PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 7, July 2012, Pages 2429–2440
S 0002-9939(2011)11409-6
Article electronically published on November 15, 2011

# ON A CHARACTERIZATION OF BILINEAR FORMS ON THE DIRICHLET SPACE

CARME CASCANTE AND JOAQUIN M. ORTEGA

(Communicated by Richard Rochberg)

Abstract. Arcozzi, Rochberg, Sawyer and Wick obtained a characterization of the holomorphic functions $b$ such that the Hankel type bilinear form $T_b(f,g)=\int_{\mathbb{D}}(I+R)(fg)(z)\overline{(I+R)b(z)}dv(z)$ is bounded on $\mathcal{D}\times\mathcal{D}$, where $\mathcal{D}$ is the Dirichlet space. In this paper we give an alternative proof of this characterization which tries to understand the similarity with the results of Maz′ya and Verbitsky relative to the Schrödinger forms on the Sobolev spaces $L^1_2(\mathbb{R}^n)$.

## 1. Introduction

Let $\mathcal{D}$ be the Dirichlet space on the unit disk $\mathbb{D}$, that is, the space of holomorphic functions on $\mathbb{D}$ such that

$$\|f\|^2_{\mathcal{D}}=\int_{\mathbb{D}}|(I+R)f(z)|^2dv(z)<+\infty,$$

where $Rf(z)=z\frac{\partial}{\partial z}f(z)$ is the radial derivative and $dv$ is the Lebesgue measure on $\mathbb{D}$. We recall that $\mathcal{D}$ coincides with the Hardy-Sobolev space $H^2_{\frac{1}{2}}$ on the unit disk.

If $b$ is a holomorphic function on $\mathbb{D}$, such that $(I+R)b\in L^1(\mathbb{D})$, let $T_b$ denote the bilinear form, defined initially for holomorphic polynomials $f,g$ by

$$T_b(f,g)=\int_{\mathbb{D}}(I+R)(fg)(z)\overline{(I+R)b(z)}dv(z).$$

The norm of $T_b$ is

$$\|T_b\|=\sup\{|T_b(f,g)|\,:\,\|f\|_{\mathcal{D}},\,\|g\|_{\mathcal{D}}\le 1\}.$$

The object of this paper is the study of the functions $b$ such that $\|T_b\|$ is finite. In fact, a characterization of the functions $b$ satisfying this condition was given in [4], where the following theorem was shown, solving a conjecture stated by R. Rochberg:

**Theorem 1.1** ([4]). *If $b$ is a holomorphic symbol on $\mathcal{D}$ such that $(I+R)b\in L^1(\mathbb{D})$. Then the following assertions are equivalent:*

(i) *The bilinear form $T_b$ is bounded on $\mathcal{D}\times\mathcal{D}$.*
(ii) *The measure $d\mu_b(z)=|(I+R)b(z)|^2dv(z)$ is a Carleson measure for the Dirichlet space $\mathcal{D}$.*

Received by the editors February 23, 2011.
2010 *Mathematics Subject Classification.* Primary 31C25, 31C15, 47B35.
*Key words and phrases.* Dirichlet spaces, Carleson measures.
The authors were partially supported by DGICYT Grant MTM2011-27932-C02-01, DURSI Grant 2009SGR 1303 and Grant MTM2008-02928-E.

©2011 American Mathematical Society
Reverts to public domain 28 years from publication

A positive measure $\mu$ on $\mathbb{D}$ is a Carleson measure for the Dirichlet space $\mathcal{D}$ if and only if there exists $C > 0$ such that for any $f \in \mathcal{D}$,

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C\|f\|_{\mathcal{D}}^2.$$

Stegenga (see [12]) characterized the above Carleson measures in terms of Riesz capacities. Namely, $\mu$ is a Carleson measure for $\mathcal{D}$ if and only if there exists $C > 0$ such that for any open set $A$ on $\mathbb{T}$, the unit circle, $\mu(T(A)) \leq CC_{\frac{1}{2},2}(A)$. Here $T(A) = (\bigcup_{\zeta \notin A} D_\alpha(\zeta))^c$ is the tent over $A$ and if $\zeta \in \mathbb{T}$, $D_\alpha(\theta) = \{z : |1 - z\overline{\zeta}| < \alpha(1 - |z|^2)\}$, $\alpha > 2$ is a nontangential region.

We also recall that if $A$ is a set on $\mathbb{T}$, then the Riesz capacity is given by

$$C_{\frac{1}{2},2}(A) = \inf\{\|f\|_{L^2(\mathbb{T})}^2 : f \geq 0,\ I_{\frac{1}{2}}[f] \geq 1 \text{ on } A\},$$

where, if $f$ is a nonnegative function on $\mathbb{T}$, we denote for $w \in \mathbb{T}$,

$$I_{\frac{1}{2}}[f](w) = \int_0^{2\pi} \frac{f(\theta)}{|1 - we^{-i\theta}|^{\frac{1}{2}}} d\theta. \tag{1.1}$$

The original proof of (i) implies (ii) in [4] is quite delicate and needs very accurate estimates. They use the capacitary characterization of Carleson measures, studying the relative sizes of both $\int_V |b'|^2$, where $V$ is a certain set in $\mathbb{D}$, and the capacity of the set $\overline{V} \cap \partial\mathbb{D}$. Then they construct an "expanded" set, $V_{exp}$, satisfying that $V$ and $\mathbb{D} \setminus V_{exp}$ are "well separated", but such that $V_{exp}$ is not very large when it is measured by $\int_{V_{exp}} |b'|^2$ or by the capacity of $\overline{V} \cap \partial\mathbb{D}$. It is then possible to construct a pair of functions $f$ and $g$ such that $|T_b(f,g)|$ can be expressed, up to an error term, as $\int_V |b'|^2$, and to show the smallness of this error term. Some of the estimates are obtained by working on Bergman trees and the associated tree capacities.

