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HOLOMORPHIC FAMILIES OF LONG $\mathbb{C}^2$'S

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ABSTRACT. We construct a holomorphically varying family of complex surfaces X_s, parametrized by the points s in any Stein manifold, such that every X_s is a long $\mathbb{C}^2$ which is biholomorphic to $\mathbb{C}^2$ for some but not all values of s.

1. The main result

A complex manifold X of dimension n is a *long* $\mathbb{C}^n$ if $X = \bigcup_{j=1}^\infty X^j$, where $X^1 \subset X^2 \subset X^3 \subset \ldots$ is an increasing sequence of open domains exhausting X such that each X^j is biholomorphic to $\mathbb{C}^n$. Clearly every long $\mathbb{C}$ is biholomorphic to $\mathbb{C}$. On the other hand, for every $n > 1$ there exists a long $\mathbb{C}^n$ which is not a Stein manifold, and in particular is not biholomorphic to $\mathbb{C}^n$. Such manifolds have been constructed recently by E. F. Wold [12] using his example of a non-Runge Fatou-Bieberbach domain in $\mathbb{C}^2$ [11], thereby solving a problem posed by J. E. Fornæss [3]. Previously Fornæss [2] used Wermer's example of a non-Runge embedded polydisc in $\mathbb{C}^3$ [10] to construct for every $n \geq 3$ an n-dimensional non-Stein complex manifold that is exhausted by biholomorphic images of the polydisc.

Recently L. Meersseman asked in a private communication whether it is possible to holomorphically deform the standard $\mathbb{C}^n$ to a long $\mathbb{C}^n$ that is not biholomorphic to $\mathbb{C}^n$. This question arose naturally in certain problems concerning deformations of foliations that he had been considering. Here we give a positive answer and show that the behavior of long $\mathbb{C}^n$'s in a holomorphic family can be rather chaotic.

Theorem 1.1. *Fix an integer $n > 1$. Assume that S is a Stein manifold, $A = \bigcup_j A_j$ is a finite or countable union of closed complex subvarieties of S, and $B = \{b_j\}$ is a countable set in $S \backslash A$. Then there exists a complex manifold X and a holomorphic submersion $\pi\colon X \to S$ onto S such that*

(i) *the fiber $X_s = \pi^{-1}(s)$ is a long $\mathbb{C}^n$ for every $s \in S$,*
(ii) *X_s is biholomorphic to $\mathbb{C}^n$ for every $s \in A$, and*
(iii) *X_s is non-Stein for every $s \in B$.*

In particular, for any two disjoint countable sets $A, B \subset \mathbb{C}$ there is a holomorphic family $\{X_s\}_{s\in\mathbb{C}}$ of long $\mathbb{C}^2$'s such that X_s is biholomorphic to $\mathbb{C}^2$ for all $s \in A$ and is non-Stein for all $s \in B$. This is particularly striking if the sets A and B are chosen to be everywhere dense in $\mathbb{C}$.

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The conclusion of Theorem 1.1 can be strengthened by adding to the set B a closed complex subvariety of X contained in $X \backslash A$. We do not know whether the same holds if B is a countable union of subvarieties of X.

Several natural questions appear:

Problem 1.2. Given a holomorphic family $\{X_s\}_{s\in S}$ of long $\mathbb{C}^n$'s for some $n > 1$, what can be said about the set of points $s \in S$ for which the fiber X_s is (or is not) biholomorphic to $\mathbb{C}^n$? Are these sets necessarily a G_δ, an F_σ, of the first, resp. of the second category, etc.?

A more ambitious project would be to answer the following question:

Problem 1.3. Is there a holomorphic family X_s of long $\mathbb{C}^2$'s, parametrized by the disc $\mathbb{D} = \{s \in \mathbb{C} \colon |s| < 1\}$ or the plane $\mathbb{C}$, such that X_s is not biholomorphic to $X_{s'}$ whenever $s \neq s'$?

We do not know of any criteria to distinguish two long $\mathbb{C}^n$'s from each other, except if one of them is the standard $\mathbb{C}^n$ and the other one is non-Stein. Apparently there is no known example of a Stein long $\mathbb{C}^n$ other than $\mathbb{C}^n$. It is easily seen that any two long $\mathbb{C}^n$'s are smoothly diffeomorphic to each other, so the gauge-theoretic methods do not apply.

