$$\Psi^{k+1}_s(K^k_s) \not\subset \tilde{X}^k_s, \quad k = j, j + 1, \ldots,$$

where the hull is with respect to the algebra of holomorphic functions on the domain $\tilde{X}^{k+1}_s$ in the fiber $X_s$.

Let $K_s = \Psi^1_s(K)$ denote the compact set in $X_s$ determined by $K$; note that $K_s \subset \tilde{X}^1_s$ and $K_s = \Psi^{k+1}_s(K^k_s)$ for any $k \in \mathbb{N}$ according to (2.2) and (4.1). The above display then gives

$$\widehat{(K_s)}_{\mathbb{O}(\tilde{X}^{k+1}_s)} \not\subset \tilde{X}^k_s, \quad k = 1, 2, \ldots.$$

Since $\tilde{X}^{k+1}_s$ is a domain in $X_s$, we trivially have $\widehat{(K_s)}_{\mathbb{O}(\tilde{X}^{k+1}_s)} \subset (K_s)_{\mathbb{O}(X_s)}$; hence the hull $(K_s)_{\mathbb{O}(X_s)}$ is not contained in $\tilde{X}^k_s$ for any $k \in \mathbb{N}$. As the domains $\tilde{X}^k_s$ exhaust $X_s$, this hull is noncompact. Hence $X_s$ is not holomorphically convex (and therefore not Stein) for any $s \in B$.

This proves Theorem 1.1 provided that we can find a sequence with the stated properties.

We begin with some initial choices of domains and maps. Pick a Fatou-Bieberbach map $\theta: \mathbb{C}^2 \xrightarrow{\cong} D \subset \mathbb{C}^* \times \mathbb{C}$ whose image $D = \theta(\mathbb{C}^2)$ is Runge in
we now construct the next map $\Phi_k(s, z) = (s, \phi^k_s(z))$. Lemma 4.1 furnishes an automorphism $\psi \in \text{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that $Y \subset \psi(\theta(U))$. By Lemma 3.1 there exists an entire curve of automorphisms $\Psi_\zeta \in \text{Aut}(\mathbb{C}^* \times \mathbb{C}) (\zeta \in \mathbb{C})$ such that $\Psi_0 = \text{Id}_{\mathbb{C}^* \times \mathbb{C}}$ and $\Psi_1$ approximates $\psi$ close enough on the compact set $\theta(K)$ so that $Y \subset \Psi_1(\theta(U))$. Hence $(0, 0) \in \hat{Y} \subset \Psi_1(\theta(K))$. Set

$$\phi^1_s(z) = \psi_{f_1(s)}(\theta(z)), \quad s \in S, \; z \in \mathbb{C}^2.$$ 

If $s \in A_1$, then $f_1(s) = 0$ and hence $\phi^1_s(z) = \Psi_0(\theta(z)) = \theta(z)$, so $\phi^1_s = \theta$. If $s = b_1$, then $f_1(s) = 1$ and hence $\phi^1_s = \Psi_1 \circ \theta$. Thus $Y \subset \phi^1_{b_1}(U)$ and the polynomial hull $\phi^1_{b_1}(K)$ contains the origin of $\mathbb{C}^2$. This gives the initial step.

Suppose that we have found maps $\Phi^1, \ldots, \Phi^k$ satisfying conditions (i)–(iii) above; we now construct the next map $\Phi^{k+1}$ in the sequence. Recall that $\phi^k_s : \mathbb{C}^2 \to \mathbb{C}^2$ is the map defined by (4.1). Set

$$U^k_s = (\theta \circ \phi^k_s)(U), \quad s \in S;$$

this is a nonempty open set contained in the compact set $\theta(K^k_s) \subset \mathbb{C}^* \times \mathbb{C}$. Lemma 4.1 gives for each $j = 1, \ldots, k+1$ an automorphism $\psi_j \in \text{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that $Y \subset \psi_j(U^k_s)$. By Lemma 3.1 there exists an entire curve of automorphisms $\Psi_\zeta \in \text{Aut}(\mathbb{C}^* \times \mathbb{C}) (\zeta \in \mathbb{C})$ such that $\Psi_0 = \text{Id}_{\mathbb{C}^* \times \mathbb{C}}$ and $\Psi_j$ approximates $\psi_j$ for every $j = 1, \ldots, k+1$. If the approximation is close enough on the compact set $\theta(K^k_{b_j})$, then $Y \subset (\Psi_j \circ \theta)(K^k_{b_j})$ and hence the origin $(0, 0) \in \mathbb{C}^2$ is contained in the polynomial hull of $(\Psi_j \circ \theta)(K^k_{b_j})$. Set

$$\phi^{k+1}_s(z) = \psi_{f_{k+1}(s)} \circ \theta(z), \quad s \in S, \; z \in \mathbb{C}^2.$$ 

If $s \in A_1 \cup \cdots \cup A_{k+1}$, then $f_{k+1}(s) = 0$ and hence $\phi^{k+1}_s = \theta$. If $s = b_j$ for some $j = 1, \ldots, k+1$, then $f_{k+1}(b_j) = j$ and hence $\phi^{k+1}_s = \Psi_j \circ \theta$; therefore the polynomial hull of the set $\phi^{k+1}_s(K^k_{b_j})$ contains the origin. Taking $\phi^{k+1}_s$ as the next map in the sequence and setting

$$\phi^{k+1}_s = \phi^{k+1}_s \circ \phi^k_s, \quad K^{k+1}_s = \phi^{k+1}_s(K^k_s)$$

we see that properties (i)–(iii) hold also for $k+1$. The induction may continue.

This completes the proof of Theorem 1.1. \hfill \Box

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