In the same paper, the formal similarity with a boundedness criterion for a bilinear form associated to the Schrödinger operator due to Maz′ja and Verbitsky, [10], is observed. Let $L_2^1(\mathbb{R}^n)$ be the homogeneous Sobolev space obtained by completing $C_0^\infty(\mathbb{R}^n)$ with respect to the quasinorm induced by the Dirichlet inner product $\langle f, g\rangle_{Dir} = \int_{\mathbb{R}^n} \nabla f \overline{\nabla g} dx$. Given $b$, the Schrödinger form $S_b$ on $L_2^1(\mathbb{R}^n) \times L_2^1(\mathbb{R}^n)$ is defined as

$$S_b(f,g) = \langle fg, b\rangle_{Dir}.$$

In Corollary 2 of [10] it is shown that $S_b$ is bounded if and only if $d\mu_b = |(-\Delta)^{1/2} b|^2 dx$ is a Carleson measure for the space $L_2^1(\mathbb{R}^n)$, that is, if and only if there exists $C > 0$ such that for any $f \in L_2^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f|^2 d\mu_b \leq C\|f\|_{L_2^1(\mathbb{R}^n)}^2.$$

The proof of this result uses completely different techniques to the ones used in the proof of Theorem 1.1 in [4]. An adequate integration by parts and the use of certain weights in the $A_2$ Muckenhoupt class are fundamental tools.

The main object of this paper is to give an alternative proof of Theorem 1.1, based on the existence of holomorphic potentials associated to extremal measures with respect to the capacity of a set and inspired in part in the methods of the characterization of the boundedness of the bilinear form $S_b$ associated to the Schrödinger operator obtained in [10]. This approach is technically much simpler and tries to approximate the answer to the question posed in [4] of knowing if there is an underlying reason for such formal similarity.

In Section 2, we recall the main properties relative to capacities and holomorphic potentials needed for the proof of the theorem. Section 3 is devoted to giving the proof of Theorem 1.1.

As usual, we will adopt the convention of using the same letter for various absolute constants whose values may change in each occurrence. Also, if there is no confusion, for simplicity we will write $A \lesssim B$ if there exists an absolute constant $M$ such that $A \leq MB$. We will say that two quantities $A$ and $B$ are equivalent if both $A \lesssim B$ and $B \lesssim A$, and, in that case, we will write $A \approx B$.

## 2. Capacitary measures and holomorphic potentials

We begin this section by recalling the main properties of the capacitary extremal measure associated to an open set on $\mathbb{T}$.

**Theorem 2.1** (Thm. 2.2.7, [1])**.** *If $A$ is an open set on $\mathbb{T}$, there exists a positive measure $\nu$ on $\mathbb{T}$, the capacitary measure for $A$, such that*

(i) *$\nu$ is supported on $\overline{A}$;*
(ii) *$\|\nu\| = \|I_{\frac{1}{2}}[\nu]\|_2^2 = C_{\frac{1}{2},2}(A)$;*
(iii) *$\mathcal{P}(w) := I_{\frac{1}{2}}[I_{\frac{1}{2}}[\nu]](w) \leq C$, $\omega \in \mathbb{T}$;*
(iv) *$\mathcal{P}(w) \geq 1$ on $A$, except for a set of capacity zero.*

If we denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbb{T}$, it is proved in Theorem 20 in [11] that for any $\varepsilon > 0$, $|E|^{\varepsilon} \lesssim C_{\frac{1}{2},2}(E)$. Consequently, we also have that $\mathcal{P}(w) \geq 1$ on $A$, almost everywhere on $\mathbb{T}$. Our next lemma gives a pointwise estimate of $I_{\frac{1}{2}}[I_{\frac{1}{2}}[\nu]]$.

**Lemma 2.2.** *There exist $C, K > 0$ such that for any positive measure $\nu$ on $\mathbb{T}$ and any $e^{i\theta} \in \mathbb{T}$,*

$$I_{\frac{1}{2}}[I_{\frac{1}{2}}[\nu]](e^{i\theta}) \leq C \int_0^{2\pi} \ln \frac{K}{|e^{i\theta} - e^{it}|} d\nu(t). \tag{2.1}$$

*Furthermore, there exist $C_1, K_1, \delta > 0$, such that for any $\theta, t$, $0 < |e^{i\theta} - e^{it}| \leq \delta$,*

$$0 < C_1 \ln \frac{K_1}{|e^{i\theta} - e^{it}|} \leq \int_0^{2\pi} \int_0^{2\pi} \frac{d\eta}{|e^{i\eta} - e^{it}|^{\frac{1}{2}} |e^{i\theta} - e^{i\eta}|^{\frac{1}{2}}}. \tag{2.2}$$

*Proof.* Fubini's theorem gives that

$$I_{\frac{1}{2}}[I_{\frac{1}{2}}[\nu]](e^{i\theta}) = \int_0^{2\pi} \int_0^{2\pi} \frac{d\eta}{|e^{i\eta} - e^{it}|^{\frac{1}{2}} |e^{i\theta} - e^{i\eta}|^{\frac{1}{2}}} d\nu(t).$$

Next,

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\eta}{|e^{i\eta}-e^{it}|^{\frac12}|e^{i\theta}-e^{i\eta}|^{\frac12}} \\
&= \int_{|e^{i\theta}-e^{i\eta}|\le \frac{|e^{i\theta}-e^{it}|}{3}} \frac{d\eta}{|e^{i\eta}-e^{it}|^{\frac12}|e^{i\theta}-e^{i\eta}|^{\frac12}} \\
&+ \int_{|e^{it}-e^{i\eta}|\le \frac{|e^{i\theta}-e^{it}|}{3}} \frac{d\eta}{|e^{i\eta}-e^{it}|^{\frac12}|e^{i\theta}-e^{i\eta}|^{\frac12}} \\
&+ \int_{|e^{i\theta}-e^{i\eta}|> \frac{|e^{i\theta}-e^{it}|}{3},\ |e^{it}-e^{i\eta}|> \frac{|e^{i\theta}-e^{it}|}{3}} \frac{d\eta}{|e^{i\eta}-e^{it}|^{\frac12}|e^{i\theta}-e^{i\eta}|^{\frac12}} = I+II+III.
\end{aligned}$$

On the region considered in the integral given by $I$, $|e^{i\eta}-e^{it}| \approx |e^{i\theta}-e^{it}|$, and consequently

$$I \lesssim \frac{1}{|e^{i\theta}-e^{it}|^{\frac12}} \int_{|e^{i\theta}-e^{i\eta}|\le \frac{|e^{i\eta}-e^{it}|}{3}} \frac{d\eta}{|e^{i\theta}-e^{i\eta}|^{\frac12}} \approx C.$$

The integral given by $II$ is estimated analogously.