To prove Theorem 1.1 we follow Wold's construction of a non-Stein long $\mathbb{C}^2$ [12], but doing all the key steps with families of Fatou-Bieberbach maps depending holomorphically on the parameter in a given Stein manifold S. (The same proof applies for any $n \geq 2$.) By using the Andersén-Lempert theory [1, 4, 8, 9] we insure that in a holomorphically varying family of injective holomorphic maps $\phi_s \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$ $(s \in S)$ the image domain $\phi_s(\mathbb{C}^2)$ is Runge for some but not all values of the parameter. In the limit manifold X we thus get fibers X_s that are biholomorphic to $\mathbb{C}^2$, as well as fibers that are not holomorphically convex, and hence non-Stein.

2. Constructing holomorphic families of long $\mathbb{C}^n$'s

Let S be a complex manifold that will be used as the parameter space. We recall how one constructs a complex manifold X and a holomorphic submersion $\pi \colon X \to S$ such that the fiber $X_s = \pi^{-1}(s)$ is a long $\mathbb{C}^n$ for each $s \in S$. (This is a parametric version of the construction in [2] or [12, §2].)

Assume that we have a sequence of injective holomorphic maps

$$\Phi^k \colon X^k = S \times \mathbb{C}^n \hookrightarrow X^{k+1} = S \times \mathbb{C}^n, \quad \Phi^k(s,z) = \big(s, \phi^k_s(z)\big), \tag{2.1}$$

where $s \in S,\ z \in \mathbb{C}^n$, and $k = 1, 2, \ldots$. Set $\Omega^k = \Phi^k(X^k) \subset X^{k+1}$. Thus for every fixed $k \in \mathbb{N}$ and $s \in S$ the map $\phi^k_s \colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ is biholomorphic onto its image $\phi^k_s(\mathbb{C}^n) = \Omega^k_s \subset \mathbb{C}^n$ and it depends holomorphically on the parameter $s \in S$. In particular, if Ω^k_s is a proper subdomain of $\mathbb{C}^n$, then ϕ^k_s is a *Fatou-Bieberbach map.* Let X be the disjoint union of all X^k for $k \in \mathbb{N}$ modulo the following equivalence relation. A point $x \in X^i$ is equivalent to a point $x' \in X^k$ if and only if one of the following hold:

(a) $i = k$ and $x = x'$,
(b) $k > i$ and $\Phi^{k-1} \circ \cdots \circ \Phi^i(x) = x'$, or
(c) $i > k$ and $\Phi^{i-1} \circ \cdots \circ \Phi^k(x') = x$.

For each $k \in \mathbb{N}$ we have an injective map $\Psi^k : X^k \hookrightarrow X$ onto the subset $\widetilde{X}^k = \Psi^k(X^k) \subset X$ which sends any point $x \in X^k$ to its equivalence class $[x] \in X$. Denoting by $\iota^k : \widetilde{X}^k \hookrightarrow \widetilde{X}^{k+1}$ the inclusion map, we have

$$\iota^k \circ \Psi^k = \Psi^{k+1} \circ \Phi^k, \quad k = 1, 2, \ldots. \tag{2.2}$$

The inverse maps $(\Psi^k)^{-1} : \widetilde{X}^k \xrightarrow{\cong} X^k = S \times \mathbb{C}^n$ provide local charts on X. It is easily verified that this endows X with the structure of a Hausdorff, second countable complex manifold. Since each of the maps Φ^k respects the fibers over S, we also get a natural projection $\pi : X \to S$ which is clearly a submersion. For every $s \in S$ the fiber X_s is the increasing union of open subsets $\widetilde{X}^k_s$ biholomorphic to $\mathbb{C}^n$. Observe that we get the same limit manifold X by starting with any term of the sequence (2.1).

The next lemma follows from the Andersén-Lempert theory [1]; cf. [12, Theorem 1.2].

Lemma 2.1. *Let $\pi : X \to S$ be as above. Assume that for some $s \in S$ there exists an integer $k_s \in \mathbb{N}$ such that for every $k \geq k_s$, the domain $\Omega^k_s = \phi^k_s(\mathbb{C}^n) \subset \mathbb{C}^n$ is Runge in $\mathbb{C}^n$. Then X_s is biholomorphic to $\mathbb{C}^n$.*

Proof. The main point is that any biholomorphic map $\mathbb{C}^n \xrightarrow{\cong} \Omega$ onto a Runge domain $\Omega \subset \mathbb{C}^n$ can be approximated, uniformly on compact sets, by holomorphic automorphisms of $\mathbb{C}^n$. This observation allows one to renormalize the sequence of biholomorphisms $(\Psi^k_s)^{-1} : \widetilde{X}^k_s \xrightarrow{\cong} \mathbb{C}^n$ for $k \geq k_s$ so that the new sequence converges uniformly on compact sets in X_s to a biholomorphic map $X_s \xrightarrow{\cong} \mathbb{C}^n$; we leave out the straightforward details. □

3. Entire families of holomorphic automorphisms

Let $\aleph_{\mathcal{O}}(X)$ denote the complex Lie algebra of all holomorphic vector fields on a complex manifold X.

