AN EXTREMAL DECOMPOSITION PROBLEM
FOR HARMONIC MEASURE

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Abstract. Let $E$ be a continuum in the closed unit disk $|z| \leq 1$ of the complex $z$-plane which divides the open disk $|z| < 1$ into $n \geq 2$ pairwise nonintersecting simply connected domains $D_k$ such that each of the domains $D_k$ contains some point $a_k$ on a prescribed circle $|z| = \rho, 0 < \rho < 1, k = 1, \ldots, n$. It is shown that for some increasing function $\Psi$, independent of $E$ and the choice of the points $a_k$, the mean value of the harmonic measures

$$
\frac{1}{n} \sum_{k=1}^{n} \Psi(\omega(a_k, E, D_k))
$$

is greater than or equal to the harmonic measure $\omega(\rho, E^*, D^*)$, where $E^* = \{z : z^* \in [-1, 0]\}$ and $D^* = \{z : |z| < 1, |\arg z| < \pi/n\}$. This implies, for instance, a solution to a problem of R. W. Barnard, L. Cole, and A. Yu. Solynin concerning a lower estimate of the quantity $\inf_{E} \max_{k=1,\ldots,n} \omega(a_k, E, D_k)$ for arbitrary points of the circle $|z| = \rho$. These authors stated this hypothesis in the particular case when the points are equally distributed on the circle $|z| = \rho$.

1. Introduction and formulation of the result

Problems concerning extremal decomposition of domains on the complex sphere go back to the works of Lavrentiev, Goluzin, Nehari, Jenkins and have a rich history and are more or less directly associated with several areas of geometric function theory (see, e.g., [J2, Kuz, S]). A majority of extremal decompositions is related to estimates for the products whose factors are powers of inner radii of pairwise nonoverlapping domains. For the case of harmonic measure the problem concerning extremal decomposition was perhaps first formulated in the paper of R. W. Barnard, L. Cole, and A. Yu. Solynin [BCS]. Let $E$ be a continuum in the closed unit disk $\overline{U}, U = \{z : |z| < 1\}$, and let it divide $U$ into two subdomains $D_1 \ni \rho$ and $D_2 \ni -\rho, 0 < \rho < 1$. Thus $U \setminus E = D_1 \cup D_2$. The authors of the paper [BCS] give the minimum of the sums

$$
\frac{1}{2}(\omega(\rho, E, D_1) + \omega(-\rho, E, D_2)),
$$

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taken over the set of all continua $E \subset \overline{U}$ that divide $U$ into a pair of domains as described above. Here $\omega(z,E,D)$ denotes the harmonic measure of the set $E \cap \partial D$ with respect to the domain $D$, evaluated at the point $z$. For the properties of the harmonic measure we refer to [A, Ch 1.1]. In the paper [BCS] also a physical motivation of the problem is given. It is of interest to observe that the extremal configuration in this problem is not symmetric with respect to the imaginary axis. Among other things the authors of the paper [BCS] discuss the problem of finding the lower bound

$$\inf_{E} \max_{k=1,\ldots,n} \omega(a^*_k, E, D_k)$$

over all continua $E$ that divide $U$ into $n$ simply connected domains

$$D_k \ni a^*_k = \rho \exp(2\pi i (k-1)/n), \quad k = 1, \ldots, n, \quad n \geq 3.$$ 

They conjectured that the extremal configuration of the minmax problem consists of $n$ circular sectors

$$D^*_k = \{ z \in U : |\arg z - 2\pi (k-1)/n| < \pi/n \}, \quad k = 1, \ldots, n,$$

and observed that under the additional assumption that the closure of $\partial D_k \cap U$ is connected for all $k = 1, \ldots, n$, this can be proved applying Jenkins’s theory of extremal decomposition [J1, S](see [BCS, p. 246]). A. Yu. Solynin has informed the authors that this result has remained unpublished.

In this paper we prove the following theorem.

**Theorem 1.1.** For a given $\rho$, $0 < \rho < 1$, let $E$ be a continuum in the closed unit disk $\overline{U}$ dividing the open disk $U$ into $n \geq 2$ pairwise nonintersecting simply connected domains $D_k$, such that each of the domains $D_k$ intersects the circle $|z| = \rho$, $k = 1, \ldots, n$. Then for arbitrary points $a_k \in D_k$ on the circle $|z| = \rho$, the following inequality holds:

$$(1.2) \quad \frac{1}{n} \sum_{k=1}^{n} \log \frac{1 + \sin(\pi \omega_k/2)}{1 - \sin(\pi \omega_k/2)} \geq -n \log \rho$$

where $\omega_k = \omega(a_k, E, D_k), k = 1, \ldots, n$. Equality holds in (1.2) if and only if

$$E = \{ z : (e^{i\theta} z)^n \in [-1, 0] \}, \quad D_k = \{ z : z e^{i\theta} \in D^*_k \}$$

and $a_k = a^*_k e^{-i\theta}, k = 1, \ldots, n$, for some fixed real $\theta$.