Next, we split the region considered in $III$ into two parts, one corresponding to the points where $|e^{i\eta}-e^{it}| > |e^{i\theta}-e^{i\eta}|$ and one where $|e^{i\eta}-e^{it}| \le |e^{i\theta}-e^{i\eta}|$. In the first case the integral given by $III$ is bounded from above by

$$\int_{\frac{|e^{i\theta}-e^{it}|}{3}<|e^{i\theta}-e^{i\eta}|\le 2} \frac{d\eta}{|e^{i\theta}-e^{i\eta}|} \lesssim \ln \frac{K}{|e^{i\theta}-e^{it}|}.$$

Finally, in order to obtain the estimate given in (2.2), we consider for $0 < |e^{i\theta}-e^{it}| \le \delta$, where $\delta>0$ is to be chosen, the region where $|e^{i\eta}-e^{it}| \ge |e^{it}-e^{i\theta}|$. In that case, we have that $|e^{i\eta}-e^{i\theta}| \le 2|e^{i\eta}-e^{it}|$, and consequently, if $\delta$ is small enough,

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\eta}{|e^{i\eta}-e^{it}|^{\frac12}|e^{i\theta}-e^{i\eta}|^{\frac12}} \\
&\succeq \int_{|\eta-t|>2|t-\theta|}^{\pi} \frac{d\eta}{|\eta-t|} \simeq \ln\frac{K_1}{|e^{i\theta}-e^{it}|} > 0. \qquad \blacksquare
\end{aligned}$$

If ln denotes the principal branch of the logarithm, we define the holomorphic potential of a finite positive measure $\mu$ on $\mathbb{T}$ by

$$\mathcal{U}(z) = \int_{\mathbb{T}} \ln \frac{K}{1-z\overline{\zeta}} d\mu(\zeta), \tag{2.3}$$

where $K>0$ is a fixed constant satisfying that $\ln\frac{K}{2} \ge 4\pi$ and big enough so that the estimate (2.1) in Lemma 2.2 can be applied. This choice of $K>0$ implies in particular that for any $z\in\mathbb{D}$, $\mathrm{Re}\,\mathcal{U}(z) \ge 2|\mathrm{Im}\,\mathcal{U}(z)|$ and $|\mathcal{U}| \approx \mathrm{Re}\,\mathcal{U}(z)$. We will define $\mathcal{V}(z) = \mathrm{Re}\,\mathcal{U}(z)$.

We remark that if we replace $K$ by a larger constant in the definition of the holomorphic potential $\mathcal{U}$, the estimates obtained in the following lemmas still remain true. For technical reasons which will appear later on, it will be convenient at some point to replace the constant $K$ previously considered by $6K$. When we need to specify the precise constant, we will write $\mathcal{V}_K$ instead of $\mathcal{V}$.

**Lemma 2.3.** *If $A\subset\mathbb{T}$ is an open set, $\nu$ is the extremal measure for $A$ and $\mathcal{U}$ is the holomorphic potential associated to $\nu$, we have:*

(i) $\|\mathcal{U}\|_{\mathcal{D}} \lesssim C_{\frac{1}{2},2}(A)^{\frac{1}{2}}$.
(ii) $|\mathcal{U}(z)| \leq C$.
(iii) $1 \lesssim \mathcal{V}(z) = \mathrm{Re}\,\mathcal{U}(z)$ *on* $T(A)$.

*Proof.* The proof will be based on the properties of the extremal measure $\nu$ and the above lemma. Let us begin with (i).

It is proved in Theorem 2.10 in [3] that

$$\|\mathcal{U}\|_{\mathcal{D}} \lesssim \|I_{\frac{1}{2}}[\nu]\|_2,$$

which, by (ii) in Theorem 2.1, coincides with $C_{\frac{1}{2},2}(A)^{\frac{1}{2}}$. In particular, $\mathcal{U} \in \mathcal{D} = H^2_{\frac{1}{2}} \subset H^2$, and consequently, it has nontangential boundary values $\mathcal{U}^*$ almost everywhere. If we denote $\mathcal{V}^* = \mathrm{Re}\,\mathcal{U}^*$, and if $P(z,\theta) = \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}$ is the Poisson kernel, we then have

$$|\mathcal{U}(z)| \approx \mathcal{V}(z) \tag{2.4}$$

$$= \int_0^{2\pi} P(z,\theta)\mathcal{V}^*(e^{i\theta})d\theta. \tag{2.5}$$

Consequently, in order to prove (ii), it is enough to show that $\mathcal{V}^*(e^{i\theta}) \lesssim 1$. Indeed, by (2.2) and (ii) and (iii) in Theorem 2.1, we have that $\mathcal{V}^*(e^{i\theta}) \lesssim C_{\frac{1}{2}2}(A) + I_{\frac{1}{2}}[I_{\frac{1}{2}}[\nu]](e^{i\theta}) \lesssim 1$.

Finally, in order to prove (iii), we observe that if $z \in T(A)$, then there exists $c < 1$ such that $B(\frac{z}{|z|}, c(1-|z|)) \subset A$. Since by (iv) in Theorem 2.1, $\mathcal{V}^* \succeq 1$ a.e. on $A$, we have then that

$$\begin{aligned}
\mathcal{V}(z) &= \int_0^{2\pi} \frac{(1-|z|^2)}{|z-e^{i\theta}|^2}\mathcal{V}^*(\theta)d\theta \\
&\geq \int_{B(\frac{z}{|z|},c(1-|z|))} \frac{(1-|z|^2)}{|z-e^{i\theta}|^2}\mathcal{V}^*(\theta)d\theta \\
&\geq C\int_{B(\frac{z}{|z|},c(1-|z|))} \frac{(1-|z|^2)}{|z-e^{i\theta}|^2}d\theta \approx \frac{1}{(1-|z|^2)}\int_{B(\frac{z}{|z|},c(1-|z|))} d\theta \approx 1. \qquad \blacksquare
\end{aligned}$$