A vector field $V \in \aleph_{\mathcal{O}}(X)$ is said to be $\mathbb{C}$-*complete*, or *completely integrable*, if its flow $\{\phi_t\}_{t\in\mathbb{C}}$ exists for all complex values $t \in \mathbb{C}$, starting at an arbitrary point $x \in X$. Thus $\{\phi_t\}_{t\in\mathbb{C}}$ is a complex one-parameter subgroup of the holomorphic automorphism group $\operatorname{Aut} X$. The manifold X is said to enjoy the (holomorphic) *density property* if the Lie subalgebra $\operatorname{Lie}(X)$ of $\aleph_{\mathcal{O}}(X)$, generated by the $\mathbb{C}$-complete holomorphic vector fields, is dense in $\aleph_{\mathcal{O}}(X)$ in the topology of uniform convergence on compact sets in X (see Varolin [8, 9]). More generally, a complex Lie subalgebra $\mathfrak{g}$ of $\aleph_{\mathcal{O}}(X)$ enjoys the density property if $\mathfrak{g}$ is densely generated by the $\mathbb{C}$-complete vector fields that it contains. This property is very restrictive on open manifolds. The main result of the Andersén-Lempert theory [1] is that $\mathbb{C}^n$ for $n > 1$ enjoys the density property; in fact, every polynomial vector field on $\mathbb{C}^n$ is a finite sum of complete polynomial vector fields (the shear fields).

Varolin proved [8] that any domain of the form $(\mathbb{C}^*)^k \times \mathbb{C}^l$ with $k + l \geq 2$ and $l \geq 1$ enjoys the density property; we shall need this for the manifold $\mathbb{C}^* \times \mathbb{C}$. (Here $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.)

Lemma 3.1. *Assume that X is a Stein manifold with the density property. Choose a distance function dist_X on X. Let $\psi_1, \ldots, \psi_k \in \operatorname{Aut} X$ be such that for each $j = 1, \ldots, k$ there exists a $\mathcal{C}^2$ path $\theta_{j,t} \in \operatorname{Aut} X$ ($t \in [0, 1]$) with $\theta_{j,0} = \operatorname{Id}_X$ and $\theta_{j,1} = \psi_j$. Given distinct points $a_1, \ldots, a_k \in \mathbb{C}^*$, a compact set $K \subset X$ and a* *number $\epsilon > 0$, there exists a holomorphic map $\Psi\colon \mathbb{C}\times X \to X$ satisfying the following properties:*

(i) $\Psi_\zeta = \Psi(\zeta,\cdot\,) \in \mathrm{Aut}\, X$ *for all* $\zeta \in \mathbb{C}$,

(ii) $\Psi_0 = \mathrm{Id}_X$,

(iii) $\sup_{x\in K} \mathrm{dist}_X\big(\Psi(a_j,x),\psi_j(x)\big) < \epsilon$ *for* $j = 1,\ldots,k$.

A holomorphic map Ψ satisfying property (i) will be called an *entire curve of holomorphic automorphisms of* X. Here Id_X denotes the identity on X.

Proof. Consider a $\mathcal{C}^2$ path $[0,1] \ni t \mapsto \gamma_t \in \mathrm{Aut}\, X$. Pick a Stein Runge domain $U \subset X$ containing the set K. Then $U_t = \gamma_t(U) \subset X$ is Runge in X for all $t \in [0,1]$. By [1] or, more explicitly, by (the proof of) [4, Theorem 1.1] there exist finitely many complete holomorphic vector fields $V_1,\ldots,V_m$ on X, with flows $\theta_{j,t}$, and numbers $c_1 > 0,\ldots,c_m > 0$ such that the composition $\theta_{m,c_m}\circ\ldots\circ\theta_{1,c_1} \in \mathrm{Aut}\, X$ approximates the automorphism $\psi = \gamma_1$ within ϵ on the set K. (The proof in [4] is written for $X = \mathbb{C}^n$, but it applies in the general case stated here. We first approximate $\gamma_t\colon U \to U_t$ by compositions of short time flows of globally defined holomorphic vector fields on X; here we need the Runge property of the sets U_t. Since X enjoys the density property, these vector fields can be approximated by Lie combinations (using sums and commutators) of complete holomorphic vector fields. This approximates γ_t for each $t \in [0,1]$, uniformly on K, by compositions of flows of complete holomorphic vector fields on X.)