Because the function

$$\Psi(x) = \log \frac{1 + \sin(\pi x/2)}{1 - \sin(\pi x/2)}$$

is strictly increasing on the interval $(0, 1)$, the inequality (1.2) implies a solution to the problem of R. W. Barnard, L. Cole, and A. Yu. Solynin [BCS], and furthermore, the points $a_k$ need not coincide with $a^*_k, k = 1, \ldots, n$. Our proof is based on the result of the first author concerning the product of inner radii of nonoverlapping domains with respect to “free” points on the given circle [D1]. We also make use of an idea of J. Krzyż about the transition from inner radius to conformal invariants [K]. If in the case of [K] the Green function was discussed, then in our case the harmonic measure is in focus. We note further that by the invariance of the harmonic measure under Möbius automorphisms of the disk $U$, the inequality (1.2) holds for all points $a_k$, located on an arbitrary hyperbolic circle in $U$ of hyperbolic radius $\log((1+\rho)/(1-\rho))$. 

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2. Proof of Theorem 1.1

Fix an integer $k, 1 \leq k \leq n$. If all the boundary points of the domain $D_k$ lie in the continuum $E$, then $\omega_k = 1$ and the left side of the inequality \[12\] is $+\infty$. In what follows we exclude this case for all $k = 1, \ldots, n$. Now let $z$ be some point of the boundary $\partial D_k$, not contained in $E$. Then $z \in T := \partial U$. In fact, in the opposite case, $z \in U \setminus E$, and therefore $z \in D_j$ for some $j, 1 \leq j \leq n$. Because of the openness of a domain, we see that $D_j \cap D_k \neq \emptyset$, which contradicts the hypothesis of the theorem. Further, because the set $E$ is closed, on the circle $T$ there is an open arc containing the point $z$ and not containing any points of $E$. We prove that all points of this arc are boundary points of the domain $D_k$. Suppose that there exists a point $z' \in \alpha \setminus \partial D_k$. Because $z' \notin E$, there exists a domain $D_{k'}$ such that $z' \in \partial D_{k'}, 1 \leq k' \leq n$. Choose sequences of points $z_m, z'_m, m = 1, 2, \ldots$, and circular arcs $\lambda_m, m = 1, 2, \ldots$, in $U$ with $z_m, z'_m$ as end points and satisfying the following condition. The points $z_m \in D_k, z_m \to z, m \to \infty; z'_m \in D_{k'}, z'_m \to z', m \to \infty$, and the arcs $\lambda_m$ to $T$ when $m \to \infty$. This last convergence means that the upper bound of the distances of the points of the arc $\lambda_m$ to the circle $T$ approaches 0 when $m \to \infty$. Because $D_k \cap D_{k'} = \emptyset$, we see that on every arc $\lambda_m$ there exists a point $e_m \in E, m = 1, 2, 3, \ldots$. The sequence of the points $\{e_m\}_{m=1}^\infty$ contains a subsequence converging to a point $e$ of the arc $\alpha$. Because $E$ is closed, the point $e$ must be in $\alpha \cap E$, a contradiction to the choice of $\alpha$. Let us denote by $\alpha_k$ the maximal open subarc of the circle $T$, containing the arc $\alpha$ and not containing the points of the set $E$. Clearly, the end points of the arc $\alpha_k$ are contained in $E$. The domain $D_k$ cannot contain on its boundary two different arcs of the aforementioned type (each corresponding to different points $z$) in view of the connectedness of the set $E$. We see that $\alpha_k = (\partial D_k) \setminus E, k = 1, 2, \ldots, n$.

We introduce the notation $B_k = D_k \cup \alpha_k \cup \{z : 1/z \in D_k\}, k = 1, \ldots, n$. From what was said above it follows that the sets $B_k, k = 1, \ldots, n$, are simply connected pairwise nonintersecting domains on the Riemann sphere $\overline{\mathbb{C}}$. By Riemann’s theorem, there exists a function $\varphi_k$ mapping conformally and univalently the corresponding domain $B_k$ onto the domain $G_k$ which is the complement of the arc $\beta_k := \{w : |w| = 1, |\arg w| \leq \theta_k\}, \varphi_k(a_k) = 0, \varphi_k'(a_k) > 0, k = 1, \ldots, n$. On the basis of symmetry, the image of the domain $D_k$ is the disk $|w| < 1$, and further $\varphi_k(a_k) = \{w \in G_k : |w| = 1\}$ and $\varphi_k(E \cap \partial D_k) = \beta_k$. Because the harmonic measure is conformally

\[\text{FIGURE 1. Left: The continuum } E \text{ divides the unit disk } U \text{ into three domains. Right: The image of the domain } D_k \text{ under } \varphi_k.\]
invariant, we have
\[ \omega_k = \omega(0, \beta_k, U_w) = \theta_k / \pi, \quad k = 1, \ldots, n. \]

We denote by \( \ell_k \) the union of a finite number of half-open arcs of the analytic curve 
\( \gamma_k := \varphi_k^{-1}((-1, 0)) \), having the following property: for every \( r, \rho \leq r < 1 \), the circle 
\( |z| = r \) intersects with \( \ell_k \) only at one point. We set \( z_k(r) = \{ z \in \ell_k : |z| = r \} \). The 
construction of the arcs \( \ell_k \) can be made in several ways. We choose one of them.