Before we state our next lemma, we give a couple of definitions. If $f$ is a function in $L^2(\mathbb{D})$, its Bergman projection is defined by

$$B[f](z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^2}dv(w), \;\; w \in \mathbb{D}.$$

If $\omega$ is a weight on $\mathbb{D}$, it was proved in [5] that $B$ is bounded on $L^2(\omega)$ if and only if $\omega$ is in the class $B_2$ defined by

**Definition 2.4.** A weight $\omega$ is in $B_2$ if there exists $C > 0$ such that for any interval $I \subset \mathbb{T}$,

$$\left(\frac{1}{|T(I)|}\int_{T(I)} \omega dv\right)\left(\frac{1}{|T(I)|}\int_{T(I)} \omega^{-1}dv\right) \leq C.$$

**Lemma 2.5.** *The function* $\mathcal{V}^2$ *is in the* $B_2$ *class.*

*Proof.* Observe that

$$\begin{aligned}\mathrm{Re}(\mathcal{U}^2) &= (\mathrm{Re}\,\mathcal{U})^2 - (\mathrm{Im}\,\mathcal{U})^2 = ((\mathrm{Re}\,\mathcal{U} + \mathrm{Im}\,\mathcal{U})(\mathrm{Re}\,\mathcal{U} - \mathrm{Im}\,\mathcal{U}) \approx \\ &(\mathrm{Re}\,\mathcal{U})^2 = \mathcal{V}^2.\end{aligned} \tag{2.6}$$

In [9] (Lemma 3 of 2.6) there is a proof of a result by Verbitsky that shows that for any $\beta \in (1, +\infty)$, the weight $(I_{\frac{1}{2}}[I_{\frac{1}{2}}[\nu]])^\beta$ is in the Muckenhoupt class $A_1$ on $\mathbb{T}$, with constants independent of the measure $\nu$. In particular, we have that the weight $(\mathcal{V}^*)^2$ is in the class $A_1$ on $\mathbb{T}$, and consequently, it is also in the $A_2$ class on $\mathbb{T}$.

It is proved in [8] that if the weight $(\mathcal{V}^*)^2$ is in the Muckenhoupt class $A_2$ on $\mathbb{T}$ and we consider the set $I_z = \{\zeta \in \mathbb{T} : |1 - \frac{z}{|z|}\overline{\zeta}| \leq c(1 - |z|^2)\}$, $c < 1$, then the weight defined by

$$(\tilde{\mathcal{V}^*})^2(z) = \frac{1}{1 - |z|}\int_{I_z} (\mathcal{V}^*)^2(\zeta)|d\zeta|$$

is in the class $B_2$.

The estimate (2.6) gives that $P[(\mathcal{V}^*)^2] \approx P[\mathrm{Re}(\mathcal{U}^2)] = \mathrm{Re}(\mathcal{U}^2) \approx (\mathcal{V})^2$.

Hence, in order to finish the proof of the lemma it is enough to check that

$$P[(\mathcal{V}^*)^2] \approx (\tilde{\mathcal{V}}^*)^2. \tag{2.7}$$

If $z = re^{i\theta_0} \in \mathbb{D}$, for any $k \geq 1$, let $I_k(z) = \{\zeta \in \mathbb{T} : |1 - e^{i\theta_0}\overline{\zeta})| \leq 2^k(1 - r)\}$ and $I_0 = \emptyset$. We then have

$$\begin{aligned}P[(\mathcal{V}^*)^2](z) &= \sum_{k\geq 0}\int_{I_{k+1}\setminus I_k} \frac{1 - |z|^2}{|1 - z\overline{\zeta}|^2}(\mathcal{V}^*)^2(\zeta)d\zeta \\ &\lesssim \sum_{k\geq 0}\frac{1}{2^{2k}(1 - r)}\int_{I_{k+1}} (\mathcal{V}^*)^2(\zeta)d\zeta.\end{aligned}$$

Since $(\mathcal{V}^*)^2$ is in $A_2$, it is well known that $(\mathcal{V}^*)^2$ is also in $A^{2-\varepsilon}$, for some $\varepsilon > 0$, and in particular, it satisfies a doubling condition of order $\tau < 2$. Thus, $\int_{I_{k+1}} (\mathcal{V}^*)^2(\zeta)d\zeta \leq C2^{k\tau}\int_{I_k} (\mathcal{V}^*)^2(\zeta)d\zeta$. Consequently, the above sum is bounded above by

$$\sum_{k\geq 0}\frac{1}{2^{k(2-\tau)}(1 - r)}\int_{I_z} (\mathcal{V}^*)^2(e^{i\theta})d\theta \approx (\tilde{\mathcal{V}}^*)^2(z).$$

The lower estimate in (2.7) is a consequence of the inequality

$$\frac{1}{|I_z|}\int_{I_z} (\mathcal{V}^*)^2(\zeta)d\zeta \lesssim P[(\mathcal{V}^*)^2](z). \qquad \blacksquare$$

## 3. Proof of Theorem 1.1

3.1. **(ii) implies (i).** The proof of this implication is direct; see [4]. For the sake of completeness, we include it here. Assume that $d\mu_b(z) = |(I + R)b(z)|^2 dv(z)$ is a

Carleson measure for the Dirichlet space $\mathcal{D}$, and let $f, g \in \mathcal{D}$. We then have:

$$\begin{aligned}
&|T_b(f,g)| \\
&= \left| \int_{\mathcal{D}} (I+R)f(z)\, g(z)\overline{(I+R)b(z)}dv(z) + \int_{\mathcal{D}} (f(z)\, Rg(z))\overline{(I+R)b(z)}dv(z) \right| \\
&\leq \left( \int_{\mathcal{D}} |(I+R)f(z)|^2 dv(z) \right)^{\frac{1}{2}} \left( \int_{\mathcal{D}} |g(z)|^2 |(I+R)b(z)|^2 dv(z) \right)^{\frac{1}{2}} \\
&\quad + \left( \int_{\mathcal{D}} |f|^2 |(I+R)b(z)|^2 dv(z) \right)^{\frac{1}{2}} \left( \int_{\mathcal{D}} |Rg(z)|^2 dv(z) \right)^{\frac{1}{2}} \lesssim \|f\|_{\mathcal{D}} \|g\|_{\mathcal{D}}.
\end{aligned}$$

3.2. **(i) implies (ii).** Assume now that $T_b$ is bounded on $\mathcal{D} \times \mathcal{D}$. We want to prove that $d\mu_b(z)$ is a Carleson measure for the Dirichlet space $\mathcal{D}$, which is equivalent to proving that there exists a constant $K$ such that for any open set $A \subset \mathbb{T}$,

$$\int_{T(A)} |(I+R)b|^2(z) dv(z) \leq K C_{\frac{1}{2},2}(A). \tag{3.1}$$

We begin with some technical lemmas needed in the proof.