Consider $t^1 = (t_1,\ldots,t_m)$ as complex coordinates on $\mathbb{C}^m$. The map

$$\mathbb{C}^m \ni (t_1,\ldots,t_m) \mapsto \Theta_1(t_1,\ldots,t_m) = \theta_{m,t_m}\circ\ldots\circ\theta_{1,t_1} \in \mathrm{Aut}\, X$$

is entire, its value at the origin $0 \in \mathbb{C}^m$ is Id_X, and its value at the point $(c_1,\ldots,c_m)$ is an automorphism that is ϵ-close to $\psi = \gamma_1$ on K.

Using this argument we find for every $j = 1,\ldots,k$ an integer $m_j \in \mathbb{N}$ and an entire map $\Theta_j\colon \mathbb{C}^{m_j} \to \mathrm{Aut}\, X$ such that $\Theta_j(0) = \mathrm{Id}_X$ and $\Theta_j(c^j_1,\ldots,c^j_{m_j})$ is ϵ-close to ψ_j on K at some point $c^j = (c^j_1,\ldots,c^j_{m_j}) \in \mathbb{C}^{m_j}$. Let $t = (t^1,\ldots,t^k)$ be the complex coordinates on $\mathbb{C}^M = \mathbb{C}^{m_1}\oplus\cdots\oplus\mathbb{C}^{m_k}$, where $t^j = (t^j_1,\ldots,t^j_{m_j}) \in \mathbb{C}^{m_j}$. The composition

$$\mathbb{C}^M \ni t \mapsto \Theta(t^1,\ldots,t^k) = \Theta^k(t^k)\circ\cdots\circ\Theta^1(t^1) \in \mathrm{Aut}\, X$$

is an entire map satisfying $\Theta(0) = \mathrm{Id}_X$ such that $\Theta(0,\ldots,0,c^j,0,\ldots,0)$ is ϵ-close to ψ_j on K for each $j = 1,\ldots,k$.

Choose an entire map $g\colon \mathbb{C} \to \mathbb{C}^M$ with $g(a_j) = (0,\ldots,c^j,\ldots,0)$ for $j = 1,\ldots,k$ and $g(0) = 0$. Then the map $\mathbb{C} \ni \zeta \mapsto \Psi(\zeta) = \Theta(g(\zeta)) \in \mathrm{Aut}\, X$ satisfies the conclusion of the lemma. □

4. Proof of Theorem 1.1

We shall need the following result from [11, §2]. This construction is due to Stolzenberg [6]; see also [7, pp. 392–396].

Lemma 4.1. *There exists a compact set $Y \subset \mathbb{C}^*\times\mathbb{C}$ (a union $Y = D_1\cup D_2$ of two embedded, disjoint, polynomially convex discs) such that*

(i) Y *is* $\mathcal{O}(\mathbb{C}^*\times\mathbb{C})$*-convex,*

(ii) *the polynomial hull* $\widehat{Y}$ *contains the origin* $(0,0) \in \mathbb{C}^2$, *and*

(iii) *for any nonempty open set* $U \subset \mathbb{C}^* \times \mathbb{C}$ *there exists a holomorphic automorphism* $\psi \in \mathrm{Aut}\,(\mathbb{C}^* \times \mathbb{C})$ *such that* $Y \subset \psi(U)$.

Property (iii) is [11, Lemma 3.1]: Since $\mathbb{C}^* \times \mathbb{C}$ enjoys the density property according to Varolin [8], the isotopy that shrinks each of the two discs $D_1, D_2 \subset Y$ to a point in U can be approximated by an isotopy of automorphisms of $\mathbb{C}^* \times \mathbb{C}$ by using the methods in [4].

Proof of Theorem 1.1. We give the proof for $n = 2$. Let $B = \{b_1, b_2, \ldots\}$ be as in the theorem. Choose a set $Y \subset \mathbb{C}^* \times \mathbb{C}$ satisfying Lemma 4.1. Pick a closed ball $K \subset \mathbb{C}^2$ (or any compact set with nonempty interior).