It follows from the paper [D1] (cf. also [D2, p. 53]) that for every \( r, \rho \leq r < 1 \), the 
equality
\[ n \sqrt{\prod_{k=1}^{n} r(B_k, z_k(r))} \leq \frac{4r}{n} \]
holds and furthermore the equality in (2.1) holds if and only if the domains 
\( B_k \) and the points 
\( z_k(r) \) coincide with the domains 
\[ \{ z : |\arg z - 2\pi(k-1)/n| < \pi / n \} \]
and the points \( r \exp(2\pi(k-1)/n) \), respectively, with a possible rotation around the 
origin. Here \( r(B, z) \) stands for the inner radius of the domain \( B \) with respect to 
\( z \); see [D2, Sec. 1]. The inequality (2.1) was proved by the piecewise 
separating symmetrization method of [D2].

By the arithmetic-geometric mean inequality this implies that
\[ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{r(B_k, z_k(r))} \geq \frac{n}{4r}. \]

Integration yields the conclusion that
\[ \frac{1}{n} \sum_{k=1}^{n} \int_{\gamma_k} \frac{|dz|}{r(B_k, z)} \geq \frac{1}{n} \sum_{k=1}^{n} \int_{\ell_k} \frac{|dz|}{r(B_k, z)} \]
\[ \geq \frac{1}{n} \sum_{k=1}^{n} \int_{\rho}^{1} \frac{dr}{r(B_k, z_k(r))} \geq \int_{\rho}^{1} \frac{1}{4r} dr = -\frac{n}{4} \log \rho. \]

On the other hand
\[ \int_{\gamma_k} \frac{|dz|}{r(B_k, z)} = \int_{-1}^{0} \frac{dw}{r(G_k, w)} \]
for all \( k = 1, \ldots, n \). For the computation of the inner radius \( r(G_k, w) \) we map the 
domain \( G_k \) by a univalent conformal mapping onto the unit disk \( |\zeta| < 1 \). It is easy 
to see that for such a map we can take a superposition of the following functions:

\[ w_1 = \frac{w - 1}{w + 1}, \quad w_2 = -i w_1 \cot \frac{\theta_k}{2}, \quad \zeta = w_2 - \sqrt{w_2^2 - 1}, \]

where the last mapping is that branch of the function, inverse of the Joukowski 
transformation, which maps \( \infty \) to zero. Because under a conformal univalent map-
ning the inner radius is multiplied by the modulus of the derivative at the corre-
sponding point, we have
\[ r(G_k, w) |w_1'(w) \cot \frac{\theta_k}{2} \zeta'(w_2)| = r(U_\zeta, \zeta) = 1 - |\zeta|^2. \]

Simple calculations give from this
\[ r(G_k, w) = \frac{1 - w}{\sin(\theta_k/2)} \sqrt{w^2 - 2w \cos \theta_k + 1}. \]
Inserting this value of the inner radius in the inequality (2.3), we obtain

\[ \int_{-1}^{0} \frac{dw}{r(G_k, w)} = \frac{\sin \theta_k}{2} \int_{-1}^{0} \frac{dw}{(1 - w)(w^2 - 2w \cos \theta_k + 1)} = \frac{1}{4} \log \frac{1 + \sin(\theta_k/2)}{1 - \sin(\theta_k/2)}. \]

The computation of the integral is carried out either with standard integration methods or using tables such as [PBM, p. 94]. Applying the relations (2.2) and (2.3) we arrive at the desired inequality (1.2).

We now assume that in the inequality (1.2) we have equality. Then the equalities also hold in (2.2) and (2.3) for all \( r, \rho \leq r < 1 \). From the equality in (2.2) it follows that the curves \( \ell_k = \gamma_k \) are segments on radial rays, emanating from the origin, \( k = 1, \ldots, n \). Equality in (2.1) shows that the angles between neighboring rays are equal to \( 2\pi/n \) and the domains \( B_k \) are sectors with angle \( 2\pi/n \) for which these rays are bisectors. This observation implies that for some real \( \theta, a_k = a_k^* e^{-i\theta} \) and \( D_k = \{ z : z e^{i\theta} \in D_k^* \} \), \( k = 1, \ldots, n \). In this case the continuum \( E \) necessarily contains the set \( E_{\theta} := \{ z : (ze^{i\theta})^n \in [-1, 0] \} \). With a simple verification we see that for \( E = E_{\theta} \) in (1.2) we have the sign of equality. It remains to observe that for other continua \( E \supset E_{\theta}, E \neq E_{\theta} \) we have a strict inequality. The theorem is proved.

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