**Lemma 3.1.** *If $T_b$ is bounded on $\mathcal{D} \times \mathcal{D}$, the symbol $b$ belongs to the Dirichlet space $\mathcal{D}$.*

*Proof.* Let $h \in \mathcal{D}$. By hypothesis,

$$\left| \int_{\mathcal{D}} (I+R)h(z)\overline{(I+R)b(z)}dv(z) \right| \lesssim \|T_b\| \|h\|_{\mathcal{D}}.$$

Duality gives then that $b \in \mathcal{D}$ and $\|b\|_{\mathcal{D}} \lesssim \|T_b\|$. ∎

**Lemma 3.2.** *Let $b \in \mathcal{D}$ and $f$ be the holomorphic function on $\mathbb{D}$ defined by $f = (I+R)^{-1} B[[(I+R)b]\chi_{T(A)}]$, where $B$ is the Bergman projection on $\mathbb{D}$. Then $f \in \mathcal{D}$ and $\|f\|_{\mathcal{D}} \lesssim \|b\|_{\mathcal{D}}$.*

*Proof.* We have that $(I+R)f = B[[(I+R)b]\chi_{T(A)}]$, and consequently, the boundedness of the Bergman projection on $L^2(\mathbb{D})$ and Lemma 3.1 give that

$$\|f\|_{\mathcal{D}} = \|B[[(I+R)b]\chi_{T(A)}]\|_{L^2(\mathbb{D})} \lesssim \|(I+R)b\chi_{T(A)}\|_{L^2(\mathbb{D})} \leq \|b\|_{\mathcal{D}}.$$ ∎

We now proceed to prove the theorem. Given an open set $A$, let $\nu$ be the extremal measure as in Theorem 2.1, let $\mathcal{U} \in \mathcal{D}$ be the holomorphic potential associated to $\nu$ and $A$, defined in (2.3), and let $f$ be the holomorphic function associated to $b$, defined in Lemma 3.2. Next, observe that

$$\begin{aligned}
T_b(\mathcal{U}, \frac{f}{\mathcal{U}}) &= \int_{\mathbb{D}} (I+R)f(z)\overline{(I+R)b(z)}dv(z) \\
&= \int_{\mathbb{D}} B[[(I+R)b]\chi_{T(A)}](z)\overline{(I+R)b(z)}dv(z) \\
&= \int_{\mathbb{D}} [(I+R)b]\chi_{T(A)}(z)\overline{(I+R)b(z)}dv(z) \\
&= \int_{T(A)} |(I+R)b(z)|^2 dv(z) = \mu_b(T(A)).
\end{aligned} \tag{3.2}$$

Since $T_b$ is bounded, by Lemma 2.3 we have

$$|T_b(\mathcal{U}, \frac{f}{\mathcal{U}})| \lesssim \|\mathcal{U}\|_{\mathcal{D}} \|\frac{f}{\mathcal{U}}\|_{\mathcal{D}} \lesssim C_{\frac{1}{2}2}(A)^{\frac{1}{2}} \|\frac{f}{\mathcal{U}}\|_{\mathcal{D}}.$$

We will show that

$$\|\frac{f}{\mathcal{U}}\|_{\mathcal{D}} \lesssim \left(\int_{T(A)} |(I+R)b(z)|^2 dv(z)\right)^{\frac12} = \mu_b(T(A))^{\frac12}. \tag{3.3}$$

Indeed, if (3.3) holds, then by (3.2), $\mu_b(T(A)) \lesssim C_{\frac12,2}(A)^{\frac12}\mu_b(T(A))^{\frac12}$, and since by Lemma 3.1, $\mu_b(T(A)) \lesssim \|b\|_{\mathcal{D}} < +\infty$, we finish the proof of (i) implies (ii).

So, we are led to show (3.3). But $(I+R)\frac{f}{\mathcal{U}} = \frac{(I+R)f}{\mathcal{U}} - \frac{fR\mathcal{U}}{\mathcal{U}^2}$. Thus it is enough to show that

$$\int_{\mathbb{D}} \frac{|(I+R)f(z)|^2}{|\mathcal{U}^2(z)|} dv(z) \lesssim \mu_b(T(A)) \tag{3.4}$$

and

$$\int_{\mathbb{D}} \frac{|f(z)R\mathcal{U}(z)|^2}{|\mathcal{U}^4(z)|} dv(z) \lesssim \mu_b(T(A)). \tag{3.5}$$

We begin with (3.4). Since $|\mathcal{U}| \approx \mathcal{V}$ and Lemma 2.5 gives that $\frac{1}{\mathcal{V}^2}$ is a weight in $B_2$, the integral corresponding to the left-hand side of (3.4) is bounded by

$$\begin{aligned}\int_{\mathbb{D}} \frac{|(I+R)f(z)|^2}{|\mathcal{U}^2(z)|} dv(z) &\le \int_{\mathbb{D}} \frac{|[B[(I+R)b]\chi_{T(A)}](z)|^2}{\mathcal{V}^2(z)} dv(z)\\ \lesssim \int_{T(A)} \frac{|(I+R)b(z)|^2}{\mathcal{V}^2(z)} dv(z) &\lesssim \int_{T(A)} |(I+R)b(z)|^2 dv(z) = \mu_b(T(A)),\end{aligned}$$

where in the last estimate we have used that by Lemma 2.3, $\mathcal{V}^2(z) \ge C$ on $T(A)$.

In order to prove the estimate (3.5), we need some technical results, which we state in the following two lemmas.