We shall inductively construct a sequence of injective holomorphic maps $\Phi^k \colon S \times \mathbb{C}^2 \hookrightarrow S \times \mathbb{C}^2$ $(k = 1, 2, \ldots)$ of the form

$$\Phi^k(s, z) = (s, \phi^k_s(z)), \quad s \in S,\ z \in \mathbb{C}^2,$$

such that, setting

$$\widetilde{\phi}^k_s = \phi^k_s \circ \phi^{k-1}_s \circ \cdots \circ \phi^1_s, \qquad K^k_s = \widetilde{\phi}^k_s(K) \subset \mathbb{C}^2, \tag{4.1}$$

the following properties hold for all $k \in \mathbb{N}$:

(i) $\Omega^k := \Phi^k(S \times \mathbb{C}^2) \subset S \times (\mathbb{C}^* \times \mathbb{C})$,

(ii) the fiber $\Omega^k_s = \phi^k_s(\mathbb{C}^2)$ is Runge in $\mathbb{C}^2$ for all $s \in A_1 \cup \cdots \cup A_k$, and

(iii) $Y \subset \mathrm{Int}\, K^k_s$ for each $s \in \{b_1, \ldots, b_k\}$. In particular, the polynomial hull of the set K^k_s contains the origin for every such s.

Suppose for the moment that we have such a sequence. Let X denote the limit manifold and let $\Psi^k \colon X^k = S \times \mathbb{C}^2 \xrightarrow{\cong} \widetilde{X}^k \subset X$ be the induced inclusions (see §2).

If $s \in \bigcup_k A_k = A$, then property (ii) insures, in view of Lemma 2.1, that the fiber X_s is biholomorphic to $\mathbb{C}^2$.

Suppose now that $s = b_j$ for some $j \in \mathbb{N}$. Property (iii) shows that for every integer $k \geq j$ the polynomial hull of the set K^k_s contains the origin of $\mathbb{C}^2$; in particular, $\widehat{K^k_s}$ is not contained in $\Omega^k_s \subset \mathbb{C}^* \times \mathbb{C}$. For the corresponding subsets of the limit manifold X_s we get in view of (2.2) that

$$\Psi^{k+1}_s\widehat{(K^k_s)} \not\subset \widetilde{X}^k_s, \qquad k = j, j+1, \ldots,$$

where the hull is with respect to the algebra of holomorphic functions on the domain $\widetilde{X}^{k+1}_s$ in the fiber X_s.

Let $K_s = \Psi^1_s(K)$ denote the compact set in X_s determined by K; note that $K_s \subset \widetilde{X}^1_s$ and $K_s = \Psi^{k+1}_s(K^k_s)$ for any $k \in \mathbb{N}$ according to (2.2) and (4.1). The above display then gives

$$\widehat{(K_s)}_{\mathcal{O}(\widetilde{X}^{k+1}_s)} \not\subset \widetilde{X}^k_s, \quad k = 1, 2, \ldots.$$

Since $\widetilde{X}^{k+1}_s$ is a domain in X_s, we trivially have $\widehat{(K_s)}_{\mathcal{O}(\widetilde{X}^{k+1}_s)} \subset \widehat{(K_s)}_{\mathcal{O}(X_s)}$; hence the hull $\widehat{(K_s)}_{\mathcal{O}(X_s)}$ is not contained in $\widetilde{X}^k_s$ for any $k \in \mathbb{N}$. As the domains $\widetilde{X}^k_s$ exhaust X_s, this hull is noncompact. Hence X_s is not holomorphically convex (and therefore not Stein) for any $s \in B$.

This proves Theorem 1.1 provided that we can find a sequence with the stated properties.

We begin with some initial choices of domains and maps. Pick a Fatou-Bieberbach map $\theta \colon \mathbb{C}^2 \xrightarrow{\cong} D \subset \mathbb{C}^* \times \mathbb{C}$ whose image $D = \theta(\mathbb{C}^2)$ is Runge in

$\mathbb{C}^2$. (Such a D can be obtained as the attracting basin of an automorphism of $\mathbb{C}^2$ that fixes the complex line $\{0\} \times \mathbb{C}$. See e.g. [5, Example 9.7] for an explicit example.) Let $U = \mathrm{Int}\, K \subset \mathbb{C}^2$; then $\theta(U) \subset D$ is a nonempty open set in $\mathbb{C}^* \times \mathbb{C}$.