**Lemma 3.3.** *There exists $C > 0$ such that for any compactly supported real-valued $\mathcal{C}^1$ function $\phi$ and any nonidentically zero positive measure $\nu$ supported on $\mathbb{T}$, if $\mathcal{U}$ is the holomorphic potential defined a.e. in $\mathbb{C}$ as in Lemma 2.3 and $\mathcal{V} = \mathrm{Re}\,\mathcal{U}$, we have that*

$$\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla\mathcal{V}(z)|^2}{(\mathcal{V}(z))^4} dv(z) \le C \int_{\mathbb{C}} \frac{|\nabla\phi(z)|^2}{(\mathcal{V}(z))^2} dv(z). \tag{3.6}$$

*Proof.* In order to prove the lemma, we will make the following reduction.

Let us consider the measures $\nu_\varepsilon = \mu * \varphi_\varepsilon$, regularizations of $\nu$, where $\varphi$ is a radial $\mathcal{C}^\infty$ function on $\mathbb{C}$, which is zero outside a neighborhood of 0 and such that $\int_{\mathbb{C}} \varphi = 1$ and $\varphi_\varepsilon(y) = \frac{1}{\varepsilon}\varphi(y/\varepsilon)$, for any $y \in \mathbb{C}$. It is enough to show the estimate (3.6) for the functions

$$\mathcal{V}^\varepsilon(z) = \int_{\mathbb{C}} \ln \frac{K}{|1 - z\bar\zeta|} d\nu_\varepsilon(\zeta),$$

with constants independent of $\varepsilon > 0$. Indeed, since $\nu_\varepsilon$ converges weakly to $\nu$, as $\varepsilon \to 0$, we have that both $\mathcal{V}^\varepsilon$ and $\nabla\mathcal{V}^\varepsilon$ converge pointwise almost everywhere to $\mathcal{V}$ and $\nabla\mathcal{V}$ respectively as $\varepsilon \to 0$. Then applying Fatou's lemma first and the Dominated Convergence Theorem (since by definition, $\frac{1}{\mathcal{V}^\varepsilon} \le C$ for any $\varepsilon > 0$), we have

$$\begin{aligned}\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla\mathcal{V}(z)|^2}{(\mathcal{V}(z))^4} dv(z) &\le \liminf_{\varepsilon\to 0} \int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla\mathcal{V}^\varepsilon(z)|^2}{(\mathcal{V}^\varepsilon(z))^4} dv(z)\\ \lesssim \liminf_{\varepsilon\to 0} \int_{\mathbb{C}} \frac{|\nabla\phi(z)|^2}{(\mathcal{V}^\varepsilon(z))^2} dv(z) &= \int_{\mathbb{C}} \frac{|\nabla\phi(z)|^2}{(\mathcal{V}(z))^2} dv(z).\end{aligned}$$

So we are left to show

$$\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla \mathcal{V}^\varepsilon(z)|^2}{(\mathcal{V}^\varepsilon(z))^4} dv(z) \lesssim \int_{\mathbb{C}} \frac{|\nabla \phi(z)|^2}{(\mathcal{V}^\varepsilon(z))^2} dv(z), \tag{3.7}$$

with constants independent of $\varepsilon > 0$. Applying integration by parts and using that the function $\phi$ is compactly supported, we then have

$$\begin{aligned}
&\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla \mathcal{V}^\varepsilon(z)|^2}{(\mathcal{V}^\varepsilon(z))^4} dv(z) \\
&= -\int_{\mathbb{C}} \phi(z)^2 \frac{\Delta \mathcal{V}^\varepsilon(z)}{(\mathcal{V}^\varepsilon(z))^3} dv(z) - 2\int_{\mathbb{C}} \phi(z)\nabla\phi(z)\cdot\nabla\mathcal{V}^\varepsilon(z)\frac{1}{\mathcal{V}^\varepsilon(z)^3} dv(z) \\
&\quad + 4\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla \mathcal{V}^\varepsilon(z)|^2}{(\mathcal{V}^\varepsilon(z))^4} dv(z).
\end{aligned}$$

Hence, since $-\Delta\mathcal{V}^\varepsilon(z) = \nu_\varepsilon$ is a positive measure,

$$\begin{aligned}
&3\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla \mathcal{V}^\varepsilon(z)|^2}{(\mathcal{V}^\varepsilon(z))^4} dv(z) \\
&= 2\int_{\mathbb{C}} \phi(z)\nabla\phi(z)\cdot\nabla\mathcal{V}^\varepsilon(z)\frac{1}{\mathcal{V}^\varepsilon(z)^3} dv(z) + \int_{\mathbb{C}} \phi(z)^2 \frac{\Delta \mathcal{V}^\varepsilon(z)}{(\mathcal{V}^\varepsilon(z))^3} dv(z) \\
&\le 2\int_{\mathbb{C}} \phi(z)\nabla\phi(z)\cdot\nabla\mathcal{V}^\varepsilon(z)\frac{1}{\mathcal{V}^\varepsilon(z)^3} dv(z).
\end{aligned}$$

Applying Hölder's inequality to the last estimate, we obtain

$$\begin{aligned}
&3\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla \mathcal{V}^\varepsilon(z)|^2}{(\mathcal{V}^\varepsilon(z))^4} dv(z) \\
&\le 2\left(\int_{\mathbb{C}} \frac{|\nabla\phi(z)|^2}{(\mathcal{V}^\varepsilon(z))^2} dv(z)\right)^{\frac{1}{2}} \left(\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla \mathcal{V}^\varepsilon(z)|^2}{(\mathcal{V}^\varepsilon(z))^4} dv(z)\right)^{\frac{1}{2}},
\end{aligned}$$

which gives that

$$\int_{\mathbb{C}} \phi(z)^2 \frac{|\nabla \mathcal{V}^\varepsilon(z)|^2}{(\mathcal{V}^\varepsilon(z))^4} dv(z) \le \left(\frac{2}{3}\right)^2 \int_{\mathbb{C}} \frac{|\nabla\phi(z)|^2}{(\mathcal{V}^\varepsilon(z))^2} dv(z),$$

and that gives (3.7) and ends the lemma. ■

The next lemma is a modification of a classical extension theorem on real Sobolev spaces (see [6]) to a weighted situation. We construct an extension to the whole plane of a function in the Sobolev space on the disc with respect to the weight $\frac{dv(z)}{\mathcal{V}(z)^2}$, with estimates of the corresponding norm. We recall that if $L > 0$, we will write $\mathcal{V}_L(z) = \int_{\mathbb{T}} \ln \frac{L}{|1 - z\overline{\zeta}|} d\nu(\zeta)$ when we want to specify the constant $L$. Namely, we have:

**Lemma 3.4.** *There exists $C > 0$, such that for any real-valued $\mathcal{C}^1$ function $\phi$ on $\overline{\mathbb{D}}$, there exists a real-valued compactly supported $\mathcal{C}^1$ extension to $\mathbb{C}$ of $\phi$, $\tilde{\phi}$, satisfying that*

$$\int_{\mathbb{C}} \frac{|\nabla\tilde{\phi}(z)|^2}{\mathcal{V}_{6K}(z)^2} dv(z) \le C\left(\int_{\mathbb{D}} \frac{|\nabla\phi(z)|^2}{\mathcal{V}_{2K}(z)^2} dv(z) + \int_{\mathbb{D}} \frac{|\phi(z)|^2}{\mathcal{V}_{2K}(z)^2} dv(z)\right). \tag{3.8}$$

*Proof.* We first extend the function $\phi$ to a neighborhood of the closed unit disc by defining the function

$$\phi^e(\rho e^{i\theta}) = 3\phi((2-\rho)e^{i\theta}) - 2\phi((3-2\rho)e^{i\theta}),$$

$1 < \rho < \frac{3}{2}$. We observe that $\lim_{\rho\to 1} \phi^e(\rho e^{i\theta}) = \phi(e^{i\theta})$, and $\lim_{\rho\to 1} \frac{\partial \phi^e}{\partial \rho}(\rho e^{i\theta}) = \frac{\partial \phi}{\partial \rho}(e^{i\theta})$, and consequently, $\phi^e$ is a $\mathcal{C}^1$ extension of $\phi$ to $|z| < \frac{3}{2}$. Let $0 < \varepsilon < \frac{1}{4}$ be fixed and let $\psi$ be a $\mathcal{C}^\infty$ function on $\mathbb{C}$ such that $0 \le \psi \le 1$, $\psi(z) = 1$ for $|z| < 1+\varepsilon$, and $\psi(z) = 0$ for $|z| > \frac{3}{2} - \varepsilon$. We consider the function $\tilde{\phi} = \psi\phi^e$, which is a $\mathcal{C}^1$ compactly supported function.

We have

$$\int_{\mathbb{C}} \frac{|\nabla\tilde{\phi}(z)|^2}{\mathcal{V}_{6K}(z)^2} dv(z) \lesssim \int_{|z|<\frac{3}{2}} \frac{|\nabla\phi^e(z)|^2}{\mathcal{V}_{6K}(z)^2} dv(z) + \int_{|z|<\frac{3}{2}} \frac{|\phi^e(z)|^2}{\mathcal{V}_{6K}(z)^2} dv(z) = I + II.$$

In order to estimate the integrals in both $I$ and $II$, we split both integrals into two pieces, one corresponding to $\mathbb{D}$, and the other corresponding to $1 \le |z| < \frac{3}{2}$. In this last region, we will apply an appropriate change of variables to transform the integrals into integrals over a subset of $\mathbb{D}$. We first observe that for any $0 < r < 1$, $1-r \le |re^{i\theta} - \zeta|$ and $|e^{i\theta} - \zeta| \le 2|re^{i\theta} - \zeta|$. Consequently, for $1 < \rho < \frac{3}{2}$, $|\rho e^{i\theta} - \zeta| \le |1-(2-\rho)| + |e^{i\theta} - \zeta| \le 3|(2-\rho)e^{i\theta} - \zeta|$. Similarly, $|\rho e^{i\theta} - \zeta| \le \frac{5}{2}|(3-2\rho)e^{i\theta} - \zeta|$.

Hence, we have that both $\ln \frac{6K}{|\rho e^{i\theta} - \zeta|} \ge \ln \frac{2K}{|(2-\rho)e^{i\theta} - \zeta|} \ge 1$ and $\ln \frac{6K}{|\rho e^{i\theta} - \zeta|} \ge \ln \frac{2K}{|(3-2\rho)e^{i\theta} - \zeta|} \ge 0$, and consequently,

$$\begin{aligned}
&\int_{1<|z|<\frac{3}{2}} \frac{|\nabla\phi^e(z)|^2}{\mathcal{V}_{3K}(z)^2} dv(z) \\
&\lesssim \int_{1<|z|<\frac{3}{2}} \frac{|\nabla\phi((2-|z|)\frac{z}{|z|})|^2}{\mathcal{V}_{6K}(z)^2} dv(z) + \int_{1<|z|<\frac{3}{2}} \frac{|\nabla\phi((3-2|z|)\frac{z}{|z|})|^2}{\mathcal{V}_{6K}(z)^2} dv(z) \\
&\lesssim \int_{1<|z|<\frac{3}{2}} \frac{|\nabla\phi((2-|z|)\frac{z}{|z|}))|^2}{\mathcal{V}_{2K}((2-|z|)\frac{z}{|z|}))^2} dv(z) + \int_{1<|z|<\frac{3}{2}} \frac{|\nabla\phi((3-2|z|)\frac{z}{|z|}))|^2}{\mathcal{V}_{2K}((3-2|z|)\frac{z}{|z|}))^2} dv(z) \\
&\lesssim \int_{\mathbb{D}} \frac{|\nabla\phi(z)|^2}{\mathcal{V}_{2K}(z)^2} dv(z).
\end{aligned}$$

This gives that

$$I \lesssim \int_{\mathbb{D}} \frac{|\nabla\phi(z)|^2}{\mathcal{V}_{2K}(z)^2} dv(z).$$

Next we deal with the integral given by $II$. Arguing as before, we have that

$$\int_{1<|z|<\frac{3}{2}} \frac{|\phi^e(z)|^2}{\mathcal{V}_{6K}(z)^2} dv(z) \lesssim \int_{\mathbb{D}} \frac{|\phi(z)|^2}{\mathcal{V}_{2K}(z)^2} dv(z). \qquad \blacksquare$$