For each $k = 1, 2, \ldots$ we choose a holomorphic function $f_k \colon S \to \mathbb{C}$ such that $f_k = 0$ on the subvariety $A_1 \cup \cdots \cup A_k$ of S and $f_k(b_j) = j$ for $j = 1, \ldots, k$. If the set $B \subset X \backslash A$ also contains a closed complex subvariety B' of X of positive dimension, we let $f_k = 1$ on B'.

We now construct the first map $\Phi^1(s, z) = (s, \phi^1_s(z))$. Lemma 4.1 furnishes an automorphism $\psi \in \mathrm{Aut}\,(\mathbb{C}^* \times \mathbb{C})$ such that $Y \subset \psi(\theta(U))$. By Lemma 3.1 there exists an entire curve of automorphisms $\Psi_\zeta \in \mathrm{Aut}\,(\mathbb{C}^* \times \mathbb{C})$ ($\zeta \in \mathbb{C}$) such that $\Psi_0 = \mathrm{Id}_{\mathbb{C}^* \times \mathbb{C}}$ and Ψ_1 approximates ψ close enough on the compact set $\theta(K)$ so that $Y \subset \Psi_1(\theta(U))$. Hence $(0, 0) \in \widehat{Y} \subset \widehat{\Psi_1(\theta(K))}$. Set

$$\phi^1_s(z) = \Psi_{f_1(s)}(\theta(z)), \quad s \in S,\ z \in \mathbb{C}^2.$$

If $s \in A_1$, then $f_1(s) = 0$ and hence $\phi^1_s(z) = \Psi_0(\theta(z)) = \theta(z)$, so $\phi^1_s = \theta$. If $s = b_1$, then $f_1(s) = 1$ and hence $\phi^1_s = \Psi_1 \circ \theta$. Thus $Y \subset \phi^1_{b_1}(U)$ and the polynomial hull $\widehat{\phi^1_{b_1}(K)}$ contains the origin of $\mathbb{C}^2$. This gives the initial step.

Suppose that we have found maps $\Phi^1, \ldots, \Phi^k$ satisfying conditions (i)–(iii) above; we now construct the next map Φ^{k+1} in the sequence. Recall that $\widetilde{\phi}^k_s \colon \mathbb{C}^2 \to \mathbb{C}^2$ is the map defined by (4.1). Set

$$U^k_s = (\theta \circ \widetilde{\phi}^k_s)(U), \quad s \in S;$$

this is a nonempty open set contained in the compact set $\theta(K^k_s) \subset \mathbb{C}^* \times \mathbb{C}$. Lemma 4.1 gives for each $j = 1, \ldots, k+1$ an automorphism $\psi_j \in \mathrm{Aut}\,(\mathbb{C}^* \times \mathbb{C})$ such that $Y \subset \psi_j(U^k_{b_j})$. By Lemma 3.1 there exists an entire curve of automorphisms $\Psi_\zeta \in \mathrm{Aut}\,(\mathbb{C}^* \times \mathbb{C})$ ($\zeta \in \mathbb{C}$) such that $\Psi_0 = \mathrm{Id}_{\mathbb{C}^* \times \mathbb{C}}$ and Ψ_j approximates ψ_j for every $j = 1, \ldots, k+1$. If the approximation is close enough on the compact set $\theta(K^k_{b_j})$, then $Y \subset (\Psi_j \circ \theta)(K^k_{b_j})$ and hence the origin $(0, 0) \in \mathbb{C}^2$ is contained in the polynomial hull of $(\Psi_j \circ \theta)(K^k_{b_j})$. Set

$$\phi^{k+1}_s(z) = \Psi_{f_{k+1}(s)} \circ \theta(z), \quad s \in S,\ z \in \mathbb{C}^2.$$

If $s \in A_1 \cup \cdots \cup A_{k+1}$, then $f_{k+1}(s) = 0$ and hence $\phi^{k+1}_s = \theta$. If $s = b_j$ for some $j = 1, \ldots, k+1$, then $f_{k+1}(b_j) = j$ and hence $\phi^{k+1}_{b_j} = \Psi_j \circ \theta$; therefore the polynomial hull of the set $\phi^{k+1}_{b_j}(K^k_{b_j})$ contains the origin. Taking ϕ^{k+1}_s as the next map in the sequence and setting

$$\widetilde{\phi}^{k+1}_s = \phi^{k+1}_s \circ \widetilde{\phi}^k_s, \qquad K^{k+1}_s = \phi^{k+1}_s(K^k_s)$$

we see that properties (i)–(iii) hold also for $k + 1$. The induction may continue. This completes the proof of Theorem 1.1. □

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