**Theorem 3.5.** *There exists a constant $C > 0$ such that for any $h$ in the Dirichlet space $\mathcal{D}$,*

$$\int_{\mathbb{D}} |h(z)|^2 \frac{|R\mathcal{V}(z)|^2}{\mathcal{V}_{6K}(z)^4} dv(z) \le C \int_{\mathbb{D}} \frac{|(I+R)h(z)|^2}{\mathcal{V}_K(z)^2} dv(z).$$

*Proof.* Since $\mathcal{U}$ is holomorphic, $|R\mathcal{U}| \le |\nabla\mathcal{V}|$. If $\rho < 1$, and $h_\rho(z) = h(\rho z)$, we apply Lemmas 3.3 and 3.4 to $\operatorname{Re} h_\rho$ and $\operatorname{Im} h_\rho$, and we obtain that

$$\int_{\mathbb{D}} \frac{|h_\rho(z) R\mathcal{V}(z)|^2}{\mathcal{V}_{6K}(z)^4} dv(z) \lesssim \int_{\mathbb{D}} \frac{|\nabla h_\rho(z)|^2}{\mathcal{V}_{2K}(z)^2} dv(z).$$

Next we have that $\mathcal{V}_K(z) \lesssim \mathcal{V}_{2K}(z)$ and consequently, by Fatou's lemma,

$$\int_{\mathbb{D}} |h(z)|^2 \frac{|\nabla \mathcal{V}(z)|^2}{\mathcal{V}_{6K}(z)^4} dv(z) \leq \liminf_{\rho\to 1} \int_{\mathbb{D}} |h_\rho(z)|^2 \frac{|\nabla \mathcal{V}(z)|^2}{\mathcal{V}_{6K}(z)^4} dv(z)$$
$$\lesssim \liminf_{\rho\to 1} \int_{\mathbb{D}} \frac{|\nabla h_\rho(z)|^2}{\mathcal{V}_{2K}(z)^2} dv(z) \lesssim \int_{\mathbb{D}} \frac{|\nabla h(z)|^2}{\mathcal{V}_K(z)^2} dv(z).$$

Finally, the pointwise estimate in Lemma 3.6 in [2] gives that if $\alpha < \beta$, there exists $C > 0$ such that for any $\zeta \in \mathbb{T}$,

$$\int_{D_\alpha(\zeta)} |\nabla h(z)|^2 (1-|z|^2)^{-1} dv(z) \lesssim \int_{D_\beta(\zeta)} |Rh(z)|^2 (1-|z|^2)^{-1} dv(z).$$

Consequently, multiplying by $1/(\mathcal{V}_K^*)^2(\zeta)$, integrating and applying Lemma 2.7 to the weight $1/(\mathcal{V}_K^*)^2(\zeta)$ (which is also in the class $A_2$), we easily obtain

$$\int_{\mathbb{D}} \frac{|\nabla h(z)|^2}{\mathcal{V}_K(z)^2} dv(z) \lesssim \int_{\mathbb{D}} \frac{|Rh(z)|^2}{\mathcal{V}_K(z)^2} dv(z).$$

Finally, a standard argument using integral representations (see, for instance, Theorem 2.8 in [7]) gives that the right-hand side of the above integral is bounded up to a constant by

$$\int_{\mathbb{D}} \frac{|Rh(z)|^2}{\mathcal{V}_K(z)^2} dv(z) \lesssim \int_{\mathbb{D}} \frac{|(I+R)h(z)|^2}{\mathcal{V}_K(z)^2} dv(z). \qquad \blacksquare$$

Now we can finish the proof of (i) implies (ii), proving the estimate (3.5). Since $|\mathcal{U}| \simeq \mathcal{V}$, Theorem 3.5 and the argument used in the proof of (3.4) give the proof. $\blacksquare$

## References


[1] D.R. Adams and L.I. Hedberg: Function spaces and potential theory. Grundlehren der Mathematischen Wissenschaften 314. Springer-Verlag, Berlin, 1996. MR1411441 (97j:46024)

[2] P. Ahern and J. Bruna: Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of $\mathbb{C}^n$. Revista Matemática Iberoamericana **4** (1988), 123–152. MR1009122 (90h:32011)

[3] P. Ahern and W. Cohn: Exceptional sets for Hardy Sobolev functions, $p > 1$. Indiana Univ. Math. J. **38** (1989), 417–453. MR997390 (90m:32013)

[4] N. Arcozzi, R. Rochberg, E. Sawyer and B.D. Wick: Bilinear forms on the Dirichlet space, Analysis & PDE **3** (2010), no. 1, 21-47. MR2663410 (2011f:31011)

[5] D. Békollé: Inégalité à poids pour le projecteur de Bergman dans la boule unité de $\mathbf{C}^n$. Studia Math. **71** (1981/82), no. 3, 305–323. MR667319 (83m:32004)

[6] A.P. Calderon: Lebesgue spaces of differentiable functions and distributions. Proc Sympos. Pure Math. Vol. IV, 33–49, American Mathematical Society, Providence, R.I. MR0143037 (26:603)

[7] C. Cascante and J.M. Ortega: Carleson measures for weighted Hardy-Sobolev spaces. Nagoya Math. J. **186** (2007), 29–68. MR2334364 (2008g:32011)

[8] C. Cascante and J.M. Ortega: Carleson measures for weighted holomorphic Besov spaces, Ark. Mat. **49** (2011), 31–59. MR2784256

[9] V.G. Maz′ya and T.O. Shaposhnikova: Theory of multipliers in spaces of differentiable functions, Monographs and Studies in Mathematics 23, Pitman, 1985. MR785568 (87j:46074)

[10] V.G. Maz′ya and I.E. Verbitsky: The Schrödinger operator on the energy space: boundedness and compactness criteria. Acta Math. **188** (2002), 263–302. MR1947894 (2004b:35050)

[11] N.G. Meyers: A theory of capacities for potentials of functions in Lebesgue classes. Math. Scand. **26** (1971), 255–292. MR0277741 (43:3474)

[12] D. Stegenga: Multipliers of the Dirichlet space. Illinois J. Math. **24** (1980), 113–139. MR550655 (81a:30027)

Departmento Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

*E-mail address*: cascante@ub.edu

Departmento Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

*E-mail address*: ortega@ub